Math 530

Homework 1

1. a) Define $\phi_a(z) = \dfrac{z-a}{1-\bar{a}\,z}$. Prove that $|\phi_a(z)| = 1$ if either $|z| = 1$ or $|a| = 1$. What exception must be made if $|z| = |a| = 1$?

b) Prove that $|\phi_a(z)| < 1$ if $|z| < 1$ and $|a| < 1$.

c) If $|a| < 1$, show that $\phi_a(z)$ is a *one-to-one* mapping of the open unit disk *onto* itself as a function of z. Write a formula for the inverse mapping.

2. Prove that $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$. What does this equality mean geometrically?

3. Suppose that f is an analytic function on an open set Ω_1 which maps into an open set Ω_2 on which g is defined and analytic. Prove that $h(z) = g(f(z))$ is analytic on Ω_1 and that $h'(z) = g'(f(z))f'(z)$. (This is the complex chain rule.)

4. Suppose $z(t) = x(t) + iy(t)$ where $x(t)$ and $y(t)$ are continuously differentiable real functions on the interval $[a, b]$. Write $z'(t) = x'(t) + iy'(t)$. Show that if f is analytic on $\mathbb{C}$, then $w(t) = f(z(t))$ is such that $w'(t) = f'(z(t))z'(t)$ on $[a, b]$. (This is another important Chain Rule.)

5. Show that a sequence of complex numbers $\{a_n\}$ converges to b if and only if $\text{Re}\ a_n \to \text{Re}\ b$ and $\text{Im}\ a_n \to \text{Im}\ b$. Also, show that $\{a_n\}$ is a Cauchy sequence if and only if $\text{Re}\ a_n$ and $\text{Im}\ a_n$ are Cauchy. Conclude that the completeness of the complex number system follows from the completeness of the reals.

6. Prove that an absolutely convergent series of complex numbers is convergent.

7. Show that the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n z^n$ is given by the supremum of the set of real numbers $r \geq 0$ with the property that there exists a bound M such that $|a_n| r^n \leq M$ for all n. (Note: M may depend on r.)