Lecture 18 Inverse fcn thm, Local mapping theorem

HWK 4 due Wed in class

Cor of OMT : $f : \Omega_1 \xrightarrow[\text{onto}]{1\text{-}1} \Omega_2$, Ω_1 domain, f analytic, $\Omega_2 = f(\Omega_1)$.

1) Then $F = f^{-1}$ is continuous by OMT.

2) and F is analytic on Ω_2

3) and $F'(f(z)) = \dfrac{1}{f'(z)}$ ← non-vanishing

False $\mathbb{R} \to \mathbb{R}$: $h(t) = t^3$: $(-1,1) \xrightarrow[\text{onto}]{1\text{-}1} (-1,1)$, $h^{-1}(\tau) = \tau^{1/3}$.

Locally 1-1 analytic fcns have non-vanishing derivatives.

$w = f(z) \qquad F(w) = z$

$w_0 = f(z_0) \qquad F(w_0) = z_0$

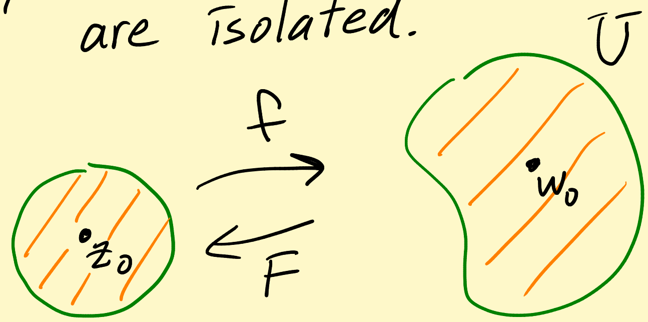
$$F \text{ cont} \Rightarrow \underbrace{F(w)}_{z} \to \underbrace{F(w_0)}_{z_0} \quad \text{as} \quad w \to w_0$$

$$DQ = \frac{F(w) - F(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \longrightarrow \frac{1}{f'(z_0)}$$

as $z \to z_0$ provided that $f'(z_0) \neq 0$.

f non-const $\Rightarrow f' \not\equiv 0$. So zeroes of f' are isolated.

($f'(z_0) = 0$, isolated)



F' exists and is non-vanishing on $U - \{w_0\}$.

F cont on $\overline{U}$.

w_0 is removable!

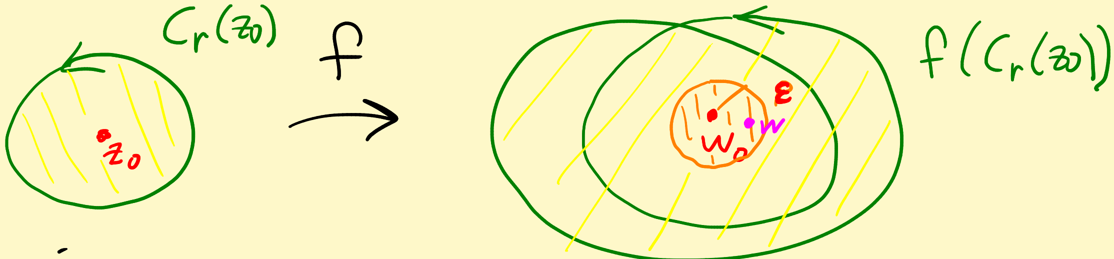
F has a removable sing at w_0!

$$F(f(z)) = z$$

$$\underbrace{F'(f(z))}_{\text{can't vanish!}}\, \underbrace{f'(z)} \equiv 1 \qquad F'(f(z)) = \frac{1}{f'(z)}$$

$$F'(w) = \frac{1}{f'(f^{-1}(w))}$$

Super inverse fcn thm!



z_0 is an isolated zero of $f(z) - w_0$ on $\overline{D_r(z_0)}$.

[Note: If $f'(z_0) \neq 0$, z_0 is a <u>simple</u> zero of $f(z) - w_0$.

<u>Standard inverse fcn thm:</u> Suppose f is analytic on a $D_R(z_0)$ and $f'(z_0) \neq 0$. Say $f(z_0) = w_0$.

Then $\exists\ \varepsilon > 0$ and a domain $U \subset D_R(z_0)$ such that $f : U \xrightarrow[\text{onto}]{1-1} D_\varepsilon(w_0)$ and f^{-1} is analytic on $D_\varepsilon(w_0)$ and $(f^{-1})'(w) = \dfrac{1}{f'(f^{-1}(w))}$.

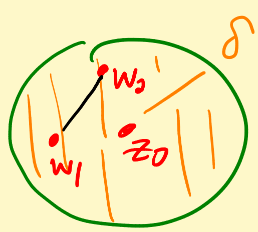
<u>Pf</u>: $H(w) = \dfrac{1}{2\pi i} \displaystyle\int_{C_r} \frac{f'(z)}{f(z) - w}\, dz$ = # times w gets hit by f

Know $H(w_0) = 1$. Everything in $D_\varepsilon(w_0)$ gets hit <u>exactly</u> once by f on $D_r(z_0)$. Let $U = f^{-1}(D_\varepsilon(w_0))$. Done ✓

<u>Another way to see $f'(z_0) \neq 0 \Rightarrow f$ locally 1-1.</u>

$f'(z) = f'(z_0) + E(z)$ where $E(z) \to 0$ as $z \to z_0$.

So $\exists\ \delta > 0$ such that $|E(z)| < \varepsilon < \dfrac{|f'(z_0)|}{2}$ on $\overline{D_\delta(z_0)}$.



$$f(w_2) - f(w_1) = \int_{L_{w_1}^{w_2}} f'(z)\, dz = \int_{L_{w_1}^{w_2}} f'(z_0) + E(z)\, dz$$

$$= f'(z_0)(w_2 - w_1) + \underbrace{\int_{L_{w_1}^{w_2}} E(z)\, dz}_{\varepsilon}$$

$$|f(w_2) - f(w_1)| \geq \Big| \underbrace{|f'(z_0)(w_2-w_1)|}_{|f'(z_0)|\,|w_2-w_1|} - \underbrace{|\mathcal{E}|}_{\text{small!}} \Big| > 0$$

$$|\mathcal{E}| \leq \underbrace{\underset{L_{w_1}^{w_2}}{\mathrm{Max}} |E(z)|}_{< \frac{|f'(z_0)|}{2}} \; \underbrace{\mathrm{Length}\left(L_{w_1}^{w_2}\right)}_{|w_2-w_1|} \qquad \text{See}$$

Local mapping theorem: Say $f(z_0) = w_0$ with mult N, meaning $f(z) - w_0$ vanishes to order N at z_0.

Step 1: Get disc $D_\rho(z_0)$ so that z_0 is only zero of $f(z) - w_0$ in $\overline{D_\rho(z_0)}$.

Step 2 $\quad f(z) - w_0 = (z - z_0)^N F(z)$ where $F(z)$ analytic on $D_\rho(z_0)$ and $F(z_0) \neq 0$.

See $F(z) \neq 0$ when $z \in \overline{D_\rho(z_0)}$.

Step 3 Know $\exists$ analytic $g(z)$ on $D_\rho(z_0)$ with

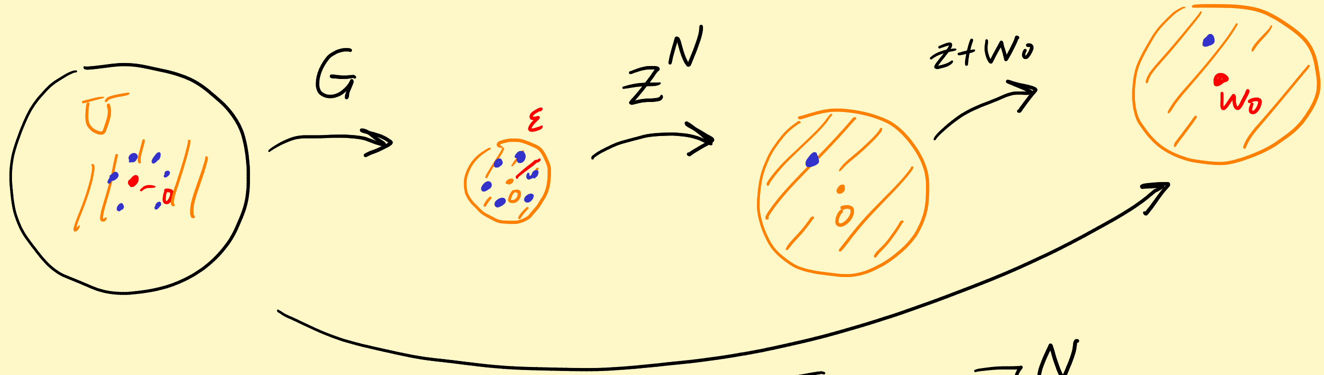
$$g(z)^N = F(z). \qquad g(z_0)^N = F(z_0) \leftarrow g(z_0) \neq 0$$

Step 4:
$$f(z) - w_0 = (z - z_0)^N \left[g(z)\right]^N$$
$$= \Big[\underbrace{(z - z_0)\, g(z)}_{G(z)} \Big]^N$$

Step 5: $G'(z) = 1 \cdot g(z) + (z-z_0)\, g'(z)$

So $G'(z_0) = g(z_0) + 0 \leftarrow$ not zero!

Step 6: Use inverse fcn thm. Get $D_\varepsilon(0)$ and domain $U \subset D_\rho(z_0)$ such that $G: U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(0)$, G analytic, G^{-1} analytic (non-vanishing derivatives).

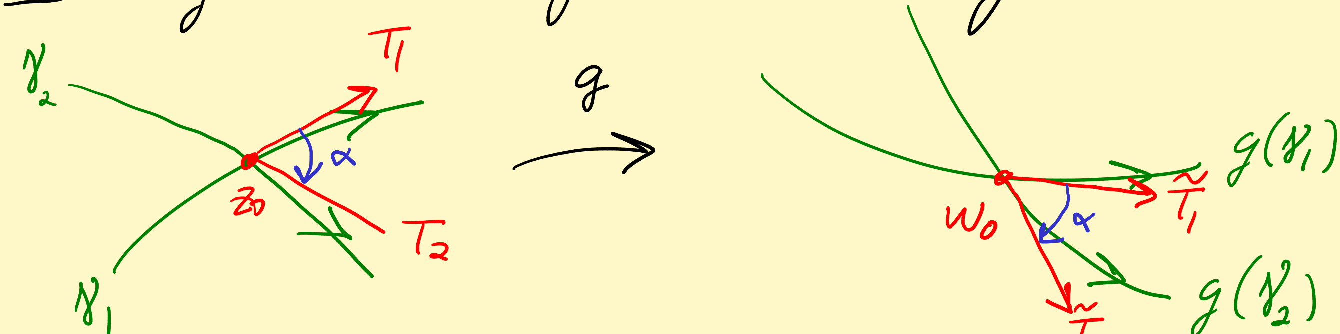


$$f(z) = w_0 + \left[G(z) \right]^N$$

See that $f: U - \{z_0\} \xrightarrow{N\text{-to-one}} D_\varepsilon(w_0) - \{w_0\}$

Big fact Analytic fcn g that is locally one-to-one preserves angles. [It is conformal.]

Pf: g 1-1 $\Rightarrow g'$ non-vanishing.



$\gamma_j : z_j(t)$, $z_j(0) = z_0$ $\qquad T_j = z_j'(0) \quad j=1,2$

$g(\gamma_j) : g(z_j(t))$. $\qquad \tilde{T}_j = g'(z_0) \underbrace{z_j'(0)}_{T_j}$

See $\tilde{T}_j = \underbrace{g'(z_0)}_{r\underbrace{e^{i\theta}}_{\text{rotation by } \theta}} T_j \qquad j=1,2$ ✓