Math 530
Practice problems for Exam 1

1. What is the radius of convergence of the power series centered at zero for the function \( \frac{1}{(z - 1 - i)^{10}} \)?

2. Prove that power series can be integrated term by term. To be precise, suppose that a power series \( \sum_{n=0}^{\infty} a_n z^n \) with radius of convergence \( R > 0 \) converges on the disc \( D_R(0) \) to an analytic function \( f(z) \). Prove that the power series \( \sum_{n=0}^{\infty} a_n \frac{1}{n+1} z^{n+1} \) also has radius of convergence \( R \) and that this series converges to an analytic anti-derivative of \( f(z) \) inside the circle of convergence.

3. Suppose that \( f \) and \( g \) are analytic in a neighborhood of \( a \). If \( f \) has a simple zero at \( a \), then
   \[
   \text{Res}_{a} \frac{g}{f} = \frac{g(a)}{f'(a)}.
   \]
   Prove a similar formula in case \( f \) has a double zero at \( a \), i.e., in case \( f \) is such that \( f(a) = 0, f'(a) = 0, \) but \( f''(a) \neq 0 \).

4. Consider the closed path which starts at the origin, follows the real axis to \( R > 0 \), then follows the circle \( Re^{i\theta} \) as \( \theta \) ranges from zero to \( 2\pi/3 \), then follows the line segment joining \( Re^{i2\pi/3} \) to the origin back to the origin. By letting \( R \to \infty \), use this path to calculate
   \[
   \int_{0}^{\infty} \frac{1}{1 + x^3} \, dx.
   \]
   Hint: Show that the integral over the circular part of the curve tends to zero.

5. Give a detailed statement and proof of the Schwarz Lemma.

6. Show that if \( f \) is an analytic mapping of the unit disk into itself such that \( f(a) = 0 \), then
   \[
   |f(z)| \leq \left| \frac{z - a}{1 - az} \right|
   \]
   for all \( z \) in the disk.

7. Show that if \( f \) is an analytic mapping of the unit disk into itself, then \( |f'(0)| \leq 1 \).

8. Suppose that \( f \) is an analytic function on the unit disc such that \( |f(z)| < 1 \) for \( |z| < 1 \). Prove that if \( f \) has a zero of order \( n \) at the origin, then \( |f(z)| \leq |z|^n \) for \( |z| < 1 \). How big can \( |f^{(n)}(0)| \) be?

9. Suppose that \( f \) is an entire function that satisfies an estimate \( |f(z)| \leq C(1 + |z|^N) \) for all \( z \) where \( C \) is a positive constant and \( N \) is a positive integer. Prove that \( f \) must be a polynomial of degree \( N \) or less.

10. Prove that if \( h_1 \) and \( h_2 \) are two analytic functions on a domain \( \Omega \) such that \( h_1^N \equiv h_2^N \) for some positive integer \( N \), then there is an \( N \)-th root of unity \( \lambda \) such that \( h_1 = \lambda h_2 \) on \( \Omega \).