MA16010 : Applied Calculus I (Traditional)
Fall 2017

COURSE WEB PAGE: http://www.math.purdue.edu/ma16010

PREREQUISITE: MA 15400 C- or better, MA 15800 C- or better, ALEKS score of 75% or above, SAT math score of 600 or above, or ACT math score of 26 or above

TEXTBOOK: No textbooks required to purchase. Course contents will be provided to students online through LON-CAPA, free of charge.

HOMEWORK ACCESS: Online homework access through LON-CAPA will be provided to students, free of charge.

Discussion Board: We will use Piazza for online discussion. A graduate assistant will moderate the discussion board on a regular basis.

Calculator: A scientific calculator with a one-line display is required. ONLY THIS TYPE OF CALCULATORS WILL BE ALLOWED. NO EXCEPTIONS. Recommended is the TI-30Xa. If in doubt, please double check with your instructor. You are allowed to use but NOT to share the approved calculators on quizzes (no calculators on the first or the last quiz) or exams.

Homework: Homework assignments will be assigned regularly. Each assignment is due at **10:00pm Eastern time** the following meeting day. There are 36 homework assignments with 2 points each. The **lowest** homework score will be dropped. If you are not able to complete your homework on time and have extenuating circumstances with valid supporting documents, please talk to your instructor.

Quizzes: There will be **15 quizzes total with 5 points each**. The **lowest** quiz score will be dropped. There will be no make-up quizzes. In extenuating circumstances, your instructor may choose to excuse you from a quiz.

Exams: There will be three midterm exams and a final exam. All exams are course-wide, multiple-choice, machine-graded exams. The three midterms are all evening exams. The time and date of the midterms can be found in the calendar. The final exam information will be given later in the semester. **The semester does not end until Saturday, Dec 16 at 9:00 pm. Individuals wanting to leave campus early will not be granted early final exams to accommodate travel plans.**

If you miss an exam for any reason, please contact your instructor immediately and explain why you missed the exam. You should be prepared to present documentation to your instructor that supports the reason for your absence. If you contact your instructor within 24 hours from the scheduled exam, your instructor will allow you to take an alternate exam either with no penalty OR with a 20 point deduction, depending on the reason for your absence. If you miss an exam with no valid reasons and you do not contact your instructor within 24 hours from the scheduled exam, you will not be allowed for an alternate exam. Not knowing the right time, date or location of an exam is not a valid reason for missing it.

Warning: If there are any special circumstances that may affect your ability to successfully complete an exam (illness, family emergency, etc.), you must discuss the situation with your instructor before taking the exam, even if you must do so right before the exam. Your instructor

will then be able to advise you on your options. Do not wait until after you take the exam to mention a situation to your instructor.

OFFICE HOURS: Most instructors hold office hours in the Math Help Rooms, MATH 205 or MATH 211. In addition to instructors from your course, instructors from other courses in the help rooms can also help you. The office hour schedule can be found on the course website. The schedule will be finalized after the first week of classes. You are strongly urged to go to office hours if you have questions. It is the best way to get individual help.

GRADES: The course grade will be based on a total of 640 points.

Homework	70
Quizzes	70
Exam 1	100
Exam 2	100
Exam 3	100
Final	200
Total	640

Final letter grades will be determined using the following grading scale.

total points > 615: A+
$563 <$ total points ≤ 615: A
$545 <$ total points ≤ 563: A-
$527 <$ total points ≤ 545: B+
$480 <$ total points ≤ 527: B
$462 <$ total points ≤ 480: B-
$444 <$ total points ≤ 462: C+
$389 <$ total points ≤ 444: C
$371 <$ total points ≤ 389: C-
$353 <$ total points ≤ 371: D+
$306 <$ total points ≤ 353: D
$288 <$ total points ≤ 306: D-
total points ≤ 288: F

ACCOMMODATIONS FOR STUDENTS WITH DISABILITIES: If you have been certified by the Disability Resource Center (DRC) as eligible for academic adjustments on exams or quizzes, please see http://www.math.purdue.edu/ada for exam and quiz procedures for your mathematics course or go to MATH 202 for paper copies.

In the event that you are waiting to be certified by the Disability Resource Center we encourage you to review our procedures prior to being certified. For all in-class accommodations, please see your instructors outside class hours-before or after class or during office hours-to share your Accommodation Memorandum for the current semester and discuss your accommodations as soon as possible.

CAMPUS EMERGENCY PROCEDURE: In the event of a major campus emergency, course requirements, deadlines and grading percentages are subject to changes that may be necessitated by a revised semester calendar or other circumstances beyond the instructor's control.

Announcements regarding campus emergencies will be sent via course-wide emails and be posted on the course web page.

ACADEMIC DISHONESTY: The Mathematics Department will not tolerate academic dishonesty of any sort. If academic dishonesty occurs, then grade penalties will be imposed, possibly to the extent of an "F" in the course. Additionally, all cases of academic dishonesty will be reported to the Office of the Dean of Students for disciplinary action (which may include probation, suspension, or expulsion). If you would like to report issues of academic integrity, you can report to the Office of the Dean of Students (purdue.edu/odos), call 765-494-8778 or email integrity@purdue.edu.

SECTION CHANGES AND DROPS: During the first week of classes, you can make section changes via Banner within myPurdue, and no signatures are required. From the second through ninth weeks of the semester, see the instructor of the section you want to enter for the required signature. If you want to drop this course during the first nine weeks of the semester, your instructor can sign your drop form. If your instructor is not available, go to MATH 835.

LAST ADD DATE: The last day you can add this course is Tues, Sep 26. Students adding at this time must take an alternate Exam 1. Students are expected to keep up with the current material while studying for the alternate Exam 1.

COURSE EVALUATIONS: On Monday of the fifteenth week of classes, you will receive an official email from evaluation administrators with a link to online course evaluations. You will have two weeks to complete this evaluation. You are strongly encouraged to participate. Your feedback is vital to maintaining and improving the quality of education at Purdue University.

EMERGENCY PREPAREDNESS SUMMARY: A document about emergency preparedness can be found on the course web page under syllabus. Here is a summary.

If an alarm is heard inside a building, immediately evaluate the building. Get a safe distance from the building. Remain outside the building until police, fire, or other emergency response personnel provide additional guidance or tell you it is safe to leave or return to the building.

If an alarm is heard outside a building, immediately seek shelter in a safe location within the closest building. These types of alarms may indicate a tornado, a civil disturbance, or release of hazardous materials in the outside air. Remain inside the building until police, fire, or other emergency response personnel provide additional guidance or tell you it is safe to leave.

In both cases above, you should seek additional clarifying information by all means possible such as Purdue University home page, email alert, TV, radio, etc.

MA 16010 Applied Calculus I

Calendar (Traditional and Distance), Fall 2017

Exam 1: Lesson 2-10 Exam 2: Lesson 11-18 Exam 3: Lesson 19-28

Date	Lesson	Topics
8/21 Mon	1	Course Information; CCI (no calculators)
8/23 Wed	2	Finding Limits Numerically; One-sided Limits
8/25 Fri	3	Finding Limits Graphically
8/28 Mon	4	Finding Limits Analytically
8/30 Wed	5	Continuity
9/1 Fri	6	The Derivative
9/4 Mon		LABOR DAY (NO CLASSES)
9/6 Wed	7	Basic Rules of Differentiation; Derivatives of the Sine and Cosine Functions; Derivative of the Natural Exponential Function
9/8 Fri	8	Instantaneous Rates of Change
9/11 Mon	9	The Product Rule
9/13 Wed	10	The Quotient Rule; Derivatives of the Other Trigonometric Functions
9/15 Fri	11	The Chain Rule
9/18 Mon	12	The Chain Rule; Derivative of the Natural Logarithmic Function
9/20 Wed	13	Higher Order Derivatives
9/22 Fri		REVIEW FOR EXAM 1
9/25 Mon		**NO CLASSES**
9/25 Mon		**EXAM 1 Time: 6:30-7:30pm Location: ELLT 116**
9/27 Wed	14	Implicit Differentiation
9/29 Fri	15	Related Rates
10/2 Mon	16	Related Rates
10/4 Wed	17	Relative Extrema and Critical Numbers
10/6 Fri	18	Increasing and Decreasing Functions and the First Derivative Test
10/9 Mon		**FALL BREAK (NO CLASSES)**
10/11 Wed		REVIEW FOR EXAM 2
10/11 Wed		**EXAM 2 Time: 8:00-9:00pm Location: ELLT 116**
10/13 Fri	19	Concavity, Inflection Points and the Second Derivative Test
10/16 Mon	20	Absolute Extrema on an Interval
10/18 Wed	21 Q	Graphical Interpretation of Derivatives
10/20 Fri	22 Q	Limits at Infinity
10/23 Mon	23	A Summary of Curve Sketching
10/25 Wed	24 Q	Optimization
10/27 Fri	25	Optimization
10/30 Mon	26 Q	Optimization
11/1 Wed	27	Antiderivatives and Indefinite Integration
11/3 Fri	28 Q	Antiderivatives and Indefinite Integration

MA 16010 Applied Calculus I

Calendar (Traditional and Distance), Fall 2017

Exam 1: Lesson 2-10 Exam 2: Lesson 11-18 Exam 3: Lesson 19-28

Date	Lesson	Topics
11/6 Mon	29 Q	Area and Riemann Sums
11/8 Wed	30 Q	Definite Integrals
11/10 Fri		REVIEW FOR EXAM 3
11/13 Mon		**NO CLASSES**
11/13 Mon		**EXAM 3 Time: 6:30-7:30pm Location: ELLT 116**
11/15 Wed	31	The Fundamental Theorem of Calculus
11/17 Fri	32	The Fundamental Theorem of Calculus
11/20 Mon		**NO CLASSES**
11/22 Wed		**THANKSGIVING VACATION (NO CLASSES)**
11/24 Fri		**THANKSGIVING VACATION (NO CLASSES)**
11/27 Mon	33	Numerical Integration
11/29 Wed	34 Q	Exponential Growth
12/1 Fri	35 Q	CCI (no calculators)
12/4 Mon	36	Exponential Decay
12/6 Wed		REVIEW FOR FINAL EXAM
12/8 Fri		REVIEW FOR FINAL EXAM
12/11-12/16		WEEK OF FINAL EXAMS

12/11 Final: 7:00 – 9:00 pm

MA 16010 - 120 / 110

Alden Bradford
bradfoa@purdue.edu
Math 605

math.purdue.edu/courses/MA16010
math.purdue.edu/academic/courses/ada

- Full Name
- Prefer to be called
- year
- Major or Area of interest
- $\sqrt{x+y} - \ln(2e^{17}) + \sin(u)$

Agenda

- Attendance / cards
- course structure:
  - 1 quiz per week (about), 15 total, 1 dropped, first friday
  - you need a calculator
  - we use LON-CAPA
  - review materials available
  - office hours to be announced by weds., or by appointment tues/thurs
- CCI (calculus concept inventory)
  - worth 2 bonus points
  - may write an essay instead

Finding limits numerically

Math Help Room hours:

Mon-Thurs 10:30-5:30

Friday 10:30-2:30

My hours:

Thursday 12:30-2:30

In[7]:=
```
f[x_] := 3 x + 4;
maketable[f, 2]
```

Out[8]=

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
4 + 3 x	9.7	9.97	9.997	9.9997	10	10.0003	10.003	10.03	10.3

In[11]:= $f[x_] := \frac{x^2 + 2x - 3}{2x - 2};$

```
maketable[f, 1]
```

Out[12]=

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$\frac{-3+2x+x^2}{-2+2x}$	1.95	1.995	1.9995	1.99995	Indeterminate	2.00005	2.0005	2.005	2.05

In[13]:= $f[x_] := \frac{\text{Tan}[x]}{x};$

```
maketable[f, 0]
```

Out[14]=

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$\frac{\text{Tan}[x]}{x}$	1.00335	1.00003	1.	1.	Indeterminate	1.	1.	1.00003	1.00335

In[15]:= $f[x_] := \frac{1}{x^2};$

```
maketable[f, 0]
```

Out[16]=

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$\frac{1}{x^2}$	100.	10 000.	$1. \times 10^6$	$1. \times 10^8$	ComplexInfinity	$1. \times 10^8$	$1. \times 10^6$	10 000.	100.

In[17]:= $f[x_] := \frac{x+1}{x-1};$

```
maketable[f, 1]
```

Out[18]=

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$\frac{1+x}{-1+x}$	-19.	-199.	-1999.	-19 999.	ComplexInfinity	20 001.	2001.	201.	21.

In[21]:= $f[x_] := \sqrt{1 - 3x};$

```
maketable[f, -1]
```

Out[22]=

x	-1.1	-1.01	-1.001	-1.0001	-1	-0.9999	-0.999	-0.99	-0.9
$\sqrt{(1-3x)}$	2.07364	2.00749	2.00075	2.00007	2	1.99992	1.99925	1.99249	1.92354

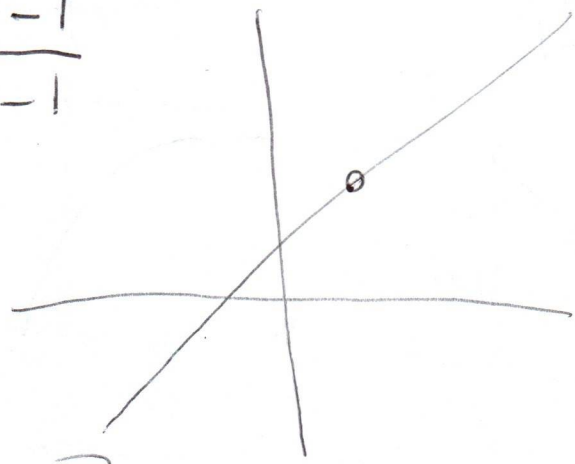
In[31]:=
```
f[x_] := 1/(4 x + 16);
maketable[f, -4]
```

Out[32]=

x	-4.1	-4.01	-4.001	-4.0001	-4	-3.9999	-3.999	-3.99	-3.9
$\frac{1}{16+4x}$	-2.5	-25.	-250.	-2500.	ComplexInfinity	2500.	250.	25.	2.5

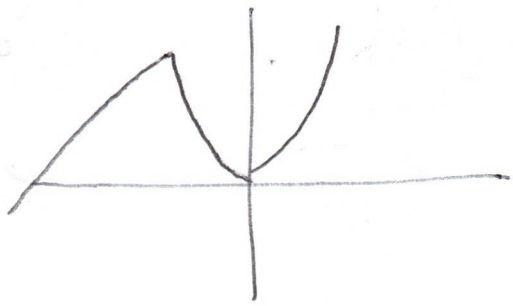
finding limits graphically (Aug 25)

$$f(x) = \frac{x^2-1}{x-1}$$



$$\lim_{x \to 1} f(x) = ?$$

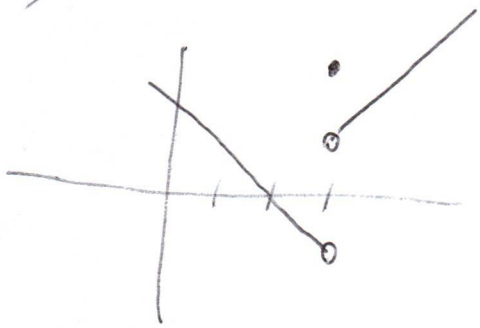
$$f(x) = \begin{cases} x^2 & \text{if } x \geq -2 \\ x+6 & \text{if } x < -2 \end{cases}$$



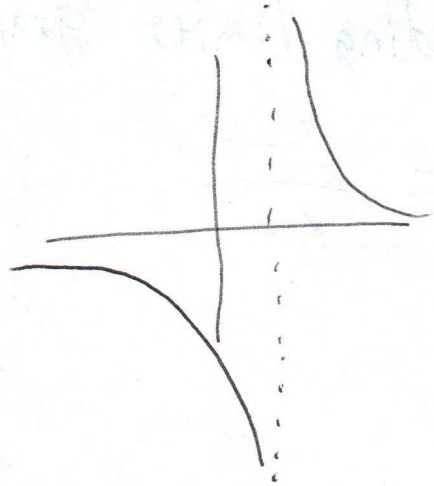
$$\lim_{x \to -2} f(x) = ?$$

$$f(x) = \begin{cases} 2-x & \text{if } x < 3 \\ x-2 & \text{if } x > 3 \\ 2 & \text{if } x = 3 \end{cases}$$

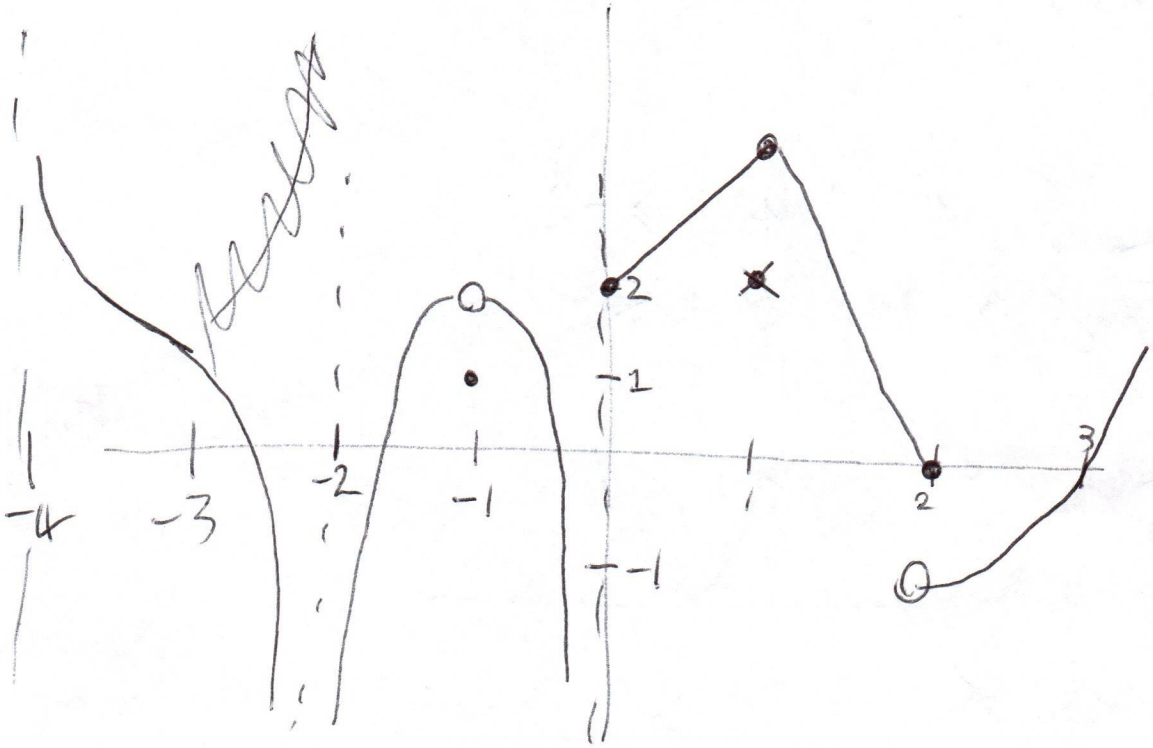
$$\lim_{x \to 3} f(x) = ?$$



$$f(x) = \frac{1}{x-1}$$



$$\lim_{x \to 1} f(x) = ?$$



Finding Limits Analytically

Quiz wednesday, bring a calculator!

$$\lim_{x \to 5} x^2 + 3x + 10 = 25 + 15 + 10 = 50$$

$$\lim_{x \to -1} \frac{x+1}{x-1} = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0$$

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - x} = \frac{1-2+1}{1-1} = \frac{0}{0}$$

$$\frac{x^2-2x+1}{x^2-x} = \frac{(x-1)(x-1)}{x(x-1)} \overset{x \neq 1}{=} \frac{x-1}{x}$$

$$\lim_{x \to 1} \frac{x-1}{x} = \frac{1-1}{1} = \frac{0}{1}$$

$$\cancel{\lim \frac{(x+6)(x^2-3)(x+2)}{(x+2)(x+3)(x+6)}}$$

$$\lim_{x \to -2} \frac{(x-3)(x+2)}{(x+2)(x+1)^2} = \lim_{x \to -2} \frac{x-3}{(x+1)^2} = \frac{-2-3}{(-2+1)^2} = \frac{-5}{(-1)^2} = -5$$

$$\lim_{x \to 3} \frac{(x-3)(x-4)}{x^3 - 3x^2} = \frac{x^2 - 7x + 12}{x^3 - 3x^2} = \frac{x-4}{x^2}$$

$$\lim_{x \to 3} \frac{x-4}{x^2} = \frac{3-4}{3^2} = -\frac{1}{9}$$

we have discussed:

$$\lim_{x \to c} f(x) = \text{number}$$

$$\lim_{x \to c} f(x) = \frac{0}{0}$$

what about $\lim_{x \to c} f(x) = \frac{0}{\text{number}}$?

what about $\lim_{x \to c} f(x) = \frac{\text{number}}{0}$?

$$\lim_{x \to 2^-} \frac{x-5}{(x-2)^2}$$

$x-2$ negative

$(x-2)^2$ positive

$\frac{1}{(x-2)^2}$ positive

$x-5$ negative

$\frac{x-5}{(x-2)^2}$ ~~positive~~ negative

$$\lim_{x \to 2^-} \frac{x-5}{(x-2)^2} = -\infty$$

$$\lim_{x \to 2^+} \frac{x-5}{(x-2)^2}$$

$x-2$ positive

$(x-2)^2$ positive

$x-5$ negative

$\frac{x-5}{(x-2)^2}$ negative

$$\lim_{x \to 2^+} \frac{x-5}{(x-2)^2} = -\infty \quad = \lim_{x \to 2} \frac{x-5}{(x-2)^2}$$

$$\lim_{x \to 3} \frac{(x-3)(x+5)}{(x-3)^2(x+2)} = \lim_{x \to 3} \frac{x+5}{(x-3)(x+2)}$$

$x+5$ positive

$x+2$ positive

$\lim_{x \to 3^-}$ $x-3$ negative $\lim_{x \to 3^-} \frac{x+5}{(x-3)(x+2)} = -\infty$

$\lim_{x \to 3^+}$ $x-3$ positive $\lim_{x \to 3^+} \frac{x+5}{(x-3)(x+2} = \infty$

lim DNE

$\lim_{x \to 3} \frac{(x-3)(x+5)}{(x-3)^2(x+2)}$ does not exist.

$\lim_{x \to 2} f(x)$

$$f(x) = \begin{cases} 3-x & \text{if } x \geq 2 \\ \frac{1+2x}{5} & \text{if } x < 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \frac{1+2(2)}{5} = \frac{5}{5} = 1$$

$$\lim_{x \to 2^+} f(x) = 3-(2) = 1$$

$$\lim_{x \to 2} f(x) = 1$$

$$\lim_{x \to \pi} f(x) \qquad f(x) = \begin{cases} 2\cos x & \text{if } x < \pi \\ 3 & \text{if } x = \pi \\ \frac{x}{\pi} - 4 & \text{if } x > \pi \end{cases}$$

$$\lim_{x \to \pi^-} f(x) = 2\cos\pi = -2$$

$$\lim_{x \to \pi^+} f(x) = \frac{\pi}{\pi} - 4 = -3$$

$$\lim_{x \to \pi} f(x) \text{ DNE}$$

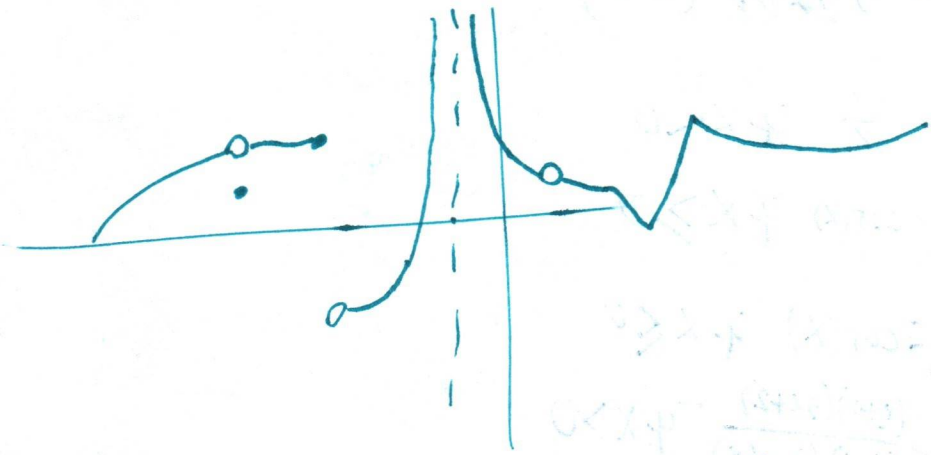
$$\lim_{x \to 1} f(x) \qquad f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x < 1 \\ 3x-1 & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x \to 1^-} x+1 = 1+1 = 2$$

$$\lim_{x \to 1^+} f(x) = 3(1) - 1 = 2$$

$$\lim_{x \to 1} f(x) = 2.$$

Continuity Aug 30



f is continuous at c if all 3 of these are true:

$f(c)$ exists

$\lim_{x \to c} f(x)$ exists

$f(c) = \lim_{x \to c} f(x)$

Is $f(x)$ continuous at $x = 4$?

$$f(x) = \begin{cases} \frac{(x-4)(x+1)}{(x-4)} & \text{if } x \neq 4 \\ 5 & \text{if } x = 4 \end{cases}$$

$$f(x) = \begin{cases} 6x - 2 & \text{if } x < 1 \\ 3x+1 & \text{if } x > 1 \\ 2 & \text{if } x = 1 \end{cases}$$

$$f(x) = \frac{(3x+1)(x-5)}{(2x+1)\cancel{(x+3)}(x-5)}$$

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ \cos(x) & \text{if } x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} -\cos(x) & \text{if } x \leq 0 \\ \frac{(x+1)(3x+2)}{(x-1)(2-6x)} & \text{if } x > 0 \end{cases}$$

CLASSROOM VISIT REPORT

TA name : Alden Bradford Date of visit : 8/30

Evaluator : Joe Chen Time of visit : 9:30

Course : 16010 Topic of class : Continuity

VG = very good ; G = good ; AD = adequate ; P = poor ; VP = very poor

1. Content

correctness	VG	G	AD	P	VP
preparedness	VG	G	AD	P	VP
consistency with textbook	VG	G	AD	P	VP
use of diagrams	VG	G	AD	P	VP
presentation : clarity	VG	G	AD	P	VP
presentation : level	VG	G	AD	P	VP
presentation : pace	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP

2. Use of the blackboard

legibility	VG	G	AD	P	VP
use of space/timely erasing	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP

3. Communication

facing the class/eye contact	VG	G	AD	P	VP
speech : loudness	VG	G	AD	P	VP
speech : clarity	VG	G	AD	P	VP
speech : speed	VG	G	AD	P	VP
speech : mannerism	VG	G	AD	P	VP
gives opportunities for questions	VG	G	AD	P	VP
asks questions	VG	G	AD	P	VP
understands questions	VG	G	AD	P	VP
answers questions	VG	G	AD	P	VP
can rephrase explanations	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP

4. Atmosphere

motivating students	VG	G	AD	P	VP
student involvement	VG	G	AD	P	VP
student attention	VG	G	AD	P	VP
other : _____	VG	G	AD	P	VP

Is a follow-up visit needed ? : no ✓ maybe ___ yes ___

Ready to teach his/her own Algebra/Trig class ? (circle one) yes maybe no unknown

PLEASE USE OTHER SIDE FOR COMMENTS

(Written comments and constructive suggestions are most helpful to instructors)

COMMENTS (Written comments and constructive suggestions are most helpful to instructors)

Arrived early. Good.

Good: asked why student ~~ca~~ had trouble w/ HW problem (limit).

Volume is good, speed is good. Sometimes your volume drops near the end of a sentence.

You explained, by saying, what it means for f(x) to be discontinuous at x=a, but didn't write it down. This is something important enough to write down.

Good diagrams. Give them a few moments to sketch before doing anything with them. Same w/ examples.

~~You explained, by referencing the diagram, what f(c) exists and~~

Good job explaining what the requirements for continuity means by referencing the diagram you drew earlier.

Writing is generally good, a little sloppy once in a while.

"any question?" more often. Not all students ~~fe~~ feel comfortable asking w/o being prompted.

Good: asked questions. Good: followed up w/ "why" after getting an answer.

Good: asked for volunteers to start an example.

- be sure to explain thoroughly the answers you got from volunteers.

In all your examples discontinuities can only happen at where the piecewise function splits. Is this always true in this course? (ok, you had one such example)

	until	workshop suggestions
returning papers	—	
go over quiz	—	
answer HW questions	9:35	
cover new material	10:02	
give quiz	end	

The Derivative

equations for lines

NO Class Mon

Slope-intercept form:

$$y = mx + b$$

point-slope form:

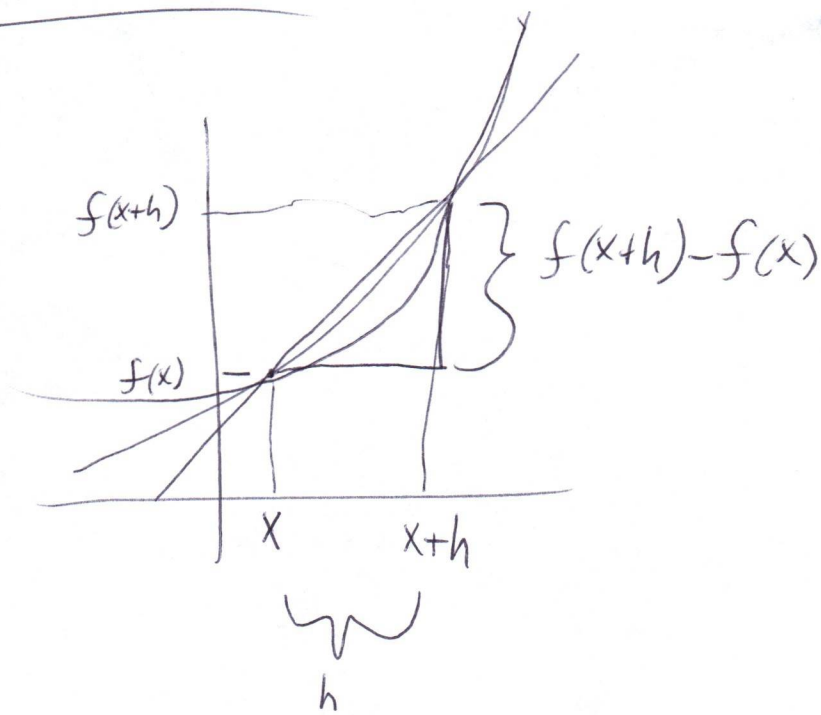
$$y - y_1 = m(x - x_1)$$

$$y = m(x - x_1) + y_1$$

Slope of a line joining two points:

$$\frac{y_2 - y_1}{x_2 - x_1}$$

tangent lines



Slope of tangent line: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{df(x)}{dx} = y' = \frac{dy}{dx}$

$y = f(x)$

ex: $f(x) = 3x^2+5$

- find the slope of the tangent line at a point x
- find the tangent line when $x=1$

ex $f(x) = \frac{1}{2}x - 3$

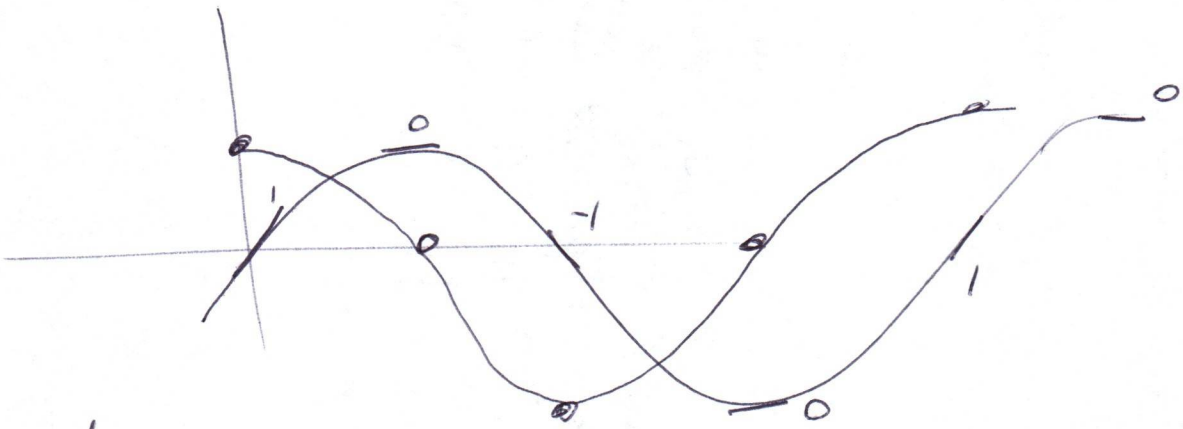
ex $f(x) = \frac{1}{x^2}$

ex $f(x) = 10$

ex $f(x) = \frac{4}{3x-2}$

ex

derivatives of sine, cosine, e^x



$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(-\sin(x)) = -\cos(x)$$

$$\frac{d}{dx}(-\cos(x)) = \sin(x)$$

$$\frac{d}{dx}e^x = e^x$$

<u>examples</u>

$$\frac{d}{dx}\left(\sin(x) + 3\cos(x) + 4e^x\right)$$

$$\cos(x) - 3\sin(x) + 4e^x$$

if $f(x) = \sqrt{x} - \sin(x)$

find $f'(\pi)$.

find the x value where y' is 2 if

$$y = 3x^{5/2}$$

$$y' = 3\left(\frac{5}{2}x^{3/2}\right) = 2$$

$$15x^{3/2} = 4$$

$$x^{3/2} = \frac{4}{15}$$

$$x^3 = \left(\frac{4}{15}\right)^2$$

$$x = \left(\frac{4}{15}\right)^{2/3}$$

The power rule

notes:

Quiz Monday

Note taker needed (§110)

$$\frac{d}{dx} x^2 = 2x$$

$$\frac{d}{dx} x^3 = 3x^2$$

$$\frac{d}{dx} x^4 = 4x^3$$

$$\frac{d}{dx} x^5 = \underline{\qquad}$$

$$\frac{d}{dx} x^{-2} = \frac{d}{dx}\frac{1}{x^2} = -\frac{2}{x^3} = -2x^{-3}$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

Question: $\frac{d}{dx} x^0 = ?$

examples:

$$\frac{d}{dx} x^{100} = 100x^{99}$$

$$\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{\frac{1}{2}}{\sqrt{x}}$$

$$\frac{d}{dx}\sqrt[10]{x^3} = \frac{d}{dx} x^{\frac{3}{10}} = \frac{3}{10}x^{-\frac{7}{10}} = \frac{3}{10\sqrt[10]{x^7}}$$

$$\frac{d}{dx}x^3\Big|_{x=2} = 3x^2\Big|_{x=2} = 12$$

constant rule

$$\frac{d}{dx} c = 0$$

constant multiple rule: $\frac{d}{dx} c f(x) = c\frac{d}{dx} f(x) = c\frac{df(x)}{dx}$

sum rule: $\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$

difference rule: $\frac{d}{dx}\left(f(x)-g(x)\right) = \frac{df(x)}{dx} - \frac{dg(x)}{dx}$

ex $\frac{d}{dx} 3x^2+5 = 3\frac{d}{dx}x^2 + \frac{d}{dx}5$

$= 3(2x) + 0$

$= 6x$

ex $\frac{d}{dx}(5x^3 - \sqrt[4]{x^3})$

$5\frac{d}{dx}x^3 - \frac{d}{dx}\sqrt{x^3}$

$5(3x^2) - \frac{d}{dx}x^{\frac{3}{2}}$

$5(3x^2) - \frac{3}{2}x^{\frac{1}{2}}$

$15x^2 - \frac{3}{2}\sqrt{x}$

ex $\frac{d}{dx}\left(\frac{x^{2.3} - x^{1/2}}{x}\right)$

$\frac{d}{dx}\left(x^{1.3} - x^{-1/2}\right)$

$1.3x^{0.3} - (-\frac{1}{2})x^{-3/2}$

Instantaneous Rate of Change

The position of an object is given by

$$s(t) = 5t^2 - 3t + 6$$

Find the position and velocity when $t = 3$.

$$s(3) = 5(3)^2 - 3(3) + 6$$
$$= 5 \cdot 9 - 9 + 6$$
$$= 45 - 9 + 6$$
$$= 42$$

$$v(t) = s'(t) = 10t - 3$$
$$s'(3) = 10(3) - 3$$
$$= 27$$

The number of rats in the hold of a pirate ship is given by

$$R(t) = 2e^t + 3t$$

where t is in weeks. How many rats will there be by the end of their six week voyage? How long until they are adding 100 rats/week?

The amount of money in my savings account is described by $M(t) = 15t^2 - \frac{10}{3}t^3 + 30$.

~~How fast is~~ where M is in dollars and t is in months since school started. How fast am I adding money one month in? four months in?

I have determined that my happiness is related to the amount of chocolate I eat by the formula

$$H(c) = 3\sin(c) - 0.7$$

where H is in smiles and c is in bars.

Find $\frac{dH}{dc}$ when I have had 1, 2, and 3 bars of chocolate. Use that info to tell me when to stop.

~~The temperature of my tea is given by~~

The population of the Hornburg Fortress during the battle ~~for~~ of Helm's Deep is given by $P(t) = 3000 - \frac{t^2}{20}$

where t is in hours since midnight. How fast are the Rohirrim getting <u>slaughtered</u> by the uruk-hai at 6:00 AM, just before Gandalf arrives with reinforcements?

Sept 11

Two ways to differentiate:

$$\frac{d}{dx}(x^2+4)(5x^4+3)$$

first way: multiply first, then differentiate

$$\frac{d}{dx}(5x^6+20x^4+3x^2+12)$$

$$30x^5+80x^3+6x.$$

second way: Product rule!

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x)+f(x)g'(x)$$

in this example:

$f(x) = x^2+4$ $\quad$ $g(x) = 5x^4+3$

$f'(x) = 2x$ $\quad$ $g'(x) = 20x^3$

$$\frac{d}{dx}(x^2+4)(5x^4+3) = 2x(5x^4+3)+20x^3(x^2+4)$$

$$= 10x^5+6x+20x^5+80x^3$$

$$= \underline{30x^5+80x^3+6x}$$

<u>another example</u>

$$\frac{d}{dx} x^{5/7}(x+1) = \left(\frac{d}{dx}x^{5/7}\right)(x+1)+\left(x^{5/7}\right)\left(\frac{d}{dx}(x+1)\right)$$

$$= \frac{5}{7}x^{-2/7}(x+1)+x^{5/7}(1)$$

$$= \frac{5}{7}x^{5/7}+\frac{5}{7}x^{-2/7}+x^{5/7}$$

More examples

$$\frac{d}{dx} 5x\cos(x) = \left(\frac{d}{dx} 5x\right)\cos x + 5x \frac{d}{dx}\cos x$$
$$= 5\cos x + 5x(-\sin x)$$
$$= 5\cos x - 5x\sin x$$

$$\frac{d}{dx} e^x x^3 = \left(\frac{d}{dx} e^x\right)x^3 + e^x \frac{d}{dx} x^3$$
$$= e^x x^3 + e^x 3x^2$$
$$= (x^3 + 3x^2)e^x$$

$$\frac{d}{dx} \cos(x)\sin(x) = \left(\frac{d}{dx}\cos x\right)\sin x + \cos x\left(\frac{d}{dx}\sin x\right)$$
$$= -\sin x \sin x + \cos x \cos x$$
$$= \cos^2 x - \sin^2 x$$

$$\frac{d}{dx} \sin x\, e^x = \cos x\, e^x + \sin x\, e^x$$
$$= (\cos x + \sin x)e^x$$

$$\frac{d}{dx} \sqrt{x}\sin x = \frac{1}{2\sqrt{x}}\sin x + \sqrt{x}\cos x$$

$$\frac{d}{dx}\left((8x+3)e^x + 2x^2\cos x\right)$$

$$\frac{d}{dx}\left(5e^x \sin x + x^{3/2} e^x\right)$$ (factor first)

Find the x values where the graph

$$y = x^3 e^x$$

has a horizontal tangent

Derivatives of Quotients

$$\frac{d}{dx}\frac{8x^5+1}{x^2+3}\ ?$$

Quotient rule

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$f(x) = 8x^5+1 \qquad g(x) = x^2+3$

$f'(x) = 40x^4 \qquad g'(x) = 2x$

$$\frac{d}{dx}\frac{8x^5+1}{x^2+3} = \frac{40x^4(x^2+3) - (8x^5+1)2x}{(2x)^2}$$

---

examples

$$\frac{d}{dx}\frac{3x^3}{x^{10}-5}$$

$$\frac{d}{dx}\frac{\sqrt[3]{x}}{1-x^2}$$

$$\frac{d}{dx}\frac{2}{3-4x}$$

$$\frac{d}{dx}\frac{2b+x}{x+x^3}, \text{ b const.}$$

$$\frac{d}{dx}\frac{\sin x}{x^2+1}$$

$$\frac{d}{dx}\frac{\sin x + \cos x}{\sin x - \cos x}$$

$$\frac{d}{dx}\frac{e^x}{\cos x}$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

$$\frac{d}{dx}\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta\cos\theta - (-\sin\theta\sin\theta)}{(\cos\theta)^2} = \frac{1}{(\cos\theta)^2} = \sec^2\theta$$

$$\frac{d}{dx}\frac{\cos\theta}{\sin\theta} = \frac{-\cos\theta(\cos\theta) + (-\sin\theta\sin\theta)}{(\sin\theta)^2} = -\frac{1}{\sin^2\theta} = -\csc^2\theta$$

$$\cancel{\frac{d}{dx}\frac{1}{\cos\theta} = \frac{\cos\theta\left(\frac{d}{d\theta}1\right) + 1(-\sin\theta)}{\cos\theta}}$$

$$\frac{d}{dx}\sec\theta = \sec\theta\tan\theta$$

$$\frac{d}{dx}\csc\theta = -\csc\theta\cot\theta$$

---

examples

$$\frac{d}{dx}\, 8\sin x\tan x$$

$$\frac{d}{dx}\, e^x\sec x$$

$$\frac{d}{dx}\, \frac{x^2}{\tan x + \cos x}$$

$$\frac{d}{dx}\, x\tan x$$

$$\frac{d}{dx}\, \frac{\sin x}{x}$$

The Chain Rule

Lucy grows twice as fast as Ben

Ben grows three times as fast as Cosmo.

How fast does Lucy grow, compared to Cosmo?

answer: $2 \times 3 = 6$ times as fast.

mathematically: $\frac{dL}{dB} = 2$, $\frac{dB}{dC} = 3$, $\frac{dL}{dC} = \frac{dL}{dB} \times \frac{dB}{dC} = 2 \times 3 = 6$

The chain rule:

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$$\frac{d}{dx} f(g(x)) = f'(g)\, g'(x)$$
$$= f'(g(x))\, g'(x)$$

---

examples

$$\frac{d}{dx} (x^2+a^2)^{3/2}, \quad a \text{ is a constant}$$

$f(g) = (g)^{3/2}$ $\qquad g(x) = x^2+a^2$

$f'(g) = \frac{3}{2} g^{\frac{1}{2}}$ $\qquad g'(x) = 2x$

$$\frac{d}{dx}(x^2+a^2)^{3/2} = \left(\frac{3}{2} g^{\frac{1}{2}}\right)(2x)$$
$$= \frac{3}{2}(x^2+a^2)^{\frac{1}{2}}\, 2x$$

$$\frac{d}{dx} (x^3-6x^2-2)^4$$

$f(g) = (g)^4$ $\qquad g(x) = x^3-6x^2-2$

$f'(g) = 4g^3$ $\qquad g'(x) = 3x^2-12x$

$$\frac{d}{dx}(x^3-6x^2-2)^4 = 4g^3(3x^2-12x) = 4(x^3-6x^2-2)^3(3x^2-12x)$$

$$\frac{d}{dx}\left(\sin(x)\right)^2 = 2\sin x \cos x$$

$$\frac{d}{dx} e^{3x} = e^{3x}\, 3 = 3e^{3x}$$

$$\frac{d}{dx} e^{5x^2+1} = e^{5x^2+1} \frac{d}{dx}(5x^2+1)$$

$$= e^{5x^2+1}(10x)$$

$$= 10x e^{5x^2+1}$$

$$\frac{d}{dx}\sin(5e^{x^3}) = \cos(5e^{x^3}) \frac{d}{dx}(5e^{x^3})$$

$$= \cos(5e^{x^3})\, 5e^{x^3} \frac{d}{dx} x^3$$

$$= \cos(5e^{x^3})\, 5e^{x^3}\, 3x^2$$

$$\frac{d}{dx}\sqrt{\frac{x+1}{x-1}} = \frac{1}{2\sqrt{\frac{x+1}{x-1}}} \frac{d}{dx}\frac{x+1}{x-1}$$

$$= \frac{1}{2\sqrt{\frac{x+1}{x-1}}} \frac{(x-1)-(x+1)}{(x-1)^2}$$

$$= \frac{\sqrt{x-1}}{2\sqrt{x+1}} \frac{-2}{(x-1)^2}$$

$$= -\frac{1}{\sqrt{x+1}\,(x-1)^{3/2}}$$

$$\frac{d}{dx}\frac{1}{\sqrt{x^3+1}} \qquad f(g)=\frac{1}{\sqrt{g}}=g^{-\frac{1}{2}} \qquad g(x)=x^3+1$$

$$f'(g)=-\frac{1}{2}g^{-\frac{3}{2}} \qquad g'(x)=3x^2$$

$$\frac{d}{dx}\frac{1}{\sqrt{x^3+1}} = -\frac{1}{2}g^{-\frac{3}{2}}\,3x^2$$

$$= -\frac{3x^2}{(x^3+1)^{3/2}}$$

$$\frac{d}{dx}\quad 3\sin\left(2\pi x+\frac{\pi}{2}\right)$$

$$f(g)=3\sin(g) \qquad g(x)=2\pi x+\frac{\pi}{2}$$

$$f'(g)=3\cos(g) \qquad g'(x)=2\pi$$

$$\frac{d}{dx}\,3\sin\left(2\pi x+\frac{\pi}{2}\right)=3\cos(g)\,2\pi$$

$$=3\cos\left(2\pi x+\frac{\pi}{2}\right)2\pi$$

Skip

$$\frac{d}{dx}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}, \quad \sigma \text{ is a constant}$$

$$f(g)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{g} \qquad g(x)=-\frac{x^2}{2\sigma^2}$$

$$f'(g)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{g} \qquad g'(x)=-\frac{x}{\sigma^2}$$

$$\frac{d}{dx}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}=\frac{1}{\sqrt{2\pi}\,\sigma}e^{g}\left(-\frac{x}{\sigma^2}\right)$$

$$=-\frac{x}{\sqrt{2\pi}\,\sigma^3}e^{-x^2/2\sigma^2}$$

The Logarithm

$$\ln(AB) = \ln(A) + \ln(B)$$
$$\ln(A^B) = B\ln(A)$$
$$\ln(A/B) = \ln(A) - \ln(B)$$
$$\ln(1) = 0$$
$$\ln(e) = 1$$
$$\ln(e^x) = x$$
$$e^{\ln(x)} = x$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\ln(5x)$$

$$\frac{d}{dx}\cos(3x) + 5\ln(x)$$

$$\frac{d}{dx}\ln(5x^2+2x+1)\tan(\pi x)$$

$$\frac{d}{dx}\ln\left(\sqrt{\frac{(x+5)^4}{2x^3}}\right)$$

"Lumberjacks make good musicians because of their natural logarithms"

$$\frac{d}{dx} \ln\left(\frac{\sqrt{x+1}}{x-1}\right)$$

$$\frac{d}{dx} \csc x \ln(x^2+1)$$

Higher - Order Derivatives

given $f(x) = x^3 + 2x^2$

find $\frac{d}{dx} f'(x)$.

$$f'(x) = 3x^2 + 4x$$

$$\frac{d}{dx} f'(x) = \frac{d}{dx}(3x^2 + 4x)$$

$$f''(x) = 6x + 4$$

find $f'''(x)$: $f'''(x) = \frac{d}{dx} f''(x)$

$$= \frac{d}{dx}(6x+4)$$

$$f^{(3)}(x) = 6$$

find $f^{(4)}(x)$: $f^{(4)}(x) = \frac{d}{dx} f^{(3)}(x)$

$$= \frac{d}{dx} 6$$

$$= 0$$

---

given $f(x) = e^x \cos x$

find $f''(0)$.

---

given $f(x) = 3\sqrt{x}$

find $f'''(9)$

---

given $f^{(6)}(x) = 24x \sin x$

find $f^{(7)}(x)$

---

position of a particle is given by

$$S(t) = \frac{3t^2}{2t+1}$$

find $V(t)$ and $a(t)$

---

velocity of a particle is given by

$$V(t) = \frac{\sin(x)}{x}$$

find $a(t)$

Implicit Differentiation

Friday: Quiz on chain rule, derivative of $\log(x)$.
Next week, exams returned, applications begin
Today, implicit differentiation!

$y-5=3(x+2)$ ← differentiate implicitly to save work

$(x-y)^2=2x^3$ ← find $\frac{dy}{dx}$

$5xy=y^2+4$ ← find $\frac{dy}{dx}$ at the point $(1,4)$

$x^2+4y^2=25$ ← find equation of a tangent line at $(3,2)$ and then at $(3,-2)$

---

find $\frac{dy}{dx}$ if

$\sin x \tan y = \ln x$

$\cos(x+3y)=8y$

$e^{5xy}=2x^2$

Related Rates

$$3x^2 + y = -xy \quad \leftarrow \text{find } \frac{dx}{dt} \text{ when } x=1,\ y=2,\ \frac{dy}{dt} = -4$$

At my old (crappy) studio apartment (don't rent from crestview) there was a leak in the ceiling dripping onto the counter, at 2mL/minute (2 cm³/minute) making a puddle, circular that was $\frac{1}{4}$ cm tall. When I found it, the puddle was 60 cm across. How fast was the radius of the puddle increasing?

$$V = \pi r^2 h = \frac{\pi}{4} r^2$$

$$\frac{dV}{dt} = \frac{\pi}{2} r \frac{dr}{dt}$$

$$2 = \frac{\pi}{2} 30 \frac{dr}{dt}$$

$$\frac{4}{30}\pi \cdot = \frac{dr}{dt} = 0.042 \text{ cm/minute}$$

procedure

1. use geometry to relate the variables
2. differentiate with respect to time (t)
3. use given information to find the rate we need

A 25-foot ladder is resting against a wall. It starts to slide, with the top moving down at 6 inches per second. I am sitting 7 feet from the wall. How fast is the base of the ladder moving when it hits me?

$$x^2 + y^2 = 25^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} = -y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{24}{7}\left(-\frac{1}{2}\right)$$

$$= \frac{24}{14}$$

$$= 1.71 \text{ feet/sec}$$

$$\frac{dy}{dt} = -\frac{1}{2}$$

$$x = 7$$

$$7^2 + y^2 = 25^2$$

$$y = 24$$

I left an ice cube sitting on the counter (oops!). Its side-length decreases at $2\ \frac{cm}{hour}$. How fast is the volume changing when the length is 1 cm?

$$V = l^3$$

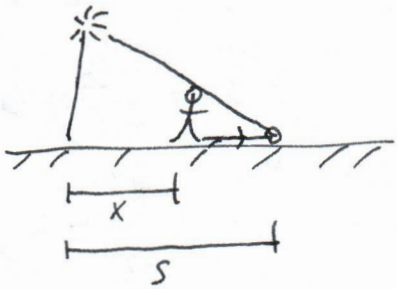
$$\frac{dV}{dt} = 3l^2\frac{dl}{dt}$$

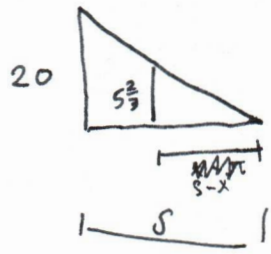
$$\frac{dV}{dt} = 3(1^2)(-2)$$

$$= -6\ \frac{cm^3}{hr}$$

My hummingbird feeder has an "ant-moat" above it, which is an inverted cone, 4 inches tall and 3 inches in diameter. Rain fills the moat at $2.5\ in^3/hour$. How fast is the height of the water increasing when the water is 3 inches deep?

I am 5'8" tall. One night (after a long evening of grading) I walked home from my office and to my surprise, the 20 foot tall street light turned on. As I walked away from it, I saw my shadow go in front of me. I walk 5 feet per second, but my shadow moves faster. How fast does the tip of my shadow move, exactly?





$$\frac{S-x}{5\frac{2}{3}} = \frac{S}{20}$$

$$20S - 20x = 5\tfrac{2}{3}S$$

$$14\tfrac{1}{3}S = 20x$$

$$\frac{dS}{dt} = \frac{20}{14\frac{1}{3}}\frac{dx}{dt}$$

$$= 6.98 \ \frac{ft}{sec}$$

$$\cancel{\frac{xS}{5\frac{2}{3}} = \frac{S}{20}}$$

$$\cancel{20x - 20S = 5\tfrac{2}{3}S}$$

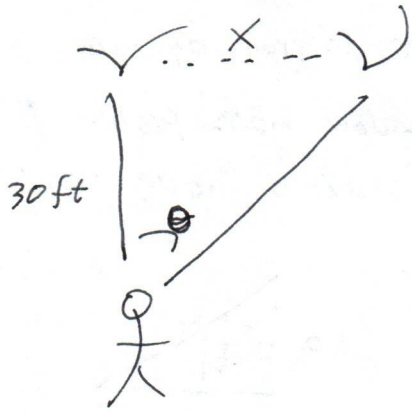
$$\cancel{20x = 5\tfrac{2}{3}S}$$

$$\cancel{20\frac{dx}{dt} = 5\tfrac{2}{3}\frac{dS}{dt}}$$

$$\cancel{20(5) = 5\tfrac{2}{3}\frac{dS}{dt}}$$

$$\cancel{\frac{dS}{dt} = \frac{100}{5\frac{2}{3}} = 17.65 \ \frac{feet}{sec}}$$

I happend to see a heron the other day. I think it passed about 30 feet above my head. I estimated it was ~~[illegible]~~ tracing an angle of ~~[illegible] radian~~ $60°$/sec when it was right above me. How fast was it flying?



$$\tan\theta = \frac{x}{30}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{30}\frac{dx}{dt}$$

$$\underset{(1)}{\sec^2(0)}(1) = \frac{1}{30}\frac{dx}{dt}$$

$$\frac{dx}{dt} = 30 \frac{ft}{sec}$$

$$\frac{60°}{sec} = \frac{\frac{60}{180}\pi \text{ rad}}{sec}$$

$$\frac{dx}{dt} = \sec^2(0)\left(\frac{\pi}{3}\right)(30)$$

$$= 10\pi \text{ ft/sec}$$

$$\approx 31.4 \text{ ft/sec}$$

Critical Points

Activity: find relative maxima & minima
find critical points

critical numbers are where $f'(x)=0$ or $f'(x)$ dne.
critical numbers are x-values.

examples
find the critical numbers!

$y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 4$

$y = 4x^2 + \frac{1}{x}$

$y = \tan^2(4x)$ on $(0, 2\pi)$

$y = 2\sin(3x) - 3\sqrt{3}\,x$ on $(0, \pi)$

$y = \frac{x^2+4}{2x}$

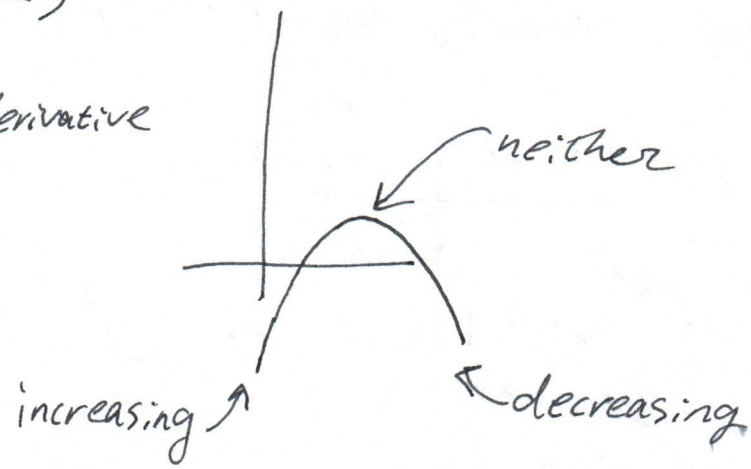
increasing, decreasing, and 1st deriv. test

example

$$y = -x^2 + 4x - 3$$

idea: look at 1st derivative

$$y' = -2x + 4$$
$$= -2(x-2)$$



notice:

$y' = 0$ when $x = 2$

$y' > 0$ when $x < 2$

$y' < 0$ when $x > 2$

on a number line,

Facts

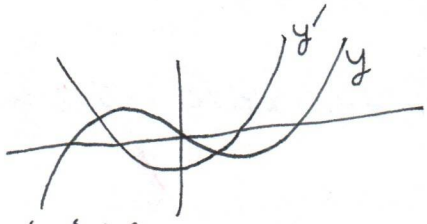
if $f'(x) > 0$ then $f(x)$ is increasing at x

if $f'(x) < 0$ then $f(x)$ is decreasing at x

if $f'(x) = 0$ then $f(x)$ is neither increasing nor decreasing

example

$$y = x^3 - x + 2$$

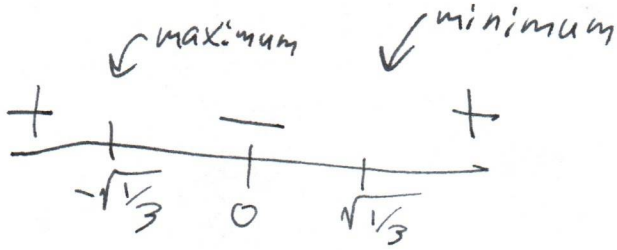


Goal: find maxima & minima

$$y' = 3x^2 - 1$$

$y' = 0 = 3x^2 - 1$

$x = \pm\sqrt{1/3}$



First Derivative Test

Step 1: find the critical points

Step 2: find where $f'(x)$ is positive and where $f'(x)$ is negative

Step 3: classify the critical points:

$f'(x) > 0 \mid f'(x) < 0$ (at c) ex. $\Rightarrow$ relative max at $x = c$

$f'(x) < 0 \mid f'(x) > 0$ (at c) ex $\Rightarrow$ relative min at $x = c$

$f'(x) > 0 \mid f'(x) > 0$ (at c) ex $\Rightarrow$ neither

$f'(x) < 0 \mid f'(x) < 0$ ex $\Rightarrow$ neither

example

apply the first derivative test to $y = -x^3$

example

apply the first derivative test to $y = 3x^5 - 2x^4$

example

apply the first derivative test to $g(x)$ if $g'(x) = (x-3)^2(x-1)^3$

A tale of 2 parabolas

$f(x) = x^2$

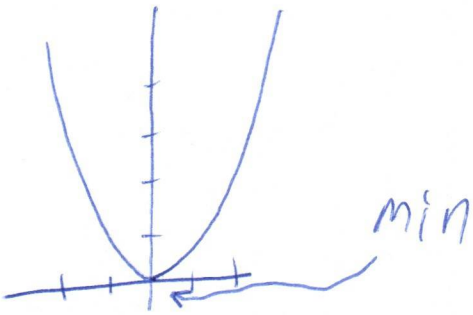
$f'(x) = 2x$

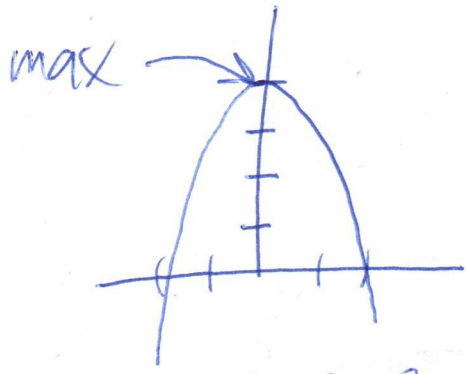
$f''(x) = 2$

$f(x) = 4 - x^2$

$f'(x) = -2x$

$f''(x) = -2$





concave up

$f''(x) > 0$

concave down

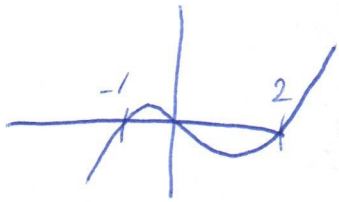
$f''(x) < 0$

Ask: what are similarities & differences between these functions?

Big idea: use these patterns to analyze other functions

example

$$f(x) = x^3 - x^2 - 2x$$
$$= x(x^2 - x - 2)$$
$$= x(x-2)(x+1)$$



- shape resembles 2 parabolas

$f'(x) = 3x^2 - 2x - 2$

$f''(x) = 6x - 2$

$f''(x) > 0$

$6x - 2 > 0$

$6x > 2$

$x > \frac{1}{3}$

concave up on $(\frac{1}{3}, \infty)$

concave down on $(-\infty, \frac{1}{3})$

inflection point at $(\frac{1}{3}, f(\frac{1}{3}))$

$= (\frac{1}{3}, -\frac{29}{27})$

example

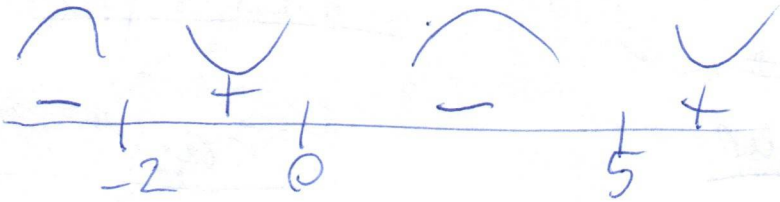
$$f(x) = \frac{x^5}{20} - \frac{x^4}{4} - \frac{5x^3}{3} + \frac{8x}{7} + 9$$

$$f'(x) = \frac{x^4}{4} - x^3 - 5x^2 + \frac{8}{7}$$

$$f''(x) = x^3 - 3x^2 - 10x$$

$$= x(x^2 - 3x - 10)$$

$$= x(x+2)(x-5)$$



$f(x)$ is concave down on $(-\infty, -2)$ and $(0, 5)$

concave up on $(-2, 0)$ and $(5, \infty)$

$f(x)$ has inflection points at $(-2, f(-2))$

$(0, f(0))$

$(5, f(5))$

example

$f(x) = x + \frac{1}{x}$

find where f is concave up/down, increasing/decreasing

example

$f(x) = x^4 - 4x^3 + 6x^2 - 36x$

find inflection points (careful!)

example

$f(x) = e^x - x^2$

find inflection points

2nd derivative test

if c is a critical point of f, then:

$f''(c) > 0$ means $f(c)$ is a relative <u>min</u>

$f''(c) < 0$ means $f(c)$ is a relative <u>max</u>

$f''(c) = 0$ tells you <u>nothing</u>! (use 1st derivative test)

<u>example</u>

find & classify critical points of $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 14x + 11$

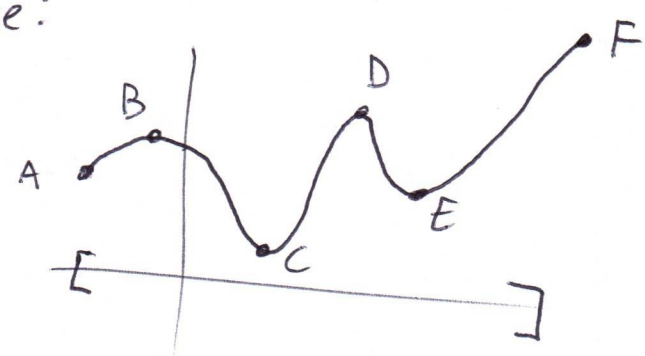
<u>example</u>

find & classify critical points of $f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 4$

Absolute Max and Min on an Interval

The Big Idea: compare fcn values at local extrema, & at endpoints.

picture:



<u>example</u> Find absolute extrema of $f(x) = 3x^2$ on $[-1, 2]$

<u>example</u> Find absolute extrema of $f(x) = \frac{x^3}{3} - 4x^2 + 12x$ on $[0, 3]$

<u>example</u> Find the maximum of $x^4 - 2x^2$ on $[-\frac{1}{2}, 2]$

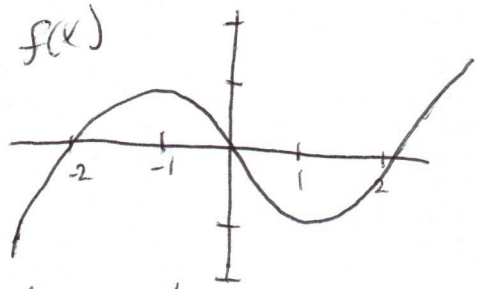
<u>example</u> find the max & min values of $x^2 e^x$ on $[-5, 5]$

<u>example</u> find the minimum value of $\frac{x^2}{x+1}$ on $(-1, 4]$

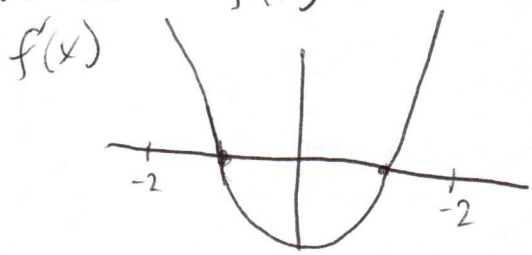
<u>example</u> find the max value of $\ln(x) - x$ on $(0, 2)$

Graphical Interpretation of Derivatives

Here is a graph of $f(x)$



Now let's draw $f'(x)$



Notice

where $f(x)$ has a critical point, $f'(x)=0$

where $f(x)$ is increasing, $f'(x)$ is positive (above the x-axis)

where $f(x)$ is decreasing, $f'(x)$ is negative (below the x-axis)

where $f(x)$ is concave up, $f'(x)$ is increasing

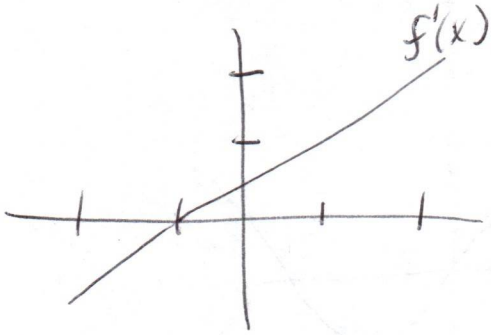
where $f(x)$ is concave down, $f'(x)$ is decreasing

where $f(x)$ has an inflection point, $f'(x)$ has a critical point

where $f(x)$ has a maximum, $f'(x)$ goes from positive to negative

where $f(x)$ has a minimum, $f'(x)$ goes from negative to positive

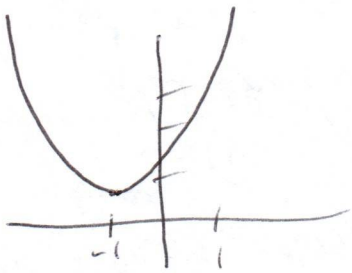
Now we go backwards

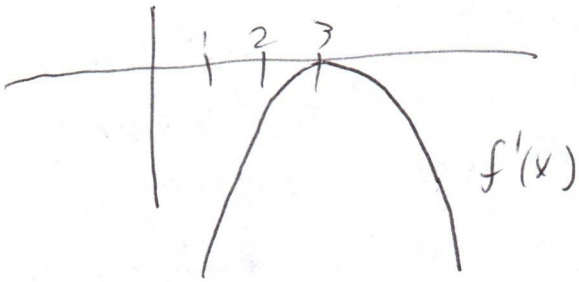


describe f(x):

- critical point at $x=-1$
- increasing on $(-1,\infty)$
- decreasing on $(-\infty,-1)$
- concave up on $(-\infty,\infty)$
- never concave down
- no inflection point
- no maximum
- minimum at $x=-1$

draw f(x):

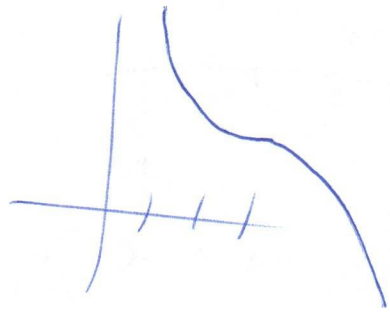




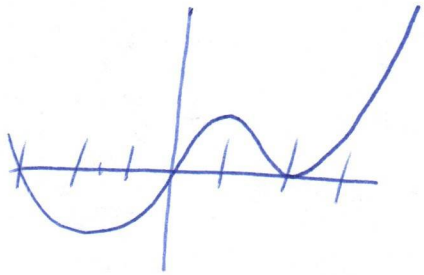
describe $f(x)$:

- critical point at $x=3$
- ~~du~~ never increasing
- decreasing everywhere (except maybe at $x=3$)
- concave up on $(-\infty, 3)$
- concave down on $(3, \infty)$
- inflection point at $x=3$
- no maximum
- no minimum

draw $f(x)$:



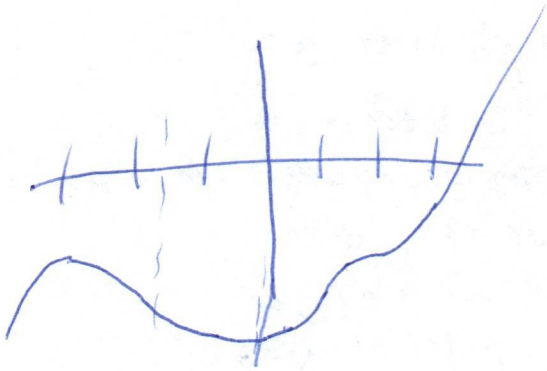
$f'(x)$ ~~#~~



describe $f(x)$:

- critical points at $x = -3, 0, 2$
- increasing on $(-\infty, -3)$ and on $(0, \infty)$
- decreasing on $(-3, 0)$
- concave ~~on~~ up on $(-\frac{3}{2}, 1)$ and on $(2, \infty)$
- concave down on $(-\infty, -\frac{3}{2})$ and on $(1, 2)$
- inflection points at $x = -\frac{3}{2}, 1$, and ~~#~~ 2
- maximum at $x = -3$
- minimum at $x = 0$

draw $f(x)$:

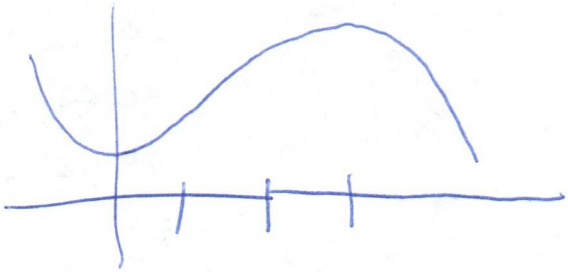


$f'(x)$:

describe $f(x)$:

- critical points at $x=0, 3$
- increasing on $(0, 3)$
- decreasing on $(-\infty, 0)$ and $(3, \infty)$
- concave up on $(-\infty, \frac{3}{2})$
- concave down on $(\frac{3}{2}, \infty)$
- inflection point at $x = 3/2$
- maximum at $x=3$
- minimum at $x=0$

draw $f(x)$:



Limits at infinity

$$\lim_{x \to \infty} 8 - \frac{5}{x}$$

$$\lim_{x \to -\infty} x^2 + 3$$

$$\lim_{x \to \infty} \frac{4x^3 + 3x - 2}{10x^3 + 6}$$

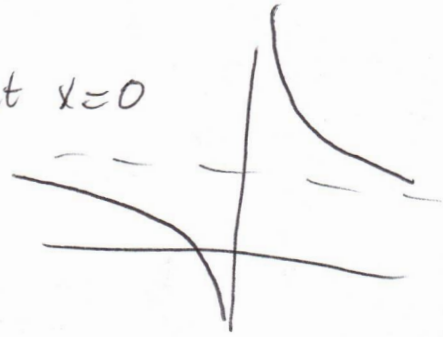
x	10	100	1000
$f(x)$	$\frac{4000+30-2}{10000+6}$	$\frac{4000000+300-2}{10000000}$	$\frac{4000000000+3000-2}{10000000000+6}$
$f(x)$	0.4026	0.40003	0.4000002996

$$\lim_{x \to \infty} \frac{4x^3 + 3x - 2}{10x^3 + 6} = \lim_{x \to \infty} \frac{4x^3}{10x^3} = \lim_{x \to \infty} \frac{4}{10} = \frac{4}{10}$$

$$\lim_{x \to -\infty} \frac{5 + x^2 - x}{2x^3 + 5x} = \lim_{x \to -\infty} \frac{x^2}{2x^3} = \lim_{x \to -\infty} \frac{1}{2x} = 0$$

Asymptotes

$\frac{1+5x}{x}$ has a <u>vertical asymptote</u> at $x=0$



$$\lim_{x \to \infty} \frac{1+5x}{x} = \lim_{x \to \infty} \frac{5x}{x} = \lim_{x \to \infty} 5 = 5$$

has a <u>horizontal asymptote</u> at $y=5$

$$\cancel{\lim_{x\to\infty} \frac{x^3+2x^2+2x+1}{2x^2+1} = \lim_{x\to\infty} \frac{x^3}{2x^2} = \lim_{x\to\infty} \frac{x}{2} = \infty}$$

has no horizontal asymptote!

$$\lim_{x\to\infty} \frac{x^3-x^2-x+2}{x^2-1} = \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$$

has vertical asymptotes at $x=1$ and $x=-1$

has <u>no</u> horizontal asymptotes

Review: Long Division

$$\begin{array}{r} 26 \\ 21\overline{)547} \\ -42\downarrow \\ \hline 127 \\ -126 \\ \hline 1 \end{array}$$

$$\frac{541}{21} = 26 + \frac{1}{21}$$

Now: Polynomial Division (same thing)

$$\frac{x^3-x^2-x+2}{x^2-1} = x-1+\frac{1}{x^2-1}$$

$$\begin{array}{r} x-1 \\ x^2-1\overline{)x^3-x^2-x+2} \\ -(x^3 \quad -x) \\ \hline 0-x^2+0 \\ -(-x^2 \quad +1) \\ \hline 1 \end{array}$$

Slant asymptote!

$$f(x) = \frac{3x^2 - 14x}{x-5} = 3x+1 + \frac{5}{x-5}$$

has a vertical asymptote at $x=5$

$$\begin{array}{r} 3x + 1 \\ x-5 \overline{\big) 3x^2 - 14x} \\ -(3x^2 - 15x) \\ \hline x \\ -(x-5) \\ \hline 5 \end{array}$$

has a slant asymptote $y = 3x+1$

$$\frac{x^4 - x^3 + 9x^2 - 9x + 4}{x^2+9}$$

has neither!

Curve Sketching

Useful properties:

x-intercepts
y-intercepts
increasing intervals/decreasing intervals
concave up/down intervals
asymptotes
inflection points

Functions to graph

$$y = \frac{3x+9}{x-2}$$

$$y = \frac{x^2-2x+5}{x^2-4}$$

$$y = \frac{4x}{x^2+6x+9}$$

$$y = \frac{x^2+8}{x-4}$$

Optimization

Two positive numbers multiply to make 64. Find their smallest possible sum:

A sheet of cardboard 10 cm by 15 cm has its corners cut and its edges turned up, to make a box. What is the largest box we can make this way?

I want to build a pen in the corner of my apartment to contain a cat. It needs to be a rectangle and it needs to contain 10 square feet. How much fencing do I need?

I have two couches, each 6 feet long. I will make a couch fort by putting them against the wall, forming a triangle. What's the biggest fort I can make this way?

Two hallways meet at a right angle, one 5 feet wide and the other 8 feet wide.

I want 10 square inches of baklava, but it's hard to cut.

I sell pies which cost \$4.50 to make. I know if I price them at \$p, then I will sell $20 - p$ pies. How much should I price them for, how many should I make?

If I use 10x pounds of fertilizer, then I will get 10x pounds more zuchini, on top of the 8 pounds I already grow. However, some zuchini will rot, leaving only $e^{-x/3}$ times as much zuchini in the field.

Find the closest points to $(2,0)$ on the curve $y^2-x^2=4$.

Integration

if $f(x) = 12x^3$, what function has $f(x)$ as its derivative?

what function has $1 + \cos x$ as its derivative?

$\int 12x^3 dx = 3x^4 + C$

$\int (1+\cos x) dx = x + \sin x + C$

where C is any constant

examples: $\int (x^2 - 2e^x + 4) dx$

$\int x(x^3+3) dx$

$\int \frac{1 + x \sin x}{x} dx$

$\int \sqrt{x}\left(5 - \frac{3}{x^2}\right) dx$

$\int \sin x (\cot x + 8) dx$

Rules:- $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ (unless $n = -1$)

$\int \cos x dx = \sin x + C$

$\int \frac{1}{x} dx = \ln x + C$

$\int \sin x dx = -\cos x + C$

$\int \sec^2 x dx = \tan x + C$

Initial Value Problems

if $y' = 6x^2 + 2e^x - 1$
and $y(0) = 8$, find $y(x)$.

---

if $y'' = -2\cos x$
and $y'(\pi) = 3$ and $y(\frac{\pi}{2}) = e^2$
~~find y(x)~~ find $y(x)$.

---

if $y'' = 12x + e^x$
and $y(0) = 4$ and $y(2) = 6$
find $y(x)$

~~The speed of~~ acceleration of my car
is given by $\frac{24}{x+5}$ feet per second per second.
If I start standing still, how fast am I going
40 seconds later?

---

find $\sum_{i=1}^{5} 2i-1$, $\sum_{i=2}^{4} 3\sin\left(\frac{i\pi}{2}\right)$

write in sigma notation:

$$(e^2-3)+(e^3-2)+\ldots+(e^{n+1}+n-4)$$

Quadrature

example: Find the signed area under the curve $y=x-1$ between $x=0$ and $x=3$

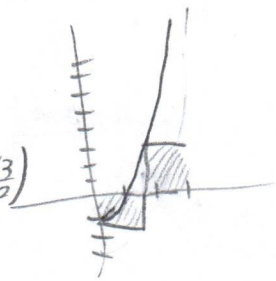
puzzle: Find the signed area under the curve $y=x^2-1$ between $x=0$ and $x=3$

Idea: break it up

Approximate the signed area with a left Riemann sum:

$-1\left(\frac{3}{2}\right)+1.25\left(\frac{3}{2}\right)$

$=3/8$



$n=2$

$n=3$

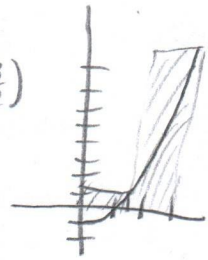
$-1(1)+0(1)+3(1)$

$=3$

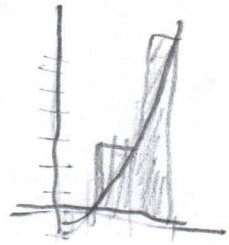
OR a right Riemann sum:

$1.25\left(\frac{3}{2}\right)+8\left(\frac{3}{2}\right)$

$=13.825$



$n=2$



$n=6$

$0(1)+3(1)+8(1)$

$=11$

use a left Riemann sum with 3 rectangles to approximate the signed area under $y=e^x$ between $x=1$ and $x=7$

Interval length?

$\Delta x = \frac{7-1}{3} = 2$

Points to check?

$x = 1, 3, 5$

or $x = 1 + (\Delta x)\cdot i, \; i = 0, 1, 2$

left riemann sum $= e^1(2) + e^3(2) + e^5(2) = \sum_{i=0}^{2} e^{1+2i}(2)$

right riemann sum?

same interval length

now check $x = 3, 5, 7$ or $x = 1 + (\Delta x)\cdot i, \; i = 1, 2, 3$

right riemann sum $= e^3(2) + e^5(2) + e^7(2) = \sum_{i=1}^{3} e^{1+2i}(2)$

What if we want to split it into 100 rectangles?

left riemann sum: $\Delta x = \frac{7-1}{100} = 0.06$

points to check: $x = 1 + 0.06\cdot i, \; i = 0, 1, \ldots, 99$

left riemann sum $= \sum_{i=0}^{99} e^{1+0.06\cdot i}\, 0.06$

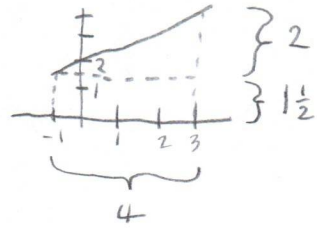
right riemann sum: points to check are $x = 1 + 0.06\cdot i$, $i = 1, 2, \ldots, 100$

right riemann sum: $\sum_{i=1}^{100} e^{1+0.06\cdot i}\, 0.06$

Definite Integrals

"It's the area under the curve!"

Example: $\int_{-1}^{3} \frac{1}{2}x + 2 \, dx$



Example: $\int_{1}^{4} -2 \, dx$

Algebra with definite integrals!

ex $\int_{-1}^{2} x^2 \, dx = 3$ and $\int_{2}^{3} x^2 \, dx = 6 + \frac{1}{3}$.

Find $\int_{-1}^{3} x^2 \, dx$.

ex $\int_{1}^{5} f(x) \, dx = 6$ and $\int_{2}^{5} f(x) \, dx = 4$.

Find $\int_{1}^{2} f(x) \, dx$.

Questions for You (work together!)

1. $f(x) = \begin{cases} -2x-2 & \text{if } x<0 \\ x+1 & \text{if } x>0 \end{cases}$

find $\int_{-2}^{2} f(x) \, dx$

2. $\int_{0}^{2} g(x) \, dx = 6$, $\int_{1}^{3} g(x) \, dx = 5$, and $\int_{0}^{3} g(x) \, dx = 9$.

Find $\int_{1}^{2} g(x) \, dx$.

More Algebra

$$\int_0^2 2+x\,dx = \int_0^2 2\,dx + \int_0^2 x\,dx$$

in general: $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$$\int_{-1}^2 2x\,dx = 2\int_{-1}^2 x\,dx$$

in general: $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

<u>examples</u>

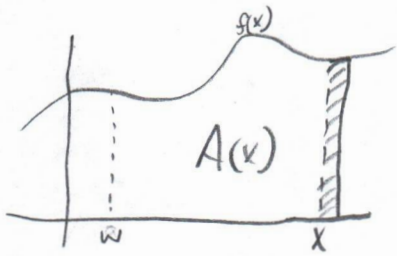
$$\int_{-5}^{20} f(x)\,dx = 3 \quad \text{and} \quad \int_{-5}^{20} g(x)\,dx = -1$$

Find $\int_{-5}^{20} (2f(x) + 8g(x))\,dx$

$$\int_{-2}^4 f(x)\,dx = 10, \quad \int_3^4 f(x) = 6, \text{ and}$$

Find $\int_{-2}^3 (2f(x)+1)\,dx$

The Fundamental Theorem of Calculus

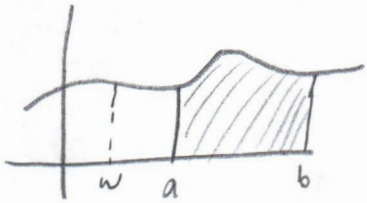


The big idea:

$A'(x) = f(x)$.

or, another way:

$A(x) = \int f(x)\,dx$



$$\int_a^b f(x)\,dx = A(b) - A(a)$$

> The fundamental theorem, version 1:
> if $F(x)$ is any antiderivative of $f(x)$,
> then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Another way to write it: $\int_a^b f(x)\,dx = F(x)\Big|_a^b$

---

ex (use the FTC)

$$\int_3^8 2\,\frac{1}{x}\,dx = 2\ln(x)\Big|_3^8 = 2\ln(8) - 2\ln(3)$$
$$= 2(\ln(8) - \ln(3))$$
$$= 2\ln(8/3).$$

$$\int_{-1}^{2} \frac{x^3 - 5}{7}\,dx$$

$$\int_0^{\frac{\pi}{2}} \sin(x)\,dx$$

ex Find the area bounded by the lines $y=0$, $x=2$, $x=4$, and the curve $y=2x^2$

Find the area bounded by the lines $y=0$, $x=-1$, $x=3$, and the curve $y=e^x+1$.

Fundamental Theorem 2

version 1:

$\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$

version 2:

$\int_a^b f'(x)\,dx = f(b) - f(a)$

The idea: add up little changes, get the total change.

Examples

The amount of bacteria on the grape I dropped behind the cabinet grows at a rate of $0.3e^{t+5}$ bacteria/day. If I drop it on a monday, how many bacteria will grow in the first week? in the second week?

A particle's velocity is given by $3t^2 - 10t$. How much does its displacement change between $t = 3$ minutes and $t = 10$ minutes?

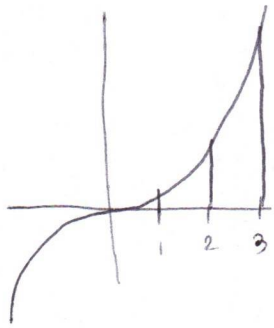
The amount of heat entering my tea kettle is $1500(60 - t)$ watts. How much heat flows into my water in its third minute of use?

At my old job, the pay rate increased depending on how long you had been there. They paid \$10 + \$1.5t/hr, where t is in years. If I worked for 10 hrs/week, how much did I earn at that job?

All About Numerical Integration

Example

approximate $\int_0^3 x^3\,dx$ by left riemann sums, right riemann sums, and the trapezoid rule using 3 subintervals. Compare with the correct value, found with FTC.



LRS:

$(\Delta x)(f(0)+f(1)+f(2))$

$1(0+1+8)$

9

RRS: $(\Delta x)(f(1)+f(2)+f(3))$

$1(1+8+27)$

36

Trap rule: $\frac{\Delta x}{2}(f(0)+2f(1)+2f(2)+f(3))$

$\frac{1}{2}(0+2+16+27)$

22.5

pretty close!

actual value: $\frac{x^4}{4}\Big|_0^3 = \frac{81}{4} = 20.25$

Notice: trap rule is the <u>average</u> of right & left riemann sums.

example

$\int_0^1 e^{-x^2}\,dx$ ← The "Gaussian Integral"

→ I can't solve this using the FTC (nobody can!)

→ approximate with trap rule, 4 subintervals

$f(x) = e^{-x^2}$

$f(0) = 1.000$

$f(\frac{1}{4}) = 0.939$

$f(\frac{1}{2}) = 0.779$

$f(\frac{3}{4}) = 0.570$

$f(1) = 0.368$

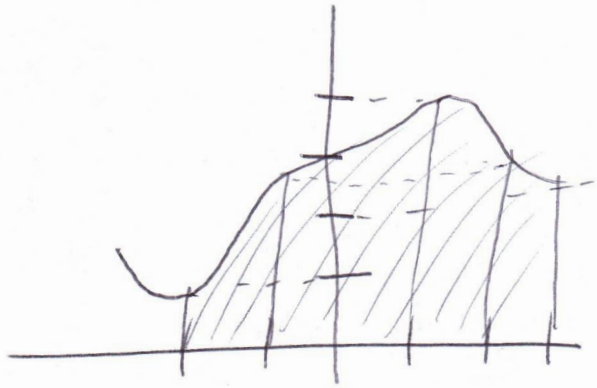
$$\int_0^1 e^{-x^2}\,dx \approx \frac{\Delta x}{2}\left(f(0) + 2f(\tfrac{1}{4}) + 2f(\tfrac{1}{2}) + 2f(\tfrac{3}{4}) + f(1)\right)$$

$$= 0.743$$

Notice: wolfram alpha gives 0.746824.

pretty darn close!

example



use trap rule with 5 subintervals to approximate the area above.

$$A \approx \frac{\Delta x}{2}\left(f(-2)+2f(-1)+2f(0)+2f(1)+2f(2)+f(3)\right)$$

$$= \frac{1}{2}(1+5+6+8+5+2)$$

$$= \frac{27}{2} = 13.5$$

Exponential Growth

A very important function:

$$y = Ce^{kt}$$

notice:

$$y' = Ce^{kt}\,k$$
$$= ky$$

The Big Idea: If anything follows the rule $\frac{dy}{dt} = ky$, then we have a solution!

<u>Example</u> $\frac{dy}{dt} = 3y$ and $y(0) = 70$. find $y(t)$.

ex ~~The number of people infected with SARS during an outbreak is proportional to the number of people~~

The rate at which people were infected with SARS during the 2003 China outbreak was proportional to the number of people already infected, with proportionality constant 0.53, where t is in days. If 3 people in a town were infected on the first day, how many were infected after a week? How long did it take the ~~pe~~ number of infected people to double?

My current savings account gives 0.1% interest. How long would it take the money in that account to double?

Suppose sam deposits $5000 in an account which will double her money in just 16 years. How much will she have in 10 years?

Exponential decay

same differential equation:

$$\frac{dy}{dt} = ky$$

but k is negative!

ex suppose $y(2) = 5$ and $y(4) = 3$, and $\frac{dy}{dt} = ky$. Find $y(10)$.

ex carbon-11 has a half-life of 20 minutes. If my sample has 30 grams of carbon-11 now, how much will there be in an hour and a half?

ex iodine 131 has a half-life of 8 days. If my sample contains 8 grams now, how much did it have 30 days ago?

ex I measured that my sample of cobalt-60 had 50 mg this time last year, and it has 43.8 mg now. What is the halflife of cobalt-60?

ex Aluminum-26 has a halflife of 717000 years. Aluminum-26 only forms in space; it decays into Mg^{26}. Some scientists found a meteorite where only 0.05% of the aluminum-26 had not decayed (the rest was Mg^{26}). How long ago did the meteorite fall?

$$t = 717000 \frac{\ln(0.0005)}{\ln(\frac{1}{2})} = 7.86 \text{ million years}$$