Lecture 36 Applications of the Poisson formula

Old Exam 2 on Home page

<u>Thm</u>: Suppose u is a continuous real valued fcn on a domain Ω that satisfies the ave property:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \quad \text{when } \overline{D_r(z_0)} \subset \Omega.$$

Then u is harmonic on Ω.

<u>Pf</u>: Being harmonic is a local property. So we can restrict attention to a disc in Ω, say $\overline{D_R(z_0)} \subset \Omega$. Can change vars $f(z) = \frac{z - z_0}{R}$. Can assume $\overline{D_1(0)} \subset \Omega$.

<u>Classic trick</u>: Let $\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta.$

Know $\tilde{u}$ is continuous on $\overline{D_1(0)}$, harmonic in $D_1(0)$, and $\tilde{u}(e^{i\theta}) = u(e^{i\theta})$.

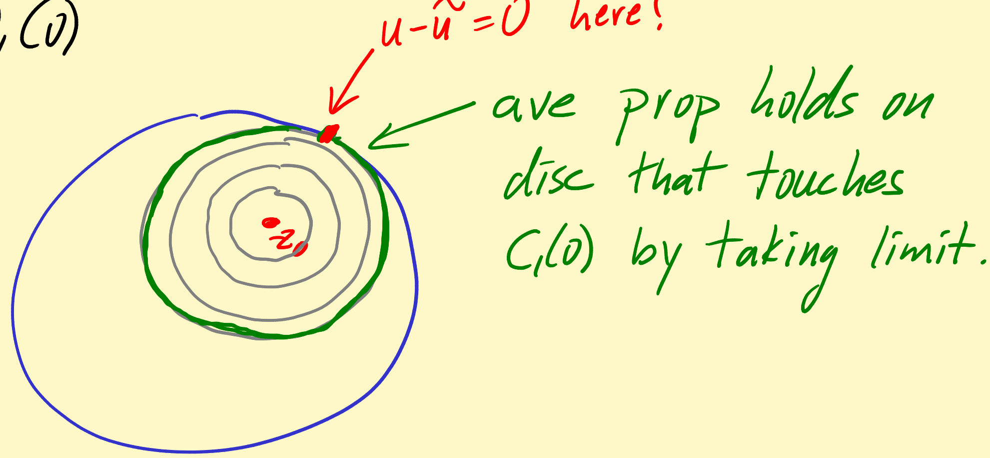
Harmonic fcns satisfy ave property. So $u - \tilde{u}$ satisfies the ave property.

<u>Claim</u>: Max princ: $\quad u(z) - \tilde{u}(z) \le \underbrace{\max_{C_1(0)} u - \tilde{u}}_{=0}$

for $z \in D_1(0)$.

<u>Pf</u>: $u - \tilde{u}$ is cont on $\overline{D_1(0)}$, so it assumes max M on set. Assume $M > 0$, and $u(z_0) - \tilde{u}(z_0) = M$.

Note: $z_0 \in D_1(0)$



$M = u(z_0) - \tilde{u}(z_0) =$ ave of fcn $\leq M$ and $< M$ at red dot. So ave $< M$ ↯.

So M must be zero.

Repeat arg using $\tilde{u} - u$ in place of $u - \tilde{u}$.

See $u - \tilde{u} \equiv 0$. So $u = \tilde{u}$ is harmonic.

Riemann removable singularity theorem for harmonic fcns.

Assume u is a real valued harmonic fcn on $D_R(z_0) - \{z_0\}$ that is bounded. Then z_0 is "removable": can give $u(z_0)$ a value making u harmonic on $D_R(z_0)$.

<u>Pf</u>: Can assume u harmonic on $D_R(0) - \{0\}$ with $R > 1$, bounded by $M > 0$; $|u(z)| \leq M$, $z \neq 0$.

Let $\tilde{u}$ = Poisson integral of u on $C_1(0)$.

So $\tilde{u} = u$ on $C_1(0)$.

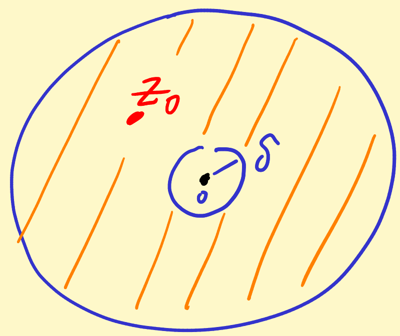
Sneaky trick: Define $v = \underbrace{u - \tilde{u}}_{\text{bndd}} + \underbrace{\varepsilon \ln|z|}_{}$, $\varepsilon > 0$

Harmonic on $\mathbb{C} - \{0\}$,
$\equiv 0$ on $C_1(0)$,
$\to -\infty$ as $|z| \to 0$.

Aha! v is harmonic on $D_1(0) - \{0\}$

<u>and</u> $v(z) \to -\infty$ as $|z| \to 0$.

Hmmm. Want a max princ.



$v(z) \to -\infty$ as $|z| \to 0$.
So $\exists\, \delta$ with $0 < \delta < 1$
such that
$v(z) < -1$ when $|z| < \delta$.
Shrink δ so that $\delta < |z_0|$.

Max princ on $\Omega = D_1(0) - \overline{D_\delta(0)}$.

$$v(z_0) \le \max_{b\Omega} v \le \max(0, -1) = 0$$

Hmmm. So $v(z) = u(z) - \tilde{u}(z) + \varepsilon \ln|z| \le 0$

True for any $\varepsilon > 0$! Let $\varepsilon \searrow 0$.

See $u(z) - \tilde{u}(z) \le 0$ on $\overline{D_1(0)} - \{0\}$.

Repeat same arg $\tilde{u} - u$ in place of $u - \tilde{u}$.

Get $\tilde{u} - u \le 0$. So $u \equiv \tilde{u}$ ← harmonic!

Setting $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$ makes u harmonic on $D_1(0)$.

__Fun fact__. Solution to D. Prob on $D_1(0)$ with polymonial bndry data $p(x,y)$ on $C_1(0)$ is a harmonic polynomial.

__Pf__: $p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

$$= \sum_{n,m=0}^{N} c_{nm} z^n \bar{z}^m .$$

Hmmm:

$$z^n \bar{z}^m = \begin{cases} 1 & \text{on } C_1(0) \quad n=m \\ \underbrace{z^n\bar{z}^n}_{=1} \bar{z}^{m-n} = \bar{z}^{m-n} & \text{on } C_1(0) \quad m>n \\ z^{n-m} \underbrace{z^m\bar{z}^m}_{=1} = z^{n-m} & \text{on } C_1(0) \quad n>m \end{cases}$$

On $C_1(0)$: $p(x,y) = c_{00} + P_1(\bar{z}) + P_2(z) = \underbrace{c_{00}+P_2(z)}_{\text{analytic}} + \underbrace{P_1(\bar{z})}_{\text{conjugate is analytic.}}$

Real and Imag parts are harmonic!

See poly harmonic extension $= \mathrm{Re}\left[\qquad \right]$.

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ also satisies poly $\to$ poly D. Prob prop.

Khavinson – Schapiro conjecture. Discs and ellipses are the only domains with this property. Open!

B, Ebenfeld, Khavinson, Schapiro: Only domains in plane such that real rational bndry data $\Longrightarrow$ rational harmonic sol^{n}s to D. Prob are <u>discs</u>.