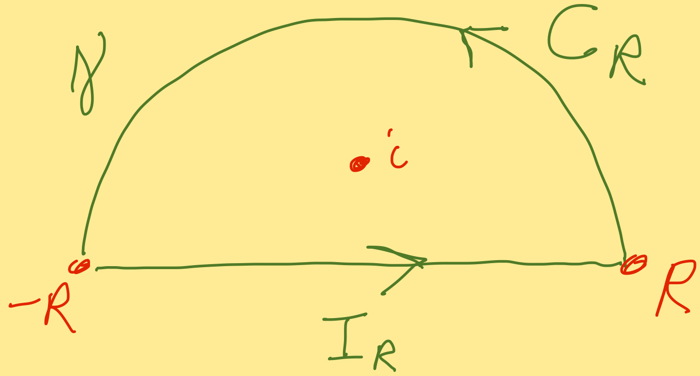
Lecture 21 Applications of the Residue thm

Fourier transform: $\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$

EX: $f(x) = \frac{1}{x^2+1}$ Key: $e^{isz} = e^{is(x+iy)} = e^{-sy} e^{isx}$

so $|e^{isz}| = e^{-sy} \begin{cases} < 1 & s>0,\ y>0 \quad \text{UHP} \\ < 1 & s<0,\ y<0 \quad \text{LHP} \end{cases}$

Case $s > 0$:



$$\left| \int_{C_R} \frac{e^{isz}}{z^2+1}\, dz \right| \le \underbrace{\left(\max_{C_R} \left| \frac{e^{isz}}{z^2+1} \right| \right)}_{\le \frac{1}{R^2-1} \text{ if } R>1} \underbrace{\text{Length}(C_R)}_{\pi R}$$

$$\underbrace{\qquad\qquad}_{\frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty}$$

Notes: $|e^{isz}| \le 1$ on C_R

$|z^2+1| \ge \underbrace{|R^2-1|}_{= R^2-1 \text{ if } |z|=R>1}$

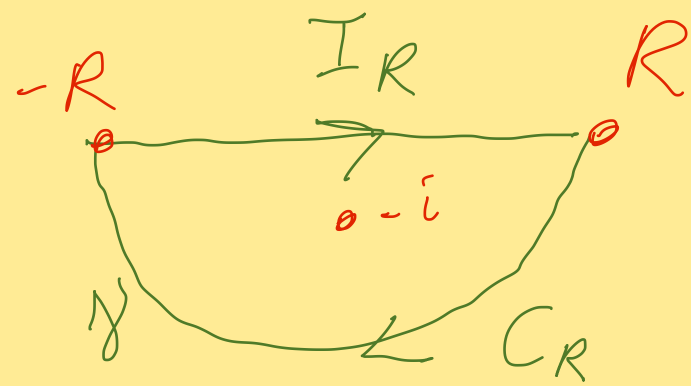
$$\int_{I_R} \frac{e^{isz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{ist}}{t^2+1}\cdot 1\,dt \quad \leftarrow \text{Want}$$

$$\underbrace{\int_\gamma \frac{e^{isz}}{z^2+1}\,dz}_{\int_{C_R}+\int_{I_R}} = 2\pi i\, \mathrm{Res}_i \frac{e^{isz}}{z^2+1} \quad \begin{matrix}\leftarrow f(z) \\ \leftarrow g(z) \\ \text{simple zero}\end{matrix}$$

$$= 2\pi i \frac{e^{isi}}{2i} = 2\pi i \frac{f(i)}{g'(i)}$$

$$\int_{C_R} \xrightarrow{} 0 \quad + \quad \int_{I_R} \to \hat{f}(s) \qquad = \pi e^{-s}$$

Case $s<0$:

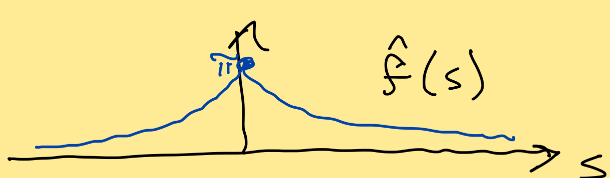


$$\underbrace{\int_\gamma \frac{e^{isz}}{z^2+1}\,dz}_{\int_{C_R}+\int_{I_R}} = -2\pi i\, \mathrm{Res}_{-i} \frac{e^{isz}}{z^2+1}$$

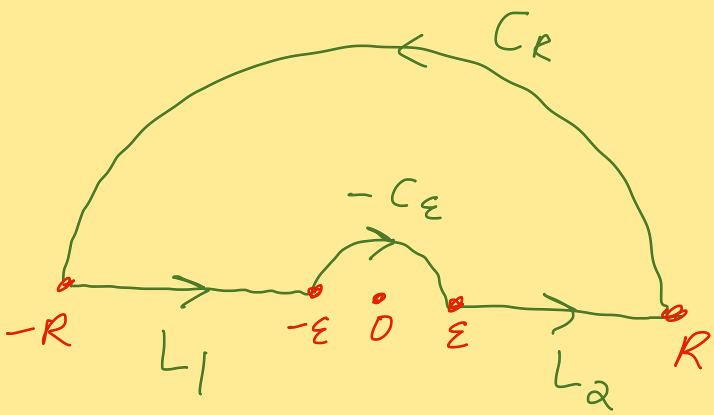
$\uparrow$ γ clockwise

$$= -2\pi i \frac{e^{is(-i)}}{2(-i)} = \pi e^{s}$$

$$\int_{C_R} \to 0 \quad + \quad \int_{I_R} \to \hat{f}(s)$$



$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \pi$$



Key: Taking $f(z) = \frac{\sin z}{z}$ doesn't work,

but $f(z) = \frac{e^{iz}}{z}$ does.

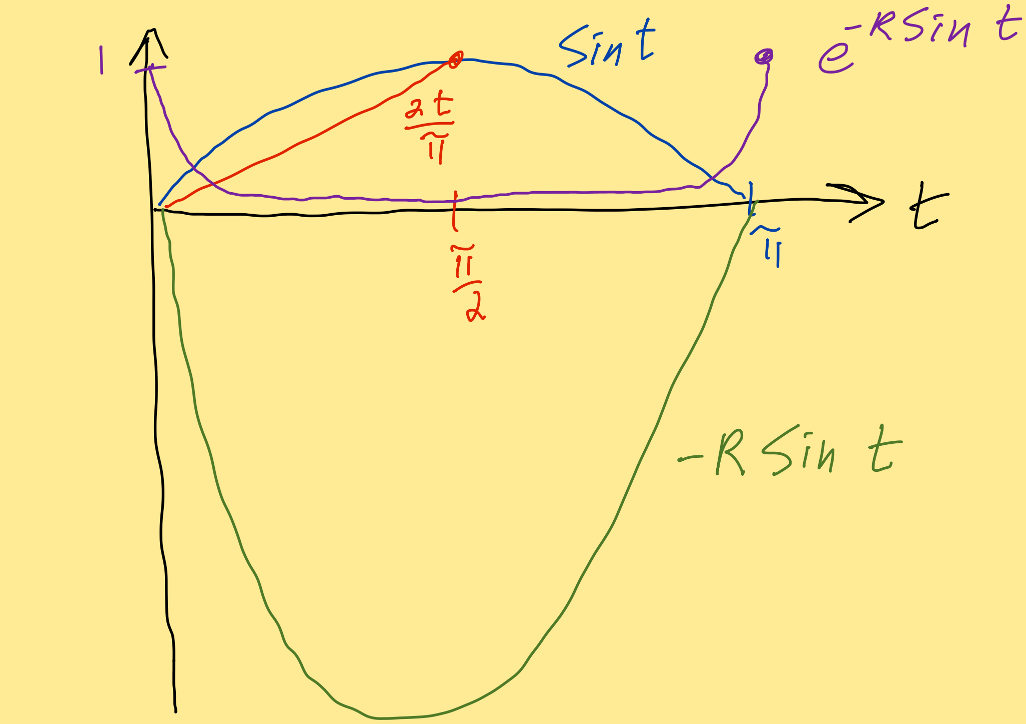
$$\left(\int_{L_1} + \int_{L_2} \right) \frac{e^{iz}}{z}\, dz = \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \underset{\text{odd}}{\frac{\cos t}{t}} + i \underset{\text{even}}{\frac{\sin t}{t}}\, dt$$

$$= 2i \int_{\varepsilon}^{R} \frac{\sin t}{t}\, dt \leftarrow \text{want!}$$

$$\int_{C_R} \frac{e^{iz}}{z}\, dz = \int_0^{\pi} \frac{e^{i(R\cos t + iR\sin t)}}{Re^{it}}\, iRe^{it}\, dt$$

$$= i \int_0^{\pi} e^{-R\sin t}\, e^{iR\cos t}\, dt$$

$$\text{So } \left| \int_{C_R} \right| \le \underbrace{\int_0^{\pi} \underbrace{e^{-R\sin t}}_{\text{symmetric about } t = \frac{\pi}{2}}\, dt}_{= 2\int_0^{\pi/2} e^{-R\sin t}\, dt}$$



$$\frac{2t}{\pi} \le \sin t \quad \text{on } \left[0, \tfrac{\pi}{2}\right], \text{ so}$$

$$-R\sin t \le -\frac{2R}{\pi} t \quad \text{on } \left[0, \tfrac{\pi}{2}\right] \text{ and}$$

$$e^{-R\sin t} \le e^{-\frac{2R}{\pi} t} \quad \text{on } \left[0, \tfrac{\pi}{2}\right] :$$

$$\left| \int_{C_R} \right| \le 2 \int_0^{\pi/2} e^{-R\sin t}\, dt$$

$$\le 2 \int_0^{\pi/2} e^{-\frac{2R}{\pi} t}\, dt$$

$$= 2 \left[-\frac{\pi}{2R} e^{-\frac{2R}{\pi} t} \right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left[1 - e^{-R}\right]$$

$$< \frac{\pi}{R} \to 0 \quad \text{as } R \to \infty.$$

Finally, near $z=0$,
$$\frac{e^{iz}}{z} = \frac{\left(1 + iz - \frac{z^2}{2!} + \cdots\right)}{z}$$

$$= \frac{1}{z} + \underbrace{\left(i - \frac{z}{2} + \cdots\right)}_{H(z) \text{ entire}}$$

$|H(z)| \le M$ on $\overline{D_1(0)}$.

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = -\int_{C_\varepsilon} \frac{1}{z}\,dz - \int_{C_\varepsilon} H(z)\,dz$$

$$= \underbrace{-\int_0^\pi \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt}_{-i\pi} - \mathcal{E}$$

where $|\mathcal{E}| \le \left(\max_{C_\varepsilon} |H|\right) \pi\varepsilon \le \pi M \varepsilon$

if $0 < \varepsilon < 1$ and so $\mathcal{E} \to 0$ as $\varepsilon \to 0$.

Finally

$$\left[\underbrace{\left(\int_{L_1} + \int_{L_2} \right)}_{2i \int_\varepsilon^R \frac{\sin t}{t}\,dt \;\to\; i \int_0^\infty \frac{\sin t}{t}\,dt} + \underbrace{\int_{C_R}}_{\to\, 0} + \underbrace{\int_{-C_\varepsilon}}_{\to\, -i\pi} \right] \frac{e^{iz}}{z}\,dz \underset{\text{Cauchy Thm}}{=} 0$$

Note: First let $\varepsilon \searrow 0$.
Then let $R \nearrow \infty$.

Get $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$ ✓