Bergman space $H^2(\Omega) \subset L^2(\Omega)$.

Fact: $H^2(\Omega)$ is a closed subspace of $L^2(\Omega)$.

Key: Averaging property:

$$\frac{1}{2\pi r} \int_{C_r(a)} f(z)\, ds = f(a)$$

$$f(a) = \frac{1}{2\pi i} \int_{C_r(a)} \frac{f(z)}{z-a}\, dz$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a}\, ire^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi r} \int_0^{2\pi} f(a+re^{i\theta}) \underbrace{r\, d\theta}_{ds}$$

Ave over disc $\overline{D_r(a)} \subset \Omega$ with respect to area:

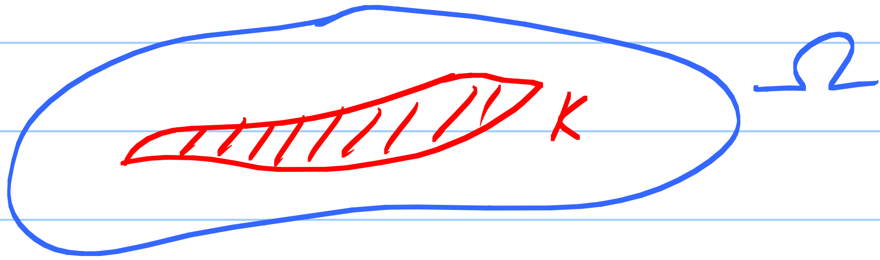
$$\iint_{D_r(a)} f(z)\, dA = \underbrace{\iint \quad r\,dr\,d\theta}_{\text{polar coord}} = \int_0^r r \underbrace{\left(\int_0^{2\pi} f\, d\theta \right)}_{2\pi f(a)} dr$$

$$= \pi r^2 f(a)$$

Cor: If h_j analytic and $h_j \xrightarrow[L^2(\Omega)]{} h$, then

h_j conv unif on compact subsets. So $\exists$ H analytic on Ω such that $H = h$ a.e.

Pf:



Let $r = \frac{1}{2}\operatorname{dist}(K, b\Omega)$. For $a \in K$, we have

$$h_j(a) = \frac{1}{\pi r^2} \iint_{D_r(a)} h_j \, dA$$

$$= \left\langle h_j, \frac{1}{\pi r^2} \chi_{D_r(a)} \right\rangle$$

So

$$|h_j(a)| \le \|h_j\| \underbrace{\left\| \frac{1}{\pi r^2} \chi_{D_r(a)} \right\|}_{\frac{1}{\sqrt{\pi}\, r}}$$

See h_j unif cauchy on K. So h_j conv unif on compacts. ✓

<u>Def^n</u>: Since H^2 is closed in L^2, the Bergman Projection: $B: L^2(\Omega) \xrightarrow[\text{orthog proj.}]{} H^2(\Omega)$ is well defined.

<u>Remark</u>: $h \mapsto \langle h, u \rangle$ $u \in L^2$, $h \in H^2$.

Riesz Rep Thm: $\exists\ g \in H^2$ such that

cont. linear fcnl $h \mapsto \langle h, u \rangle = \langle h, g \rangle$

where $g \in H^2$. $g = Bu$.

__Bergman kernel:__ $K_a = B\left(\frac{1}{\pi r^2} \chi_{D_r(a)}\right)$

where $\overline{D_r(a)} \subset \Omega$.

__Fact:__ $\langle h, K_a \rangle = h(a)$.

__Remark:__ I showed point evaluation at a is a cont. lin. fcnl on H^2. Riesz Rep Thm gives $K_a \in H^2$ so that

$$\langle h, K_a \rangle = h(a).$$

__Next:__ Cook up $\varphi \in C_0^\infty(\Omega)$ so that $K_a = B\varphi$.

Suppose $\overline{D_r(a)} \subset \Omega$. Let φ_a be a fcn in $C_0^\infty(D_r(a))$, $\varphi_a \geq 0$, radially symmetric, and $\iint \varphi \, dA = 1$. Now

$$\iint h \varphi_a \, dA \underset{\text{polar coord}}{=} \int_0^r r \underbrace{\left(\int_0^{2\pi} h \varphi_a \, d\theta \right)}_{h(a) \int_0^{2\pi} \varphi_a \, d\theta} dr$$

$$= h(a) \underbrace{\iint \varphi_a \, dA}_{1} = h(a).$$

So $K_a = B\varphi_a$.

Orthogonal complement of $H^2(\Omega)$.

Lemma: Suppose Ω has C^∞ smooth boundary and $\varphi \in C^\infty(\overline{\Omega})$ and $\varphi|_{b\Omega} = 0$.

Then $\left\langle \frac{\partial \varphi}{\partial z}, h \right\rangle = 0$ for all $h \in H^2(\Omega)$.

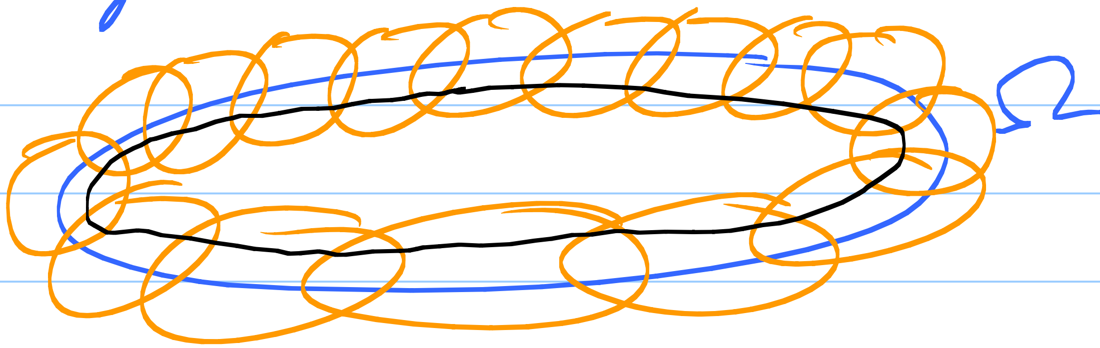
Remark: Easy if $h \in A^\infty(\Omega)$. Then

$$\left\langle \frac{\partial \varphi}{\partial z}, h \right\rangle = \iint_\Omega \frac{\partial \varphi}{\partial z} \, \bar{h} \left(\frac{-1}{2i} \, dz \wedge d\bar{z} \right)$$

$$\overset{\text{"int. by parts"}}{=} -\iint_\Omega \varphi \underbrace{\frac{\partial}{\partial z} \bar{h}}_{=0} \left(-\frac{1}{2i} \, dz \wedge d\bar{z} \right)$$

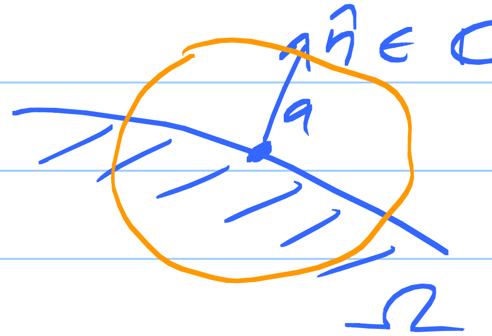
$$= \frac{-1}{2i} \iint \frac{\partial}{\partial z} (\varphi \bar{h}) \, dz \wedge d\bar{z} \overset{\text{Green's}}{=} \frac{-1}{2i} \int_{b\Omega} \varphi \bar{h} \, d\bar{z} = 0$$

Trick for general $h \in H^2(\Omega)$:



Partition of unity subordinate to cover ψ_j.

$$\frac{\partial \varphi}{\partial z} = \sum \frac{\partial}{\partial z}(\varphi \psi_j).$$ Reduced to a φ supported near $a \in b\Omega$.



$h \in H^2(\Omega)$.

$h_\varepsilon(z) = h(z - \varepsilon \hat{n})$. ← analytic, smooth up to $b\Omega$ near a.

$$\iint \frac{\partial \varphi}{\partial z} \bar{h}\, dA = \lim \underbrace{\iint \frac{\partial \varphi}{\partial z} \bar{h}_\varepsilon\, dA}_{=0}.$$ Done!

<u>Theorem</u>: $B = I - 4 \frac{\partial}{\partial z} G \frac{\partial}{\partial \bar{z}}$ on $C^\infty(\bar{\Omega})$.

<u>Pf</u>: Given $u \in C^\infty(\bar{\Omega})$.

Recall $\Delta(Gv) = v$ and $Gv\big|_{b\Omega} = 0$.

$$\underbrace{u - 4\frac{\partial}{\partial z}\left(G\frac{\partial u}{\partial \bar z}\right)}_{\perp \text{ to } H^2(\Omega) \text{ and in } C^\infty(\bar\Omega).}$$

Hit it with $\frac{\partial}{\partial \bar z}$:

$$\underbrace{\frac{\partial u}{\partial \bar z} - \underbrace{4\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}}_{\Delta}\left(G\frac{\partial u}{\partial \bar z}\right)}_{\frac{\partial u}{\partial \bar z}} = 0$$

$$h = B\Big(\underbrace{u - 4\frac{\partial}{\partial z}G\frac{\partial u}{\partial \bar z}}_{h}\Big) = Bu - 0$$

Note: We showed $G: C^\infty(\bar\Omega) \circlearrowleft$ in smooth, simply conn. domains. True in multiply conn. too.

Cor: $K_a \in C^\infty(\bar\Omega)$.