$$\overline{S_a} = \frac{1}{i} L_a T$$

$$h(a) = \langle h, S_a \rangle = \int h \, \overline{S_a} \, ds = \int h \left(\tfrac{1}{i} L_a T\right) ds$$

$$= \frac{1}{i} \int h L_a \, dz \underset{\substack{\uparrow \\ \text{Res Thm}}}{=} 2\pi i \left(\frac{1}{i} \cdot \frac{1}{2\pi i} h(a) \right) = h(a)$$

Big Fact: $L(z,a) \neq 0$ if $z \neq a$, $(z,a) \in \overline{\Omega} \times \Omega$.

Lemma: $L(z,a) = \frac{1}{2\pi i} \cdot \frac{1}{(z-a)} + \ell(z,a)$

where $\ell \in C^\infty(\overline{\Omega} \times \overline{\Omega})$.

Pf: $\overline{S_a} = \frac{1}{i} L_a T = \frac{1}{i} \Big(\underbrace{\frac{1}{2\pi i (z-a)}}_{G_a} + \underset{\substack{\uparrow \\ \ell(z,a)}}{\ell_a} \Big) T$

$$i \, \overline{S_a} \, \overline{T} = G_a + \ell_a$$

$\uparrow$ $\perp H^2(b\Omega)$.

Take P of eqn:

$$0 = P G_a + \underbrace{P \ell_a}_{\ell_a}$$

$$\ell_a = -P G_a$$

$$P(I+A) \overset{KS}{=} \mathcal{C}$$

$$P = \mathcal{C} - PA$$

$$\ell_a = -\underbrace{\mathcal{C}\, G_a}_{=0} + PA\, G_a$$

$$\mathcal{C}\, G_a = \frac{1}{2\pi i}\int \frac{1}{w-z}\cdot\frac{1}{2\pi(w-a)}\, dw = 0 \text{ by Res Thm.}$$

Next:

$$A\, G_a = \int_{w\in b\Omega} A(z,w)\,\frac{1}{2\pi(w-a)}\,\underbrace{ds}_{\overline{T}\,\underbrace{T\,ds}_{dw}}$$

$$= \frac{1}{2\pi i}\int_{w\in b\Omega} \frac{i\,A(z,w)\,\overline{T}(w)}{w-a}\, dw$$

$$= \mathcal{C}\Big(\underbrace{i\,A(z,\cdot)\,\overline{T}(\cdot)}_{C^\infty \text{ smooth}}\Big)(a) \in C^\infty(\bar{\Omega}) \text{ in } a.$$

Can diff under $P\mathcal{C}$ in z variable to see

$$\ell \in C^\infty(\bar{\Omega}\times\bar{\Omega}).$$

<u>Pf of fact:</u> Suppose $L(a,b)=0$ where $a,b\in\Omega$ and $a\neq b$.
$[=-L(b,a)=0]$

Then $L(z,a)L(z,b)$ is in $A^\infty(\Omega)$ because the poles cancel the zeroes.

<u>Trick:</u> $L(z,a)\,L(z,b)\,T(z) = \overline{S(z,a)S(z,b)T(z)}$

Hmmm: $L(z,a)L(z,b)T(z) \perp H^2$ and $\overline{H^2}$.

<u>Fact</u>: On a simply connected domain, this implies $L_a L_b T \equiv 0$.

<u>Why</u>: $0 = \int L_a L_b T \bar{h}\, ds = \int L_a L_b \bar{h}\, dz$

$$\underset{\text{Green's}}{=} \iint_\Omega \frac{\partial}{\partial \bar{z}}\left[L_a L_b \bar{h} \right] d\bar{z} \wedge dz$$

$$= \iint_\Omega L_a L_b \overline{h'} \left(2i\, dx \wedge dy \right)$$

Take h = anti-deriv of $L_a L_b$!

See $L_a L_b \equiv 0$.

Back to proof of Fact on a simply conn. domain.

$L_a L_b \equiv 0 \implies L_a \equiv 0$ or $L_b \equiv 0$.

But that's crazy. [$S_a \not\equiv 0$ and $\bar{S}_a = \frac{1}{i} L_a T$.]

Conclude that $L(a,b) \neq 0$.

<u>Next</u>: See that $L(a,z) \neq 0 \quad a \in b\Omega,\ z \in \Omega$.

Let $a_i \in \Omega$ with $a_i \to a \in b\Omega$.

Hurwicz Thm! $L(a_i, z)$ are non-vanishing analytic and converge unif on compacts to $L(a,z)$.

So $L(a,z) \equiv 0$ in z or never zero.

But $L(a,z) = \frac{1}{2\pi i(a-z)} + \underbrace{\ell(a,z)}_{\text{Bdd}}$ pole behavior at $z=a$

Can't be that $L(a,z) \equiv 0$.

Thm: $L(z,a) \not\equiv 0$ on $\bar{\Omega} \times \Omega$ if $z \neq a$.

True always. Proved for Ω simp. conn.

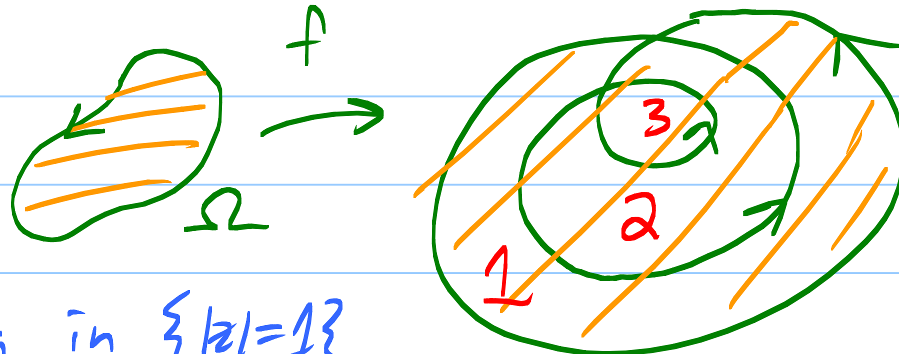
Thm: Ω C^∞ smooth simply connected domain. Riemann Mapping Fuction is $f_a = \frac{S_a}{L_a}$ and is in $C^\infty(\bar{\Omega})$.

Pf: $\bar{S}_a = \frac{1}{i} L_a T$. Argument Princ. shows

$S(z,a)$ is nonvanishing in z on $\bar{\Omega}$.

Define $f_a = \frac{S_a}{L_a}$. $\Leftarrow$ $\in A^\infty(\Omega)$, $|f_a| = 1$ on $b\Omega$, Has one simple zero at a. $(S_a(a) > 0)$.

Argument Princ can be used to see f_a is the R. Map Fcn.



f_a hits everything in $\{|z|=1\}$ exactly once! f_a is 1-1 and onto $D_1(0)$.

Remark: on $D_1(0)$
$$S(z,w) = \frac{1}{2\pi(1-z\bar{w})}$$
$$L(z,w) = \frac{1}{2\pi i(z-w)}$$

Remark: Can prove general R. Mapping Thm by exhausting Ω by C^∞ smooth simp. conn. domains. (Do bounded Ω case first. Careful with $\Omega = \mathbb{C}$ case.)

Next: Solve Dirichlet Prob in $C^\infty(\bar{\Omega})$.

Idea:
$$\varphi = h + \overline{H T}$$
$$\varphi = h + \bar{H}\left(\frac{-iS_a}{\bar{L}_a}\right)$$
$$\frac{\varphi}{S_a} = \frac{h}{S_a} - i\,\underbrace{\frac{\bar{H}}{\bar{L}_a}}_{}$$

$$\boxed{\begin{aligned} \bar{S}_a &= \tfrac{1}{i} L_a T \\ -\frac{iS_a}{\bar{L}_a} &= \bar{T} \end{aligned}}$$

harmonic!

Hmmm. Should have taken $\varphi = \varphi S_a$!