Lecture 33 Proof of the Riemann Mapping Theorem

Thm Ω simply connected domain $\neq \mathbb{C}$. There is $f : \Omega \xrightarrow[\text{onto}]{1\text{-}1} D_1(0)$ analytic.

Pf: **Lemma**: $\exists\, g : \Omega \to D_1(0)$ analytic and one-to-one.

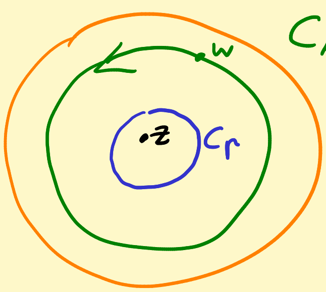
Pf: Last time. (Need simp connected for complex $\sqrt{\ }$!)

Lemma: If f_n analytic on domain Ω and $f_n \twoheadrightarrow f$, then $f_n^{(k)} \twoheadrightarrow f^{(k)}$ too!

Way false $\mathbb{R} \to \mathbb{R}$: $h_N(t) = \sum_{n=1}^{N} \frac{1}{2^n} \sin(2^n t)$

or $\frac{1}{n}\sin(2^n t) \twoheadrightarrow 0$, but derivatives don't.

Pf of lemma:



$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w) - f(w)}{(w-z)^{k+1}}\, dw \right|$$

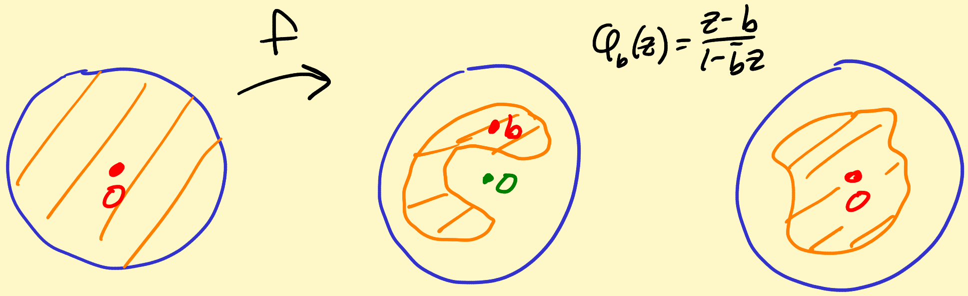
$$\leq \frac{k!}{2\pi} \underbrace{\left(\operatorname*{Max}_{C_R} |f_n - f| \right)}_{\to 0 \text{ because } C_R \text{ is compact.}} \frac{1}{(R-r)^{k+1}} \cdot (2\pi R)$$

Given compact $K \subset \Omega$, cover by blue circles.

Lemma $f : D_1(0) \circlearrowleft$ analytic. Then $|f'(0)| \leq 1$.

If $|f'(0)|=1$, then $f(z)=\lambda z$ for λ const, $|\lambda|=1$.

Pf:



Schwarz: $|(\varphi_b \circ f)'(0)| \le 1$

$$\underbrace{|\varphi_b'(b)|}_{\frac{1}{1-|b|^2}}\,|f'(0)| \le 1$$

$$|f'(0)| \le 1-|b|^2 \le 1 \checkmark$$

If $|f'(0)|=1$, then b must $=0$. $\varphi_0(z)=z$.

Schwarz part 2 $\Rightarrow$ $f(z)=\lambda z$. ✓

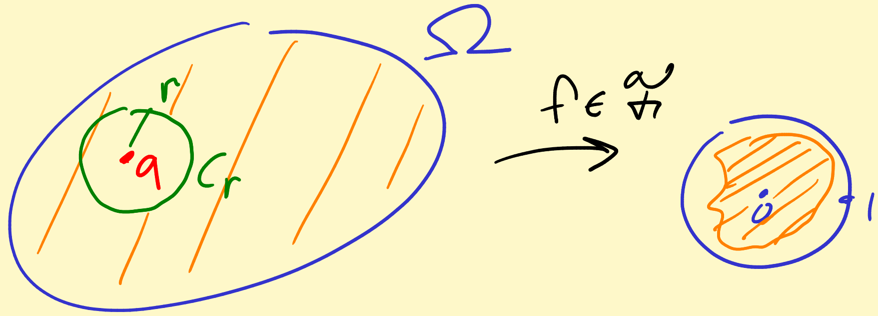
Pf of RMT: Let $\tilde{\mathcal{F}}$ = family of analytic fcns $f:\Omega \to D_1(0)$ that are one-to-one.

Lemma from last time $\Rightarrow$ $\tilde{\mathcal{F}} \neq \emptyset$.

Pick $a \in \Omega$. Let $M = \sup\{|f'(a)| : f \in \tilde{\mathcal{F}}\}$.

Step 1: Claim $0 < M < \infty$

f 1-1 $\Rightarrow f'(0) \neq 0$. ✓



Cauchy estimate: $|f'(a)| \le \dfrac{\max_{C_r}|f|}{r} < \dfrac{1}{r}$

So $M < \frac{1}{r}$.

<u>Remark</u>: Can let $r \nearrow \operatorname{dist}(a, b\Omega)$. Get

$$|f'(a)| \le \frac{1}{\operatorname{dist}(a, b\Omega)}$$

<u>Step 2</u> Take seq $f_n \in \tilde{\mathcal{F}}$ such $|f_n'(a)| \to M$.

Fcns in $\tilde{\mathcal{F}}$ are all bnd by 1. So Montel's $\Longrightarrow$

$\exists$ subseq $f_{n_k} \rightrightarrows F$ on Ω. Note F is analytic.

a Riemann map!

<u>Step 3</u> Today's lemma $\Longrightarrow$ $f_{n_k}'(a) \to F'(a)$.

So $|F'(a)| = M$.

Note: $|F'(a)| = M \neq 0$. So F is <u>not</u> constant.

Hurwicz #2 $\Longrightarrow$ F is 1-1.

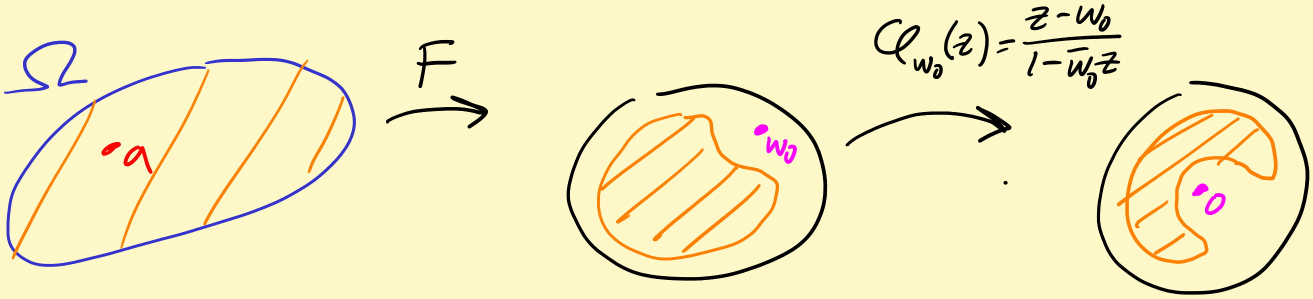
<u>Step 4</u> $F \in \tilde{\mathcal{F}}$. Just need to see $F : \Omega \to D_1(0)$.

$$f_{n_k} : \Omega \to D_1(0) \qquad |f_{n_k}(z)| < 1$$

$$f_{n_k}(z) \to F(z) \;:\; |F(z)| \leq 1.$$

Aha! If $|F(z)| = 1$ for some $z \in \Omega$, then Max Mod Thm $\Longrightarrow F \equiv$ const. It isn't const. So $|F(z)| < 1$. ✓

<u>Step 5</u> F is <u>onto</u>. Suppose not.

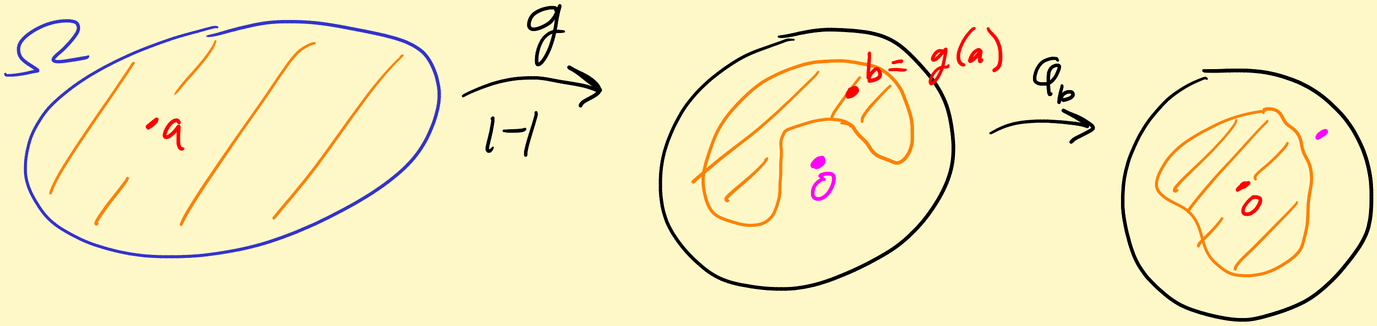


Notice that $\varphi_{w_0} \circ F$ is <u>non-vanishing</u>. So $\exists$ analytic square root $g(z)$ with $g(z)^2 = \varphi_{w_0}(F(z))$.

Note: $|g(z)|^2 < 1 \Longrightarrow |g(z)| < 1$. So $g : \Omega \to D_1(0)$.

<u>Claim:</u> g 1-1. So $g \in \tilde{\mathcal{F}}$.

<u>Why</u>:
$$g(z_1) = g(z_2)$$
$$g(z_1)^2 = g(z_2)^2$$
$$\varphi_{w_0}(F(z_1)) = \varphi_{w_0}(F(z_2)) \qquad (\varphi_{w_0} \text{ 1-1})$$
$$\Longrightarrow F(z_1) = F(z_2) \Longrightarrow z_1 = z_2 \; ✓$$



Drum roll. $\varphi_b \circ g \in \widetilde{\mathcal{F}}$.

Claim: $|(\varphi_b \circ g)'(a)| > M$ ↯ So no w_0. F was onto.

Write: $\tilde{F} = \varphi_b \circ g$

Check that $F = G \circ \tilde{F}$

where $G = \underbrace{\varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1}}_{\text{analytic. } D_1(0)}$, $s(z) = z^2$

Lemma $\Rightarrow$ $|G'(0)| \leq 1$. Hmmm. G is <u>not</u> 1-1

because of square piece. So $G(z) \not\equiv \lambda z$.

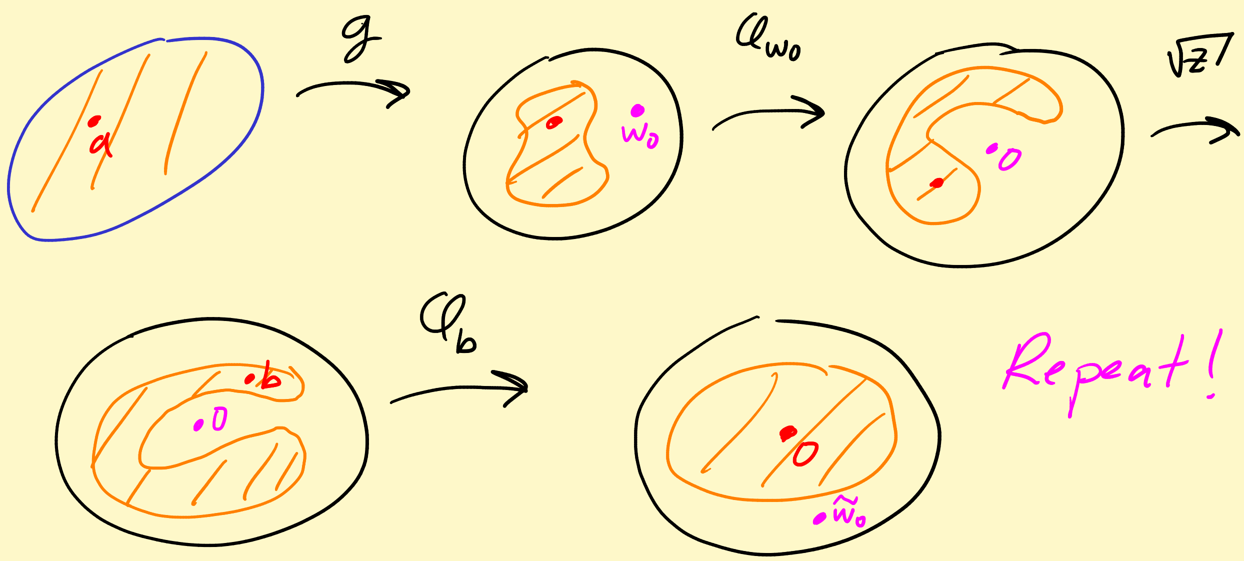
So $|G'(0)| < 1$.

Finally, $F'(a) = G'(\underbrace{\tilde{F}(a)}_{0})\, \tilde{F}'(a)$

$$\underbrace{|F'(a)|}_{M} = \underbrace{|G'(0)|}_{\substack{<1, \\ \neq 0}}\, |\tilde{F}'(a)|$$

See $|\tilde{F}'(a)| > M$. Done! F is onto!

Koebe's constuctive pf of RMT.



Repeat! $\tilde{w}_0$

Remark: The Riemann map associated to $a \in \Omega$

is the F with $F(a) = 0$, and $F'(a) \in \mathbb{R}^+$.

Unique.