Math 530  
Homework 4  

1. For what values of $z$ is the series $\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$ convergent? Same question for $\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}$.  

2. If $f$ is analytic on the unit disc and $|f(z)| \leq 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that the Cauchy Estimates will yield.  

3. Show that the successive derivatives of an analytic function at a point $a$ can never satisfy $|f^{(n)}(a)| > n!n^n$.  

4. Suppose that $f$ is analytic on a disk $D_\epsilon(0)$ and satisfies the differential equation $f'' = f$. Prove that $f$ is given by $A \cosh z + B \sinh z$, where $A$ and $B$ are constants.  

5. If $f(z) = \sum a_n z^n$, express $\sum n^3 a_n z^n$ in terms of $f$ and its derivatives.  

6. Prove that an entire function $f$ such that $\text{Re } f(z) > 0$ for all $z$ must be constant.  

7. Prove that there is no analytic function $f$ on the unit disk such that $f(1/n) = 2^{-n}$ for $n = 2, 3, 4, \ldots$.  

8. Show how the Basic Polynomial Estimate and the Maximum Principle imply the Fundamental Theorem of Algebra.  

9. Show that $\int_0^\infty \sin(x^2) \, dx = \int_0^\infty \cos(x^2) \, dx = \frac{\sqrt{2\pi}}{4}$ by integrating $e^{-z^2}$ around the counterclockwise boundary of $\{z = re^{i\theta} : 0 < r < R, 0 < \theta < \pi/4\}$ and letting $R \to \infty$. 