Lecture 17 Argument principle, complex inverse fcn theorem

Theorem If F is a non-vanishing analytic fcn on a convex open set (or on any domain where the Cauchy theorem holds) then $F = e^G$ for some analytic fcn G there. Consequently, F has analytic N-th roots too: $H = e^{G/N}$ is an N-th root because $H^N = e^G = F$.

Argument principal for a disc: Suppose f analytic on $D_R(z_0)$ and $0 < r < R$ and f has no zeroes on $C_r(z_0) = \{z : |z - z_0| = r\}$. Then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = \begin{pmatrix} \#\text{zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity} \end{pmatrix}$$

Def^n: If g has a pole at a with principal part

$$\frac{A_N}{(z-a)^N} + \cdots + \frac{A_1}{z-a},$$

then A_1 is <u>residue</u> of g at a. Write $A_1 = \mathrm{Res}_a\, g$.

Key fact: If f has a zero of multiplicity m at a, then the principal part of $\frac{f'}{f}$ at a

is $\frac{m}{z-a}$, i.e., $\frac{f'}{f}$ has a simple pole (order one)

and $Res_a \frac{f'}{f} = m$.

Pf of Arg princ: Assume hyp. Claim: f has finitely many zeroes in $D_r(z_0)$. [If infinitely many on $D_r(z_0)$, so ∞ many on $\overline{D_r(z_0)}$ ← compact, zeroes would have a limit pt in $\overline{D_r(z_0)}$. Certainly not on $C_r(z_0)$. Limit pt in $D_r(z_0) \Rightarrow f \equiv 0$ there ↯.]

Say $a_1, \dots, a_N$ are the zeroes, mult of zero at a_j is m_j. Now

$$\underbrace{\frac{f'}{f} - \sum_{j=1}^{N} \frac{m_j}{z-a_j}}_{\text{removable sing at each } a_j} = \underbrace{F}_{\text{analytic on an open disc } D_\rho(z_0),\ r<\rho<R.}$$

$$\int_{C_r} \frac{f'}{f}\,dz - \sum_{j=1}^{N} m_j \underbrace{\int_{C_r(z_0)} \frac{1}{z-a_j}\,dz}_{2\pi i} = \underbrace{\int_{C_r(z_0)} F\,dz}_{\substack{=0 \\ \text{by Cauchy's thm}}}$$

So $\frac{1}{2\pi i}\int_{C_r} \frac{f'}{f}\,dz = \sum_{j=1}^{N} m_j$ ⟵ # zeroes of f inside $C_r(z_0)$ counted with mult.

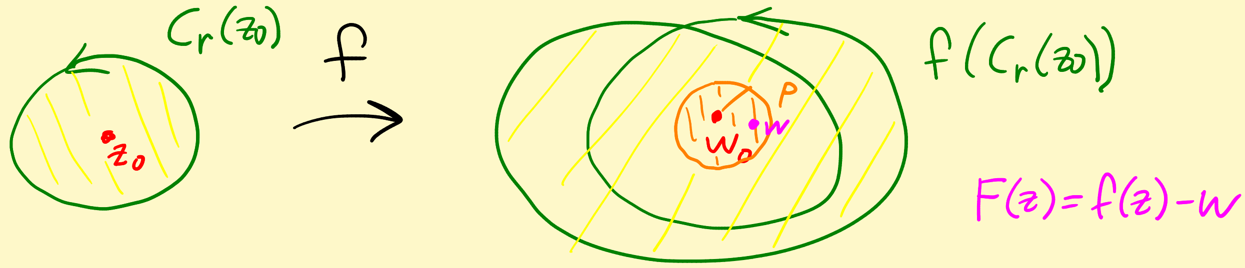
Pf of the O.M.T.: f analytic on a domain Ω and not constant. Want to see $f(\Omega)$ is open.

Suppose $w_0 \in f(\Omega)$. Then $\exists\ z_0 \in \Omega$ with $f(z_0) = w_0$.

$f(z) - w_0$ is not constant, so its zero at z_0 is isolated. So $\exists\ r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$ and z_0 is the only zero of $f(z) - w_0$ in $\overline{D_r(z_0)}$.

Let N = order of zero of $f(z) - w_0$ at z_0

[We say $f(z_0) = w_0$ with multiplicity N.]



$C_r(z_0)$: $z(t) = z_0 + re^{it}$
$0 \le t \le 2\pi$

$f(C_r(z_0))$: $f(z(t))$
$0 \le t \le 2\pi$

$|f(z(t)) - w_0|$ is non-vanishing on $[0, 2\pi]$, so it has a minimum $m > 0$. Pick ρ with $0 < \rho < m$.

Claim: $D_\rho(w_0)$ gets hit by f on $D_r(z_0)$.

Key: Arg prince

$$\left(\begin{array}{c}\#\text{ times } f \text{ hits } w \\ \text{in } C_r(z_0)\end{array}\right) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z) - w}\, dz = H(w)$$

<u>Claim</u>: $H(w)$ is analytic on $D_\rho(w_0)$ and <u>integer</u> valued, so constant! $H(w_0) = N \geq 1$. So $H(w) \equiv N \geq 1$.

Every w gets hit. ✓ w_0 arbitrary $\Rightarrow f(\Omega)$ open.

<u>Why</u>: $$H(w) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)-w}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(z(t))}{\underbrace{f(z(t))}_{\zeta(t)}-w}\, z'(t)\,dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\zeta'(t)}{\zeta(t)-w}\,dt$$

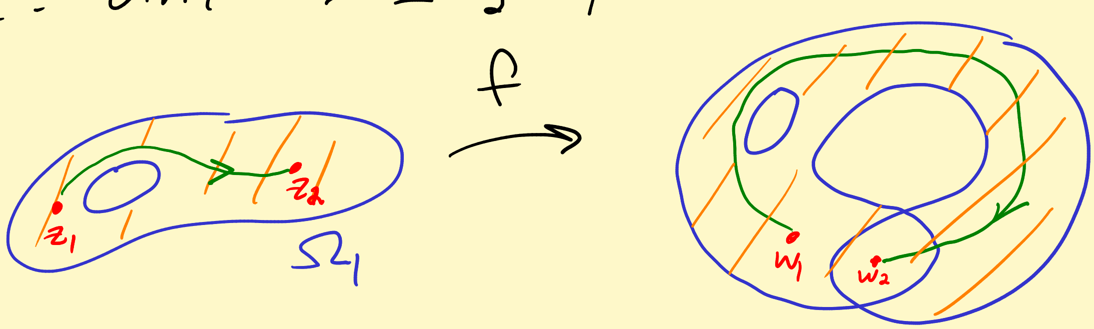
$$= \frac{1}{2\pi i}\int_{f(C_r(z_0))} \frac{1}{\zeta - w}\,d\zeta$$

HWK 2: #1 $\Rightarrow H(w)$ analytic. ✓

<u>Corolaries of the OMT</u>

<u>Cor</u>. $f : \Omega_1 \to \Omega_2$ analytic on a domain Ω_1 and $\Omega_2 = f(\Omega_1)$, f non-constant, then Ω_2 is a <u>domain</u>.

<u>Pf</u>: OMT $\Rightarrow \Omega_2$ open.



✓

Cor of OMT: $f : \Omega_1 \xrightarrow[\text{onto}]{1\text{-}1} \Omega_2$, Ω_1 domain, f analytic, $\Omega_2 = f(\Omega_1)$.

1) Then $F = f^{-1}$ is continuous by OMT.

2) and F is analytic on Ω_2

3) and $F'(f(z)) = \frac{1}{f'(z)}$ ← non-vanishing

Fact: Locally 1-1 analytic fcns have non-vanishing derivatives.

Pf: Last time: $w = f(z)$ $\quad F(w) = z$

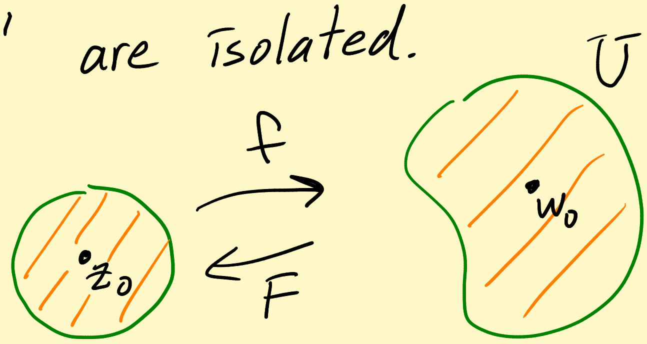
$w_0 = f(z_0)$ $\quad F(w_0) = z_0$

$$F \text{ cont} \Rightarrow \underbrace{F(w)}_{z} \to \underbrace{F(w_0)}_{z_0} \text{ as } w \to w_0$$

$$DQ = \frac{F(w) - F(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \to \frac{1}{f'(z_0)}$$

as $z \to z_0$ provided that $f'(z_0) \neq 0$.

Claim: f non-const $\Rightarrow f' \not\equiv 0$. So zeroes of f' are isolated.



F' exists and is non-vanishing on $U - \{w_0\}$.

F cont on U.

w_0 is removable!

$$F(f(z)) = z \quad \text{on}$$

$$\underbrace{F'(f(z))\, f'(z)}_{\text{neither can vanish!}} = 1$$