L_p SPACES. V2.0

Problem 1. Let ν be a measure on the Borel sets f the positive real line $[0,\infty)$ such that $\phi(t) = \nu[0,t)$ is finite for every $t > 0$. Now let $(X, \mathcal{F}, \mu)$ be a measure space and f any non-negative measurable function on X. Then

$$\int_X \phi\big(f(x)\big)\, d\mu = \int_0^\infty \mu\{x \in X : f(x) > t\}\, d\nu.$$

Problem 2. Verify that for every measurable function f, and $0 < p < \infty$,

$$\int_X |f|^p\, d\mu = \int_0^\infty p\, t^{p-1} \mu\{|f| > t\}\, dt.$$

Problem 3 (The Layer cake representation)**.** Verify that for every non-negative measurable function f,

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)\, dt.$$

Problem 4. Suppose f and g are two non-negative functions satisfying the following inequality: There exists a constant C such that for all $\varepsilon > 0$ and $\lambda > 0$,

$$\mu\{x \in X : f(x) > 2\lambda,\, g(x) \leq \varepsilon\lambda\} \leq C\varepsilon^2\, \mu\{x \in X : f(x) > \lambda\}.$$

Prove that

$$\int_X f(x)^p\, d\mu \leq C_p \int_X g(x)^p\, dx$$

for any $0 < p < \infty$ for which both integrals are finite, where C_p is a constant depending on C and p.

Problem 5. When does equality hold in Minkowski's inequality? The answer is different for $p = 1$ and for $1 < p < \infty$. What about $p = \infty$?

Problem 6. Let $I = [0, \pi]$. Show that $\int_I x^{-1/4} \sin x\, dx \leq \pi^{3/4}$.

Problem 7. Let $I = [0, \pi]$ and $f \in L_2(I)$. Is it possible to have simultaneously $\int_I \big(f(x) - \sin x\big)^2\, dx \leq 4/9$ and $\int_I \big(f(x) - \cos x\big)^2\, dx \leq 1/9$?

Problem 8. Let I be a bounded interval on $\mathbb{R}$. By means of an example, show that in general

$$\bigcap_{0<p<q} L_p(I) \neq L_q(I), \quad 0 < q \leq \infty.$$

Problem 9. Suppose $f \in L_p(\mu)$, $g \in L_q(\mu)$, $h \in L_r(\mu)$, $1 < p, q, r < \infty$, $1/p + 1/q + 1/r = 1$. Prove that $fgh \in L(\mu)$ and that $\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r$.

Problem 10 (Spring'06)**.** Show that $L_\infty(0,1) \subset \cap_{p\geq 1} L_p(0,1)$. Is equality true?

Problem 11. Show that if for some $0 < p < \infty$ $f \in L_p(\mu) \cap L_\infty(\mu)$, then for all $p < q < \infty$, $f \in L_q(\mu)$ and $\|f\|_q \leq \|f\|_p^{p/q} \|f\|_\infty^{1-p/q}$.

Problem 12 (Spring'06)**.** Show that if $f \in L_p[0,1] \cap L_r[0,1]$, with $p < r$, then $f \in L_s[0,1]$ for all $p \leq s \leq r$.

> **Hint:** The result is also true for a general measure space. Prove that $\|f\|_s \leq \|f\|_p^{1-\eta} \|f\|_r^{\eta}$, where $0 < \eta < 1$ is given by $1/s = (1-\eta)/p + \eta/r$.

Problem 13. Prove that if $\mu(X) < \infty$ and $f \in L_p \cap L_\infty$ for some $p < \infty$ so that $f \in L_q$ for all $q > p$, then $\|f\|_\infty = \lim_q \|f\|_q$.

Problem 14. Suppose $\mu(X) = 1$, and $f \in L_p$ for some $p > 0$, so that $f \in L_q$ for $0 < q < p$. Prove the following statements:

(i) $\log\|f\|_q \geq \int \log|f|$.
(ii) $\int \frac{|f|^q - 1}{q} \geq \log\|f\|_q$, and $\int \frac{|f|^q-1}{q} \to \int \log|f|$ as $q \to \infty$.
(iii) $\lim_{q\to 0} \|f\|_q = \exp\left(\int \log|f|\right)$.

Problem 15. Prove that if $\lim_n \|f_n\|_p = 0$, $1 \leq p \leq \infty$, then there exists a subsequence $\{f_{n_k}\}$ and a non-negative function $h \in L_p(\mu)$ such that $|f_{n_k}| \leq h$ a.e., and $\lim_k f_{n_k} = 0$ a.e.

Problem 16. Prove the following statements: Suppose $1 \leq p < \infty$. If $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure, and hence some subsequence converges to f a.e. On the other hand, if $f_n \to f$ in measure and $|f_n| \leq g \in L_p$ for all n, then $\|f_n - f\|_p \to 0$.

Problem 17. Prove that if $f, f_n \in L_p(\mu)$, $g, g_n \in L_q(\mu)$, $\|f_n - f\|_p \to 0$, and $\|g_n - g\|_q \to 0$, $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$, then $\|f_n g_n - fg\|_1 \to 0$.

Problem 18. Suppose $f, f_n \in L_p(\mu)$, $n \in \mathbb{N}$ satisfy $\lim_n f_n = f$ a.e., and $\lim_n \|f_n\|_p = \|f\|_p$, $0 < p < \infty$. Prove that $\lim_n \|f_n - f\|_p = 0$. Is the conclusion still true if we replace a.e. convergence by convergence in measure?

Problem 19. Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $0 < r < p$ and $\{f_n\}$ a sequence of $L_p(\mu)$ functions such that $\|f_n\|_p \le k$ for all n, and $\lim_n f_n = f$ a.e. Prove that $\lim_n \|f_n - f\|_r = 0$. Prove that the conclusion my fail with $\mu(X) = \infty$.

Problem 20 (Spring'05)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 \le p < \infty$. If $\{f_n, f\} \in L_p(\mu)$ and $\int_X f_n\, g\, d\mu \to \int_X f g\, d\mu$ for every $g \in L_{p'}(\mu)$, $1/p + 1/p' = 1$, show that

$$\|f\|_p \le \liminf_n \|f_n\|_p.$$

Problem 21 (Spring'05)**.** Let $f_n\colon I \to \mathbb{R}^+$ be non-decreasing on $I = [a, b]$ with $\|f\|_\infty \le M < \infty$, $n \in \mathbb{N}$. Assume that $\{f_n\}$ converges on a dense subset of I. Show that $\{f_n\}$ converges at every point of I except perhaps a countable set.

Problem 22 (Fall'06)**.** suppose that $f_n \in L_1[0, 1]$, $n \in \mathbb{N}$ is such that $\lim_n f_n(x) = f(x)$ a.e. x.

(i) Suppose that $\lim_n |f_n(x)|^{1/p} = |f(x)|^{1/p}$ *uniformly* on $[0, 1]$. Prove that then $\lim_n |f_n|^{1/p} = |f|^{1/p}$ in $L_p[0, 1]$, i.e. prove that $\lim_n \left\| |f_n|^{1/p} - |f|^{1/p} \right\|_p = 0$.

(ii) Prove that the conclusion in (i) still holds if instead of the uniform convergence of $|f_n|^{1/p}$, we assume that $\lim_n f_n = f$ in $L_1[0, 1]$, i.e. $\lim_n \|f_n - f\|_1 = 0$.

Problem 23 (Spring'07)**.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $f_n\colon X \to [0, \infty)$ be a sequence of measurable functions and suppose that $\|f_n\|_p \le 1$, $1 < p < \infty$, and that $f_n \to f$ a.e. Prove:

(i) $f \in L_p(\mu)$.

(ii) $\|f_n - f\|_1 \to 0$ as $n \to \infty$.

Problem 24. Show that each function $f \in L_p(\mu)$, $0 < p < \infty$ satisfies the following property: $\lim_\lambda \lambda^p \mu\{|f| > \lambda\} = 0$.

Problem 25 (Fall'05)**.** Let f be Lebesgue measurable on $[0, 1]$ with the property that $\|f\|_2 = 1$ and $\|f\|_1 = 1/2$. Prove that

$$\frac{1}{4}(1 - \lambda)^2 \le m\big\{x \in [0, 1] : |f(x)| \ge \lambda/2\big\},$$

for all $0 \le \lambda \le 1$.

Problem 26 (Spring'05)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space, $f\colon X \to \mathbb{R}$ measurable, and let $1 \le p_1 < p_2 < \infty$. Assume there exist constants $0 < c_1, c_2 < \infty$ such that

$$\mu\{x : |f(x)| > y\} \le \frac{c_j}{y^{p_j}}; j = 1, 2,$$

for every $y > 0$. Show that $f \in L_p(\mu)$, $p_1 < p < p_2$.