Lecture 41 Mittag-Lefler, Weierstraß, infinite products

Read 2.4

Cor Mero fcns on $\mathbb{C} = \{f/g : f, g \text{ entire}, g \not\equiv 0\}$.

Pf Suppose F mero on $\mathbb{C}$. Then it has a discrete set of poles. Weier $\Rightarrow$ $\exists$ entire g with zero at poles of F with mult of zero = order of pole of F.

$g \cdot F$ has removable sing at poles of F, so $gF = f$, f entire. ✓

Cor Combine M-L and Weier to be able to prescribe finitely many terms in power series on seq $a_n \to \infty$.

Infinit products $\prod_{n=1}^{\infty} p_n = \lim_{N \to \infty} \prod_{n=1}^{N} p_n$

$p_n \in \mathbb{C}$, $p_n \neq 0$.

Lemma: If $\prod_{1}^{N} p_n$ converges to $L \neq 0$ then $p_n \to 1$ as $n \to \infty$.

Pf $$p_N = \frac{\prod_{1}^{N} p_n}{\prod_{1}^{N-1} p_n} \longrightarrow \frac{L}{L} = 1 \checkmark$$

Traditional to write $\prod_{1}^{\infty} (1 + a_n)$ instead.

Lemma $\prod_{1}^{\infty}(1+a_n)$ converges to $L \neq 0$

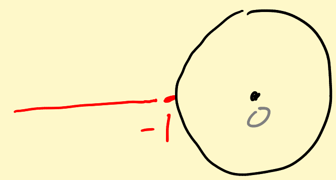
$\iff \sum_{n=1}^{\infty} \operatorname{Log}(1+a_n)$ converges.

Principal branch of log

Pf ($\Leftarrow$) Easy because $e^{\sum^{N} \operatorname{Log}(1+a_n)} = \prod^{N}(1+a_n)$.

($\Rightarrow$) Read 2.4

Fact $\operatorname{Log}(1+z)$ analytic on $D_1(0)$



and has a simple zero at $z=0$.

And $\left.\frac{d}{dz}\operatorname{Log}(1+z)\right|_{z=0} = 1$

So $\quad (1-\varepsilon)|a_n| \le |\operatorname{Log}(1+a_n)| \le (1+\varepsilon)|a_n| \quad (*)$

Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $(*)$ holds when $|a_n| < \delta$.

Def^n $\prod_{1}^{\infty}(1+a_n)$ converges absolutely when

$\sum_{n=1}^{\infty}|a_n| < \infty$.

Note: $(*)$ plus lemma $\Rightarrow$ absolute conv $\Rightarrow$ conv.

Fact: If $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on compact subsets of a domain Ω, f_n analytic on Ω, then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely and uniformly on compact subsets of Ω to an analytic.

EX: $f(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \leftarrow$ entire, simple zeroes at pts in $\mathbb{Z}$.

Remark: $\prod(z-a_n) \leftarrow$ not very useful

$\prod\left(1-\frac{z}{a_n}\right) \leftarrow$ better. If $a_n \to \infty$ fast enough.

Weierstraß big idea

$$1=(1-z)e^{-\mathrm{Log}(1-z)}$$

$$-\mathrm{Log}(1-z)=z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots$$

$$P_N(z)=z+\frac{z^2}{2}+\cdots+\frac{z^N}{N}$$

Can make $1 \approx (1-z)e^{P_N(z)}$ as close as desired on $D_{1/2}(0)$ by taking N big.

Weier: $f(z)=\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right)e^{P_{N_n}\left(\frac{z}{a_n}\right)}\right]^{m_n} \leftarrow$ mult of zero at a_n

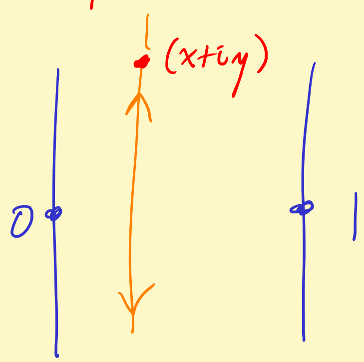
Famous formulas

$$\sin \pi z = \pi z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(1-\frac{z}{n}\right) e^{z/n}$$

$$= \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

both mero on $\mathbb{C}$ with double poles at pts in $\mathbb{Z}$.
Principal parts same! Both periodic with period $= 1$.



$$H(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2}$$

both $\to 0$
unif as $y \to \pm\infty$
$0 \le x \le 1$.

Bounded on strip.
Principal parts cancel.

Get bounded entire fcn. Liouville's $\Rightarrow$ constant.
Constant must be zero because of zero limits along vert lines.

<u>Practice problem</u>: Suppose f, g are analytic near a and g has a <u>double</u> zero at a. Find a formula for $\operatorname{Res}_a \frac{f}{g}$ involving value of f and g and their derivatives at a. [like $\operatorname{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}$ when g has a <u>simple</u> zero at a.]