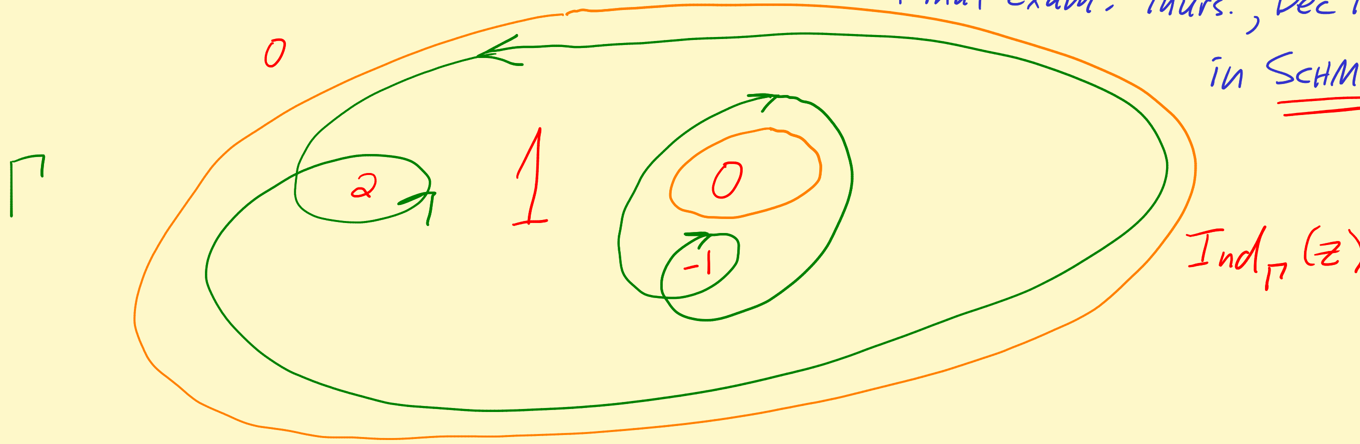
Lecture 39 The general residue theorem, argument principle, Rouché's thm

Final exam: Thurs., Dec 14, 1-3pm in SCHM 307



Cauchy thm: $\Gamma \sim 0$ in a domain Ω, f analytic on Ω

a) $\int_\Gamma f\, dz = 0$

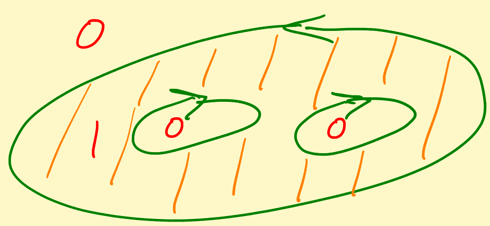
b) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$, $a \in \Omega - \mathrm{tr}(\Gamma)$

Pf that (b) $\Rightarrow$ (a). Suppose F analytic on Ω.

Pick $a \in \Omega - \mathrm{tr}(\Gamma)$. Let $f(z) = (z-a)F(z)$.

(b) yields $\frac{1}{2\pi i}\int_\Gamma \frac{(z-a)F(z)}{z-a}\,dz = 0 \cdot \mathrm{Ind}_\Gamma(a)$ ✓

Word: A cycle Γ is called "simple" if $\mathrm{Ind}_\Gamma(z) = 0$ or 1 for all z.



▨ = "inside" of Γ

rest = "outside"

Cauchy thm: f is analytic inside and on simple cycle Γ.

Then $f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$ for a "inside" Γ.

Note: Set $\mathrm{tr}(\Gamma) \cup \{z : \mathrm{Ind}_\Gamma(z) \neq 0\}$ is closed and bounded, hence compact.

<u>Def^n</u>: We say f is analytic on compact K provided $\exists$ F analytic on an open set Ω containing K and $F=f$ on K.

New improved Cauchy thm for simply connected domains Ω:

$\left(\begin{array}{l}\gamma \text{ closed} \\ \text{curve in } \Omega \\ f \text{ analytic on } \Omega\end{array}\right)$ 1) $\int_\gamma f\,dz = 0$ ← not new. $\gamma \sim \{z_0\}$ ← homotopic

new → 2) $f(a)\,\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$, $a \in \Omega - tr(\gamma)$.

<u>Pf</u>: $\gamma \sim 0$ because $\int_\gamma \frac{1}{z-a}\,dz = 0$ for $a \in \mathbb{C} - \Omega$ by Cauchy's thm.

↑ homologous

<u>General residue thm</u> Ω domain, Γ cycle in Ω, $\Gamma \sim 0$, f analytic on Ω <u>except</u> at finitely many pts $\{a_k\}_{k=1}^N$ in $\Omega - tr(\Gamma)$ where f is singular.

Then $$\int_\Gamma f\,dz = 2\pi i \sum_{n=1}^N \mathrm{Ind}_\Gamma(a_n)\,\mathrm{Res}_{a_n} f$$

<u>Pf</u>: Let $A = \{a_1, a_2, \ldots, a_N\}$ ← distinct

$$\tilde{\Omega} = \Omega - A$$

$$n_j = \mathrm{Ind}_\Gamma(a_j)$$

Def^n: $n_j C_\varepsilon(a_j)$: $z(t) = a_j + \varepsilon e^{i n_j t}$

Go around $C_\varepsilon(a_j)$ n_j-times.

Easy fact: $\mathrm{Ind}_{n_j C_\varepsilon(a_j)}(a_j) = n_j$ and

$$\int_{n_j C_\varepsilon(a_j)} f\,dz = n_j \underbrace{\int_{C_\varepsilon(a_j)} f\,dz}_{2\pi i \,\mathrm{Res}_{a_j} f}$$

Pf cont'd. Can take $\varepsilon > 0$ so small that

1) Each $\overline{D_\varepsilon(a_j)} \subset \Omega$
2) $\overline{D_\varepsilon(a_j)} \cap tr(\Gamma) = \emptyset$
3) $\overline{D_\varepsilon(a_j)} \cap \overline{D_\varepsilon(a_k)} = \emptyset$, $j \neq k$.

Aha! $\tilde{\Omega} = \Omega - A$

$$\tilde{\Gamma} = \Gamma \cup \left[\cup_{j=1}^N -n_j C_\varepsilon(a_j) \right]$$

$n_j = \mathrm{Ind}_\Gamma(a_j)$

$\mathrm{Ind}_{\tilde{\Gamma}}(a) = 0$ for outside Ω by Cauchy thm.

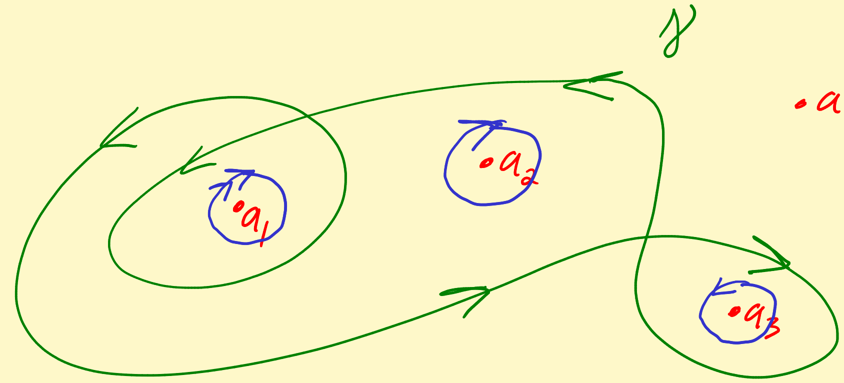
$$\mathrm{Ind}_{\tilde{\Gamma}}(a_j) = \underbrace{\mathrm{Ind}_\Gamma(a_j)}_{n_j} + (-n_j) + 0\text{'s}$$

$\leftarrow$ for other $-n_k C_\varepsilon(a_k)$'s.

$$= 0$$

General Cauchy thm implies

$$0 = \int_{\tilde{\Gamma}} f \, dz = \int_{\Gamma} f \, dz + 2\pi i \sum_{j=1}^{N} (-n_j) \operatorname{Res}_{a_j} f \checkmark$$



Remark: Not necessary to assume A is finite because $tr(\Gamma) \cup \{z : \operatorname{Ind}_\Gamma(z) \neq 0\}$ is compact. If A is discrete, but infinite, it couldn't have a limit pt in compact set. $\sum$ looks like it might be infinite, but only finitely many terms are $\neq 0$.

Def^n f is called meromorphic on a domain Ω if f is analytic on $\Omega - A$ where A is discrete and f has poles at pts in A.

Remark: Get general argument principle

$$\left[\operatorname{Res}_a \frac{f'}{f} = \begin{cases} \text{order of zero at a zero} \\ -\text{order of pole at a pole} \end{cases} \right]$$

and a general Rouché's thm: If f and g are analytic "inside" and on a simple cycle Γ and $|f-g|<|f|$ on $tr(\Gamma)$, then f and g have same # zeroes "inside" Γ counted with mult.

Last reading assignment on home page: 5.2, 5.4, 5.8, 5.10

Fun thing to read $\underbrace{5.7}_{\text{covering maps}}$, $\underbrace{5.9}_{\text{Picard's thm}}$