INTEGRATION. V2.0

1. One-liners

Problem 1. True of false: If f is a non-negative function defined on $\mathbb{R}$ and $\int_{\mathbb{R}} f\,dx < \infty$, then $\lim_{|x|\to\infty} f(x) = 0$.

Problem 2. Let $f, f_n \in L(\mathbb{R})$, $n \in \mathbb{N}$ and suppose that

$$\int_{\mathbb{R}} |f_n(x) - f(x)|\,dx \le n^{-2}, \text{ for all } n.$$

Prove that $f_n \to f$ a.e.

Problem 3. Let $(X, \mathcal{F}, \mu)$ be a measure space and $f \in L(\mu)$. Show that the set $\{f \neq 0\}$ is σ-finite.

Problem 4. Prove the following Chebychev-like inequality: If $I = [0,1]$ and $f \in L(I)$ is non-negative and has integral 1, then

$$\int_{\{f>\eta\}} f(x)\,dx \ge 1 - \eta, \text{ all } 0 < \eta < 1.$$

Problem 5. Prove the following variant of Fatou's Lemma: If $\{f_n\}$ is a sequence of non-negative measurable functions which converges to f a.e. and $\int_X f_n\,d\mu \le M < \infty$ for all n, then f is integrable and $\int_X f\,d\mu \le M$.

Problem 6. Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{f_n\}$ be a non-increasing sequence of non-negative measurable functions which converges to f. Show that $\lim_n \int_X f_n\,d\mu = \int_X f, d\mu$ provided that $f_1 \in L(\mu)$ and that the conclusion may fail if $f_1 \notin L(\mu)$.

Problem 7. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of integrable functions that converges to a function f uniformly on X. Show that f is also integrable and that $\lim_n \int_X f_n\,d\mu = \int_X f\,d\mu$. Is a similar result true if $\mu(X) = \infty$?

Problem 8. Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions that converges to f a.e. If $f \in L(\mu)$, show that $\lim_n \int_X |f_n|\,d\mu = \int_X |f|\,d\mu$ implies $\lim_n \int_X |f_n - f|\,d\mu = 0$ and that the conclusion may fail if f is not integrable.

2. Advanced Problems

Problem 9. Let f be a non-negative measurable function defined on $\mathbb{R}$. Prove that if $\sum_{n=-\infty}^{\infty} f(x+n)$ is integrable, then $f=0$ a.e.

Problem 10. Suppose f is integrable on $\mathbb{R}^n$ and for a fixed $h \in \mathbb{R}^n$ let $g(x) = f(x+h)$ be a translate of f. Show that g is also integrable and that $\int_{\mathbb{R}^n} g\,dx = \int_{\mathbb{R}^n} f\,dx$.

Problem 11. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, and f a non-negative real-valued function defined on X. Prove that a necessary and sufficient condition that $\lim_n \int_X f^n\,d\mu$ should exist as a finite number is that $\mu\{f > 1\} = 0$.

Problem 12. Let $r_1, r_2, \ldots, r_n, \ldots$ be an enumeration of the rational numbers in $I = [0,1]$, and let $f(x) = \sum_{\{n: x > r_n\}} 2^{-n}$. Compute $\int_I f(x)\,dx$.

Problem 13. Prove that the sum $\sum_{n=0}^{\infty} \int_0^{\pi/2} \left(1 - \sqrt{\sin x}\right)^n \cos x\,dx$ converges to a finite limit, and find its value.

Problem 14. Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions such that $\sum_{n=1}^{\infty} \int_X |f_n|\,d\mu < \infty$. Show that $\sum_{n=1}^{\infty} f_n$ converges absolutely a.e. and $\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$. In particular, also $\lim_n f_n = 0$ a.e.

Problem 15. Let $(X, \mathcal{F}, \mu)$ be a measure space and assume $\{f_n\}$ is a sequence of non-negative measurable functions that converges to f a.e. If $\lim_n \int_X f_n\,d\mu = \int_X f\,d\mu < \infty$, is it true that $\lim_n \int_E f_n\,d\mu = \int_E f\,d\mu$ for every $E \subset \mathcal{F}$?

Problem 16. Let $\{f_n\}$ be a sequence of non-negative integrable functions such that $\lim_n \int_X f_n\,d\mu = 0$. If $g \in L(\mu)$ has the property that $gf_n \in L(\mu)$ for all n, does it follow that $\lim_n \int_X gf_n\,d\mu = 0$?

Problem 17. Let f, g, f_n, g_n be integrable functions, $n \in \mathbb{N}$. If $\lim_n f_n = f$ a.e., $|f_n| \le g_n$ for all n, and $\lim_n \int_X g_n\,d\mu = \int_X g\,d\mu$, is it also true that $\lim_n \int_X f_n\,d\mu = \int_X f\,d\mu$?

Problem 18. Let f be a real-valued measurable function defined on $[a,b]$ such that $\int_a^b f^n\,dx = c$ for $n = 2,3,4$. Show that $f = \chi_A$ a.e. for some measurable set $A \subset [a,b]$.

Problem 19. If $f \in L_1[0,1]$, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $m(A) < \delta$ implies that $\int_A |f(x)|\,dx < \varepsilon$.

Problem 20. Show that if $f \in L(\mu)$, then $\lim_{\lambda \to \infty} \int_{\{|f| > \lambda\}} |f|\,d\mu = 0$.

Problem 21. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, and f a measurable extended real-valued function defined on X. Show that $f \in L(\mu)$ if and only if $\sum_{k=1}^{\infty} \mu\{|f| \geq k\} < \infty$.

Problem 22 (see Problem 40)**.** Discuss the following statements (prove or give a counter-example):

(i) Convergence a.e. implies convergence in L_1.
(ii) Convergence in L_1 implies convergence a.e.
(iii) Convergence in L_1 implies convergence in measure.
(iv) Convergence in measure implies convergence a.e.
(v) Convergence a.e. implies convergence in measure.

3. Qual Problems

Problem 23. [Jan'00] Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and $\varphi \in L_1(\mathbb{R})$. Show that

$$\lim_{t \to \infty} \int_E \varphi(x+t)\, dx = 0.$$

Problem 24. [Aug'00] Let $(X, \mathcal{F}.\mu)$ be a finite measure space. Let f_n be a sequence of measurable functions with $f_1 \in L_1(\mu)$ and with the property that

$$\mu\{x \in X : |f_n(x)| > \lambda\} \leq \mu\{x \in X : |f_1(x)| > \lambda\}$$

for all n and all $\lambda > 0$. Prove that

$$\lim_n \frac{1}{n} \int_X \big(\max_{1 \leq k \leq n} |f_k| \big)\, d\mu = 0.$$

Problem 25. [Aug'00] Let f be a continuous function on $[-1, 1]$. Find

$$\lim_n n \int_{-1}^{1} f(x)\big(1 - n|x|\big)\, dx.$$

Problem 26. [Aug'01] Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite measure space, $f : \Omega \to \mathbb{R}$ measurable. Suppose there is a $c \in \mathbb{R}$ such that for all $X \subset \Omega$ of finite measure, $|\int_X f\, d\mu| \leq c$ holds. Prove that $f \in L_1(\Omega, \mathcal{A}, \mu)$.

Problem 27. [Aug'01] Let $g \colon [a, b] \to \mathbb{R}$ be Lebesgue measurable, and suppose $\int_a^b g\psi\, dx = 0$ for all continuous $\psi \colon [a, b] \to \mathbb{R}$. Prove that $g = 0$ a.e.

Problem 28. [Jan'02] Let $f_n \colon X \to [0, \infty)$ be a sequence of measurable functions on the measure space $(X, \mathcal{F}, \mu)$. Suppose there is a positive constant M such that the functions $g_n(x) = f_n(x)\chi_{\{f_n \leq M\}}$

satisfy $\|g_n\|_1 \le An^{-4/3}$ and for wich $\mu\{x \in X : f_n(x) > M\} \le Bn^{-5/4}$, where A and B are positive constants independent of n. Prove that

$$\sum_{n=1}^{\infty} f_n(x) < \infty \text{ a.e.}$$

Problem 29. [Jan'02] Let $\{f_n\}$ be a sequence of non-negative functions in $L_1[0,1]$ with the property that $\int_0^1 f_n(t)\,dt = 1$ and $\int_{1/n}^1 f_n(t)\,dt \le 1/n$ for all n. Define $h(x) = \sup_n f_n(x)$. Prove that $h \ne L_1[0,1]$.

Problem 30. [Aug'02] Let $f \in L_1[0,1]$ and let $F(x) = \int_0^x f(t)\,dt$. If E is a measurable subset of $[0,1]$, show that

(i) $F(E) = \{F(x) : x \in E\}$ is measurable.
(ii) $m\{F(E)\} \le \displaystyle\int_E |f(t)|\,dt$.

Problem 31 (see problem 14, Jan'03)**.** Assume that f_n is Lebesgue measurable for $n \in \mathbb{N}$, $f_n \ge 0$, and $\sum_{n=1}^{\infty} \int f_n(x)\,dx < \infty$. Show that $f_n \to 0$ a.e.

Problem 32. In each case find $\lim_n \int_0^{\infty} f_n(x)\,dx$ and justify your answer.

(i) $f_n(x) = x^{-1/2} \cos\left(\frac{x+1}{n}\right)\chi_{[1,n-1]}$.
(ii) $f_n(x) = x^{-1/2} \sin\left(\frac{x+1}{n}\right)\chi_{[n,2n]}$.
(iii) $f_n(x) = x^{-1/2} \sin\left(1 + \frac{x}{n}\right)\chi_{(0,1)}$.

Problem 33. [Jan'04] Let $f \in L_1(\mathbb{R})$ satisfy

$$f(x) = 0 \text{ if } |x| > 1, \tag{1}$$

$$\int_{\mathbb{R}} f(x)x^k\,dx = 0, k \in \mathbb{N}. \tag{2}$$

(i) Prove that $f = 0$ a.e.
(ii) Does your argument apply if (1) is replaced with the milder condition

$$|x^k f(x)| \to 0 \text{ as } |x| \to \infty \text{ for every } k \in \mathbb{N}. \tag{3}$$

Justify your answer.

Problem 34. [Aug'04] Show that the following limit exists

$$\lim_n n \int_{1/n}^1 \frac{\cos(x + 1/n) - \cos x}{x^{3/2}}\,dx.$$

Problem 35. [Aug'04] A Lebesgue integrable function $f\colon \mathbb{R} \to \mathbb{R}$ has the property that

$$\int_E f(x)\,dx = 0$$

for all Lebesgue measurable sets $E \subset \mathbb{R}$ with $m(E) = \pi$. Prove or disprove that $f = 0$ a.e.

Problem 36. [Aug'05] Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and in $L_1(\mathbb{R})$. Suppose that

$$\int_a^b f(x)\,dm(x) \geq 0 \text{ for all } a, b \in \mathbb{R}, a \leq b.$$

Prove that $f \geq 0$ a.e.

Problem 37. [Aug'05] Prove that the following limit exists

$$\lim_n \int_0^\infty \frac{e^{-x}\cos x}{nx^2 + \frac{1}{n}}\,dx,$$

and find it, justifying all your steps.

Problem 38. [Aug'05] Let $f\colon [0,1] \to \mathbb{R}$ be Lebesgue measurable with $f > 0$ a.e. Let $\{E_n\}$ be a sequence of measurable sets in $[0,1]$ with the property that $\lim_n \int_{E_n} f(x)\,dx = 0$. Prove that $\lim_n m(E_n) = 0$.

Problem 39. [Jan'06] Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set with positive and finite measure.

(i) Let χ_A be the characteristic function of A, and set $\phi(x) = \int_{\mathbb{R}^n} \chi_A(y)\chi_A(x+y)\,dy$. Prove that ϕ is continuous.
(ii) Use (i) to show that the set $A - A$ contains a neighborhood of the origin.

Problem 40. [Jan'06] Prove or give a counter-example to the following: Let $f_n \in L_1[0,1]$, $n \in \mathbb{N}$ and suppose that $f_n \to 0$ in $L_1[0,1]$. Then $f_n \to 0$ a.e.

Problem 41. [Aug'06] Suppose that f_n, $n \in \mathbb{N}$ is a sequence of integrable functions on $[0,1]$ such that *(a)* $\lim_n f_n(x) = 0$ for all $x \in [0,1]$, and *(b)* $\int_0^1 f_n(x)\,dx = 0$ for all n. Does it follow that $\lim_n \int_0^1 |f_n(x)|\,dx = 0$? Either give a proof or a counter-example.

Problem 42. [Aug'06] Find the following limits and prove your answers:

(i) $\displaystyle\lim_{t\to 0^+} \int_0^1 \frac{e^{-t\ln x} - 1}{t}\,dx.$

(ii) $\lim_n \int_1^{n^2} \frac{n\cos(x/n^2)}{1+n\ln n}\,dx.$

Problem 43 (Spring'07)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of nonnegative measurable functions with the property that $g_n \in L_1(\mu)$ for every n, and $g_n \to g \in L_1(\mu)$. Let $\{f_n\}$ be another sequence of nonnegative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) If $F_n \leq g_n$ a.e. for every n, prove that

$$\limsup_n \int_X f_n\,d\mu \leq \int_X \limsup_n f_n\,d\mu.$$

(ii) If $f_n \to f$ a.e. and if $f_n \leq g_n$ a.e. for all n, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Problem 44. [Jan'07] Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\{g_n\}$ be a sequence of non-negative measurable functions with the property that $g_n \in L_1(\mu)$ for every n, and $g_n \to g$ in $L_1(\mu)$. Let $\{f_n\}$ be another sequence of non-negative measurable functions on $(X, \mathcal{F}, \mu)$.

(i) If $f_n \leq g_n$ a.e. for every n, prove that

$$\limsup_n \int_X f_n\,d\mu \leq \int_X \limsup_n f_n\,d\mu.$$

Hint: Start by considering a subsequence f_{n_r} such that $\lim_r \int_X f_{n_r}\,d\mu = \limsup_n \int_X f_n\,d\mu$, and let $g_{n_{r_s}}$ be a subsequence of g_{n_r} such that $g_{n_{r_s}} \to g$ a.e.

(ii) If $f_n \to f$ a.e. and if $f_n \leq g_n$ a.e. for all n, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.