Math 530

Final Exam

1. *(40 pts.)* Use branches of the complex log function to compute $\int_\gamma \frac{1}{z}\,dz$ if γ is a path parametrized by

$$z(t) = r(t)e^{it}, \qquad 0 \le t \le 3\pi,$$

where $r(t)$ is a positive real valued function such that $r(0) = 2$ and $r(3\pi) = 3$. Explain how you arrived at your answer.

2. *(30 pts.)* Suppose that f is analytic in a neighborhood of the closed unit disc and that $f(z)$ is never in the set $\{x \in \mathbb{R} : x \ge 0\}$ when $|z| = 1$. Show that f has no zeroes in the unit disc.

3. *(30 pts.)* In this problem, the only theorem you are allowed to use is the Residue Theorem. Also, if you claim that a certain term tends to zero, you must prove that it does. Calculate

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^4+1)}\,dx.$$

4. *(30 pts.)* How many roots of the equation $z^4 - 6z + 3 = 0$ fall in the annulus $\{z : 1 < |z| < 2\}$?

5. *(40 pts.)* State the Schwarz Lemma. Prove the following corollary of the Schwarz Lemma: If f is an analytic map of the unit disk into itself and $f(0) \ne 0$, then $|f'(0)| < 1$.

6. *(30 pts.)* Suppose that $a_1 = -1$, $a_2 = 1$, and $a_3 = 2i$ and that f is a function that is analytic on $\mathbb{C} - \{a_1, a_2, a_3\}$ that has essential singularities at the three points. Suppose also that

$$\int_{C_1(a_n)} f\,dz = \sqrt{n} \quad \text{for } n = 1, 2, 3,$$

where $C_1(z_0)$ denotes the circle of radius 1 about z_0 parametrized in the counter clockwise sense. Draw a closed curve γ such that

$$\mathrm{Ind}_\gamma a_1 = -1, \qquad \mathrm{Ind}_\gamma a_2 = 1, \quad \text{and} \quad \mathrm{Ind}_\gamma a_3 = 2.$$

Explain how to define a cycle Γ so that the General Cauchy Theorem on the domain $\Omega = \mathbb{C} - \{a_1, a_2, a_3\}$ can be used to compute

$$\int_\gamma f\,dz.$$

Find the value of the integral and explain your reasoning. (You are not allowed to use the General Residue Theorem here. If you failed to draw such a γ, you may assume that such a γ exists.)