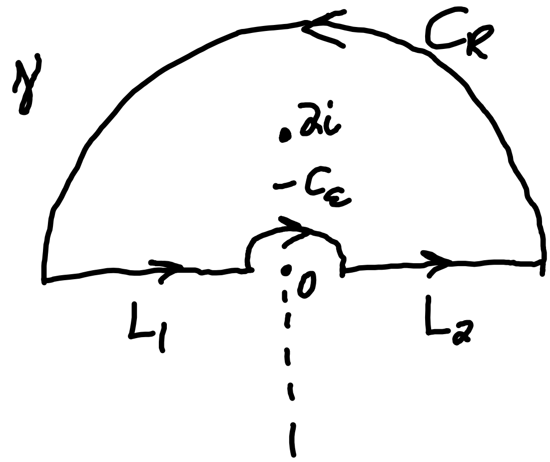
MA 530 Exam 2 solutions



$\sqrt{z} = e^{\frac{1}{2}\log z}$

where $\log z = \operatorname{Ln}|z| + i\theta$

$\theta \in \arg z$, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

Note: $\log z = \operatorname{Log} z$ on γ

$$f(z) = \frac{\left(1/\sqrt{z}\right)}{z^2+4} \quad \begin{matrix} \leftarrow G(z) \\ \leftarrow H(z) \leftarrow \text{simple zero at } 2i \end{matrix}$$

$$\operatorname{Res}_{2i} f(z) = \frac{\left(1/\sqrt{2i}\right)}{2(2i)} = \frac{G(2i)}{H'(2i)}$$

$$= \frac{\left(1/e^{\frac{1}{2}(\operatorname{Ln}2 + i\frac{\pi}{2})}\right)}{4i}$$

$$= \frac{e^{-\operatorname{Ln}\sqrt{2}}\, e^{-i\frac{\pi}{4}}}{4i} = \frac{\frac{1}{\sqrt{2}}}{4i}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= \frac{1}{i}\left(\frac{1}{8} - i\frac{1}{8}\right) = -\frac{1}{8} - \frac{1}{8}i$$

$$|z^2+4| \geq \left||z|^2 - |-4|\right| = \left||z|^2 - 4\right| = \begin{cases} R^2 - 4 & \text{if } |z| = R > 2 \\ 4 - \varepsilon^2 & \text{if } |z| = \varepsilon < 2 \end{cases}$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{1/\sqrt{R}}{R^2-4}\cdot \pi R \to 0 \text{ as } R \to 0$$

$$\left|\int_{-C_\varepsilon} f(z)\,dz\right| \le \frac{1/\sqrt{\varepsilon}}{4-\varepsilon^2}\cdot \pi\varepsilon \to 0 \text{ as } \varepsilon \to 0$$

<u>Note</u>: $|w^2| = |w|^2$. Let $w=\sqrt{z}$ to see that $|z| = |\sqrt{z}|^2$, so $|\sqrt{z}| = |z|^{1/2}$.

$-L_1$: $z(t) = -t$, $\varepsilon \le t \le R$

$$\int_{L_1} f(z)\,dz = -\int_{-L_1} f(z)\,dz = -\int_\varepsilon^R \frac{\left(1/\sqrt{-t}\right)}{(-t)^2+4}(-1)\,dt$$

$$= \int_\varepsilon^R \frac{1/e^{\frac{1}{2}(\operatorname{Ln}|-t| + i\pi)}}{t^2+4}\,dt = \underbrace{e^{-i\frac{\pi}{2}}}_{-i}\int_\varepsilon^R \frac{1/\sqrt{t}}{t^2+4}\,dt$$

$$\int_{L_2} f(z)\,dz = \int_{\varepsilon}^{R} \frac{1/\sqrt{t}}{t^2+4}\,dt$$

Residue theorem: $\displaystyle\int_{\gamma} f(z)\,dz = 2\pi i \operatorname{Res}_{2i} f(z)$

$$\Big(\underbrace{\int_{L_1} + \int_{L_2}}\; + \int_{C_R} + \int_{-C_\varepsilon}\Big) f(z)\,dz = 2\pi i\left(-\frac{1}{8} - \frac{1}{8} i\right)$$

$$\downarrow {\scriptstyle \varepsilon\to 0,\ R\to\infty} \qquad \downarrow \qquad \downarrow$$

$$(1-i)I + 0 + 0 = \pi \cdot \frac{1}{4}(1-i)$$

$$I = \frac{\pi}{4}$$

2.

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad \varphi_{-a}(z) = (\varphi_a^{-1})(z) = \frac{z+a}{1+\bar{a}z}$$

$F : D_1(0) \to D_1(0)$ and $F(0) = 0$. Schwarz $\Rightarrow$

$$|F(w)| \leq |w|$$

$$|f(\underbrace{\varphi_{-a}(w)}_{z})| \leq |\underset{\substack{\uparrow \\ w = (\varphi_{-a})^{-1}(z) = \varphi_a(z)}}{w}|$$

So $|f(z)| \leq \left|\dfrac{z-a}{1-\bar{a}z}\right|$.

Check that $\varphi'_{-a}(z) = \dfrac{1-|a|^2}{(1+\bar{a}z)^2}$

Schwarz $\Rightarrow$ $|F'(0)| \leq 1$

$$|f'(\underbrace{\varphi_{-a}(0)}_{a})| \underbrace{|\varphi'_{-a}(0)|}_{1-|a|^2} \leq 1$$

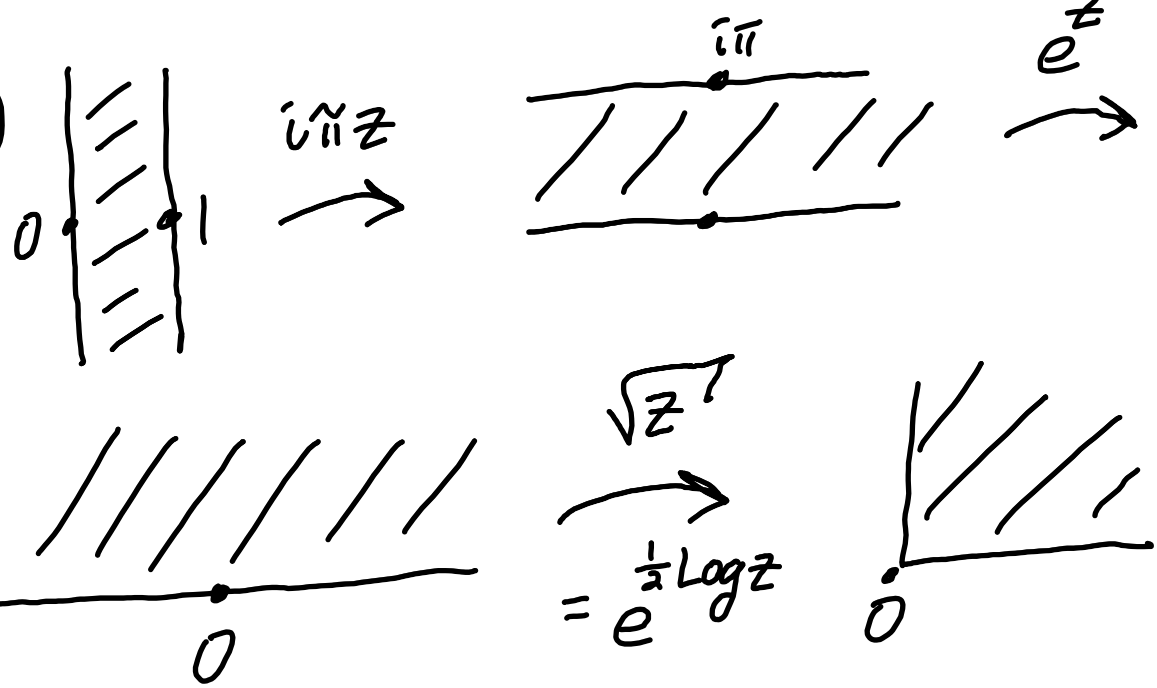
So $|f'(a)| \leq \dfrac{1}{1-|a|^2}$.

This shows that $\text{Sup}\{|f'(a)| : \text{all such } f\}$

$\leq \frac{1}{1-|a|^2}$. Since $\varphi_a'(a) = \frac{1-|a|^2}{(1-\underbrace{\bar{a}a}_{|a|^2})^2} = \frac{1}{1-|a|^2}$,

and $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ is in the class, we see that the Sup is attained.

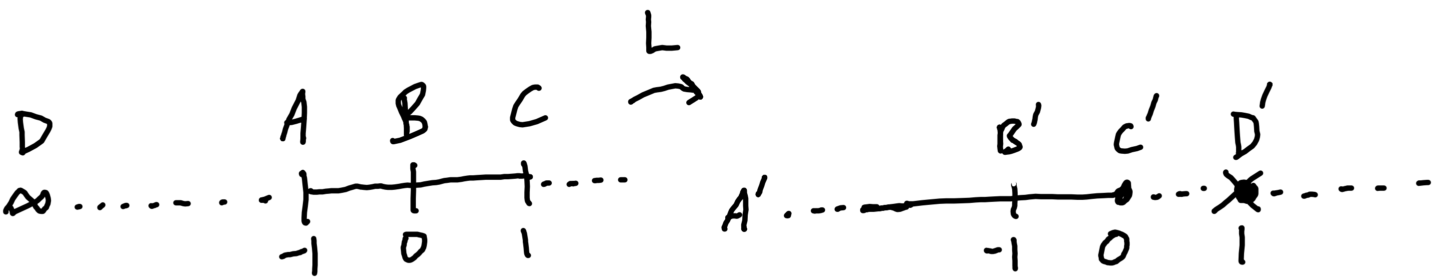
3, a)

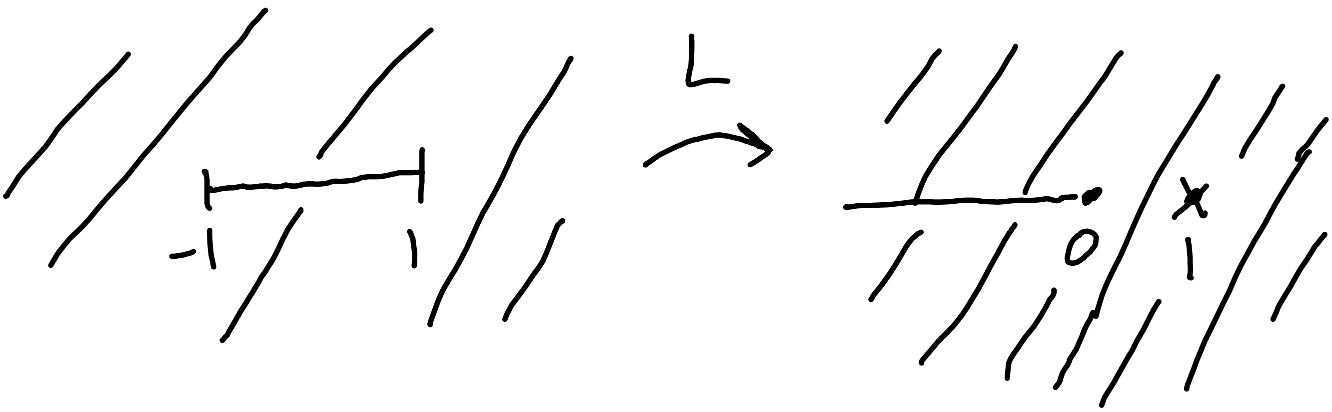


b) $L(z) = \frac{z-1}{z+1}$ is a LFT.

$L(-1) = \infty$, so $\mathbb{R} \longrightarrow$ line

$L(1) = 0$, $L(0) = -1$, $L(\infty) = 1 \leftarrow$ missed by L on $\mathbb{C}$





$$L: \mathbb{C} - [-1,1] \xrightarrow[\text{onto}]{1\text{-}1} \underbrace{\mathbb{C} - \big((-\infty,0] \cup \{1\}\big)}_{\Omega}$$

$L: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ 1-1 onto and $L(\infty) = 1$,

so 1 is missed by L on $\mathbb{C}$.

L maps $(-1,1]$ to $[-\infty,0]$.

Ω is not simply connected.

4. Let $f(z) = z^n$ and $g(z) = z^n - P(z)$

Rouché's: $|f(z) - g(z)| = |P(z)| < 1 = |z^n|$ on $C_1(0)$. f has n zeroes inside $C_1(0)$ counted with multiplicity. Hence, so does $-g(z) = a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$.

But $-g$ has $n-1$ or fewer zeroes on $\mathbb{C}$ if not all the a_k are zero, and infinitely many zeroes if all the a_k are zero. It cannot have exactly n zeroes inside $C_1(0)$. This contradiction proves that no such $P(z)$ exists.

Proof #2: Cauchy estimate for $P^{(n)}(0)$ yields that

$$\underbrace{|P^{(n)}(0)|}_{n!} \le \frac{n!\, Max\{|P(z)| : |z|=1\}}{1^n} < (n!)\cdot 1 \quad ↯$$

Proof #3: Let $Q(z) = z^n \cdot P(\frac{1}{z})$

$$= 1 + a_{n-1} z + \cdots + a_0 z^n.$$

Notice that $Q(0) = 1$ and

$$|Q(e^{i\theta})| = \underbrace{|e^{in\theta}|}_{1}\, |P(\tfrac{1}{e^{i\theta}})| = |P(e^{-i\theta})| < 1$$

for $0 \le \theta \le 2\pi$, violating the maximum principle on $D_1(0)$. ↯