Review

Final Exam: Thurs. Dec 14, 1:00–3:00 pm in SCHM 307

Closed book, no notes, no calculators or computers...

Prob Suppose f, g analytic near a and g has a double zero at a. Find a formula for $\operatorname{Res}_a f/g$.

Sol^n: $g(a)=0,\ g'(a)=0,\ g''(a)\neq 0$

$$g(z) = a_2(z-a)^2 + \cdots = (z-a)^2\underbrace{\left[a_2 + a_3(z-a) + \cdots\right]}_{G(z)}$$

$$a_n = \frac{g^{(n)}(a)}{n!}$$

$$G(a) = a_2, \quad \frac{G'(a)}{1!} = a_3$$

$$G(a) = \frac{g''(a)}{2!}, \quad G'(a) = \frac{g'''(a)}{3!}$$

$$\frac{f(z)}{g(z)} = \frac{f(z)}{(z-a)^2 G(z)} = \frac{1}{(z-a)^2}\underbrace{\left[\frac{f(z)}{G(z)}\right]}_{A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots}$$

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \text{power series}$$

$$\operatorname{Res}_a \frac{f}{g} = A_1 = \frac{1}{1!}\frac{d}{dz}\left(\frac{f(z)}{G(z)}\right)\Bigg|_{z=a} = \frac{f'(a)G(a) - f(a)G'(a)}{G(a)^2}$$

Hated formula: $\displaystyle \operatorname{Res}_a \frac{f}{g} = \lim_{z\to a} \frac{1}{1!}\frac{d}{dz}\left[(z-a)^2\frac{f(z)}{g(z)}\right]$

4. How many zeroes does the polynomial

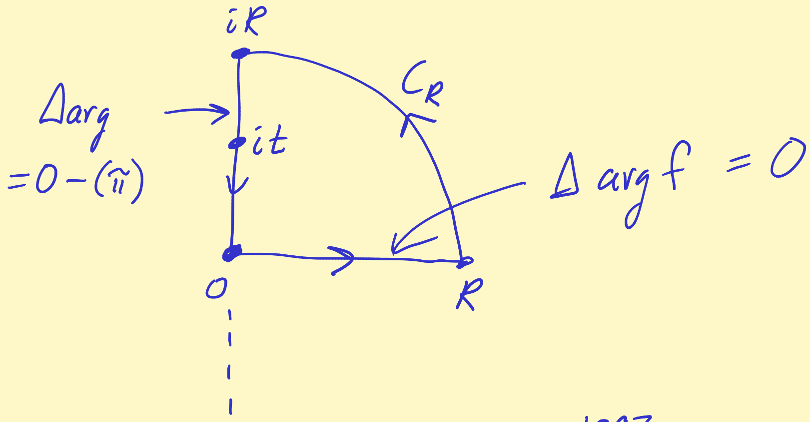
$$z^{1998} + z + 2001 = f(z)$$

$it{:} \quad = -t^{1998} + it + 2001$

have in the first quadrant? Explain your answer.

$-t^{1998} + it + 2001$

$\arg \xrightarrow[R\to\infty]{} \pi$



$$\Delta \arg_{C_R} = \mathrm{Im} \int_{C_R} \frac{f'}{f}\, dz = \int_{C_R} \underbrace{\frac{1998 z^{1997} + 1}{z^{1998} + z + 2001}}_{\sim \frac{1998}{z}}\, dz$$

5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \leq t \leq 2\pi$. Suppose that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

Hint: $F(z) = \exp\left(\frac{1}{3} \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw \right)$

Residue thm: F well defined. Show F' exists.

$$\frac{F'}{F} = \frac{1}{3} \frac{f'}{f}$$

F^3 is almost $= f$.

Trick. $\frac{d}{dz}\left(\frac{f}{F^3}\right) \equiv 0$. Fix F by a constant factor.

6. Suppose that $\{a_k\}_{k=1}^{N}$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that

$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

has an analytic antiderivative on $\underbrace{\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}}_{\Omega}$.

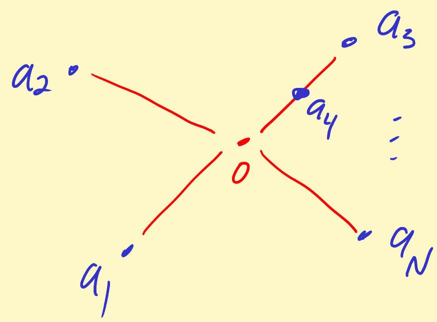
Residue thm.

fcn h has antiderivative on Ω $\iff$ $\int_\gamma h\, dz = 0$

$\forall$ closed γ in Ω.

7. Suppose $a_1, a_2, \ldots, a_N$ are distinct nonzero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \underbrace{\prod_{k=1}^{N}(z - a_k)}_{P(z)}.$$



Try $$f(z) = \exp\left(\frac{1}{N}\int_{\gamma_a^z} \frac{P'(w)}{P(w)}\,dw\right)$$

Similar to above $(N=3)$.