Lesson 2
Jan 8 2018

Copy This:

Alden Bradford
bradfoa@purdue.edu
math.purdue.edu/MA 16020
loncapa.purdue.edu
piazza.com
math.purdue.edu/~bradfoa

Office Hours: Tu 2:30-3:30, Th 1:30-2:30

You Must Bring:

A pencil/pen
A compliant calculator
lined loose leaf paper
A notebook

Ground Rules

NO computers
It's okay to come in a little late
- if you are more than 10 minutes late, come let me know why

First assignment due Tomorrow at 10:00 pm
- usually due at 7:00 am
First quiz on Wednesday
- question 1: What is your instructor's first name?

* Tour the website, mentioning:
  - SA
  - Piazza
  - calculator policy
  - Expand folders in LONCAPA

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and now, some calculus

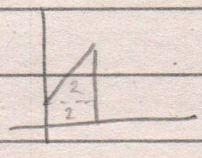
The antiderivative of $f(x)$, written as $\int f(x)dx$, is another function whose derivative is $f(x)$.

ex:

$$\int \cos(x)dx = \sin(x) + C$$
$$\int 8x^2 dx = \frac{8}{3}x^3 + C$$
$$\int \frac{1}{x} dx = \ln(|x|) + C$$

A definite integral, written as $\int_a^b f(x)dx$, is the signed area under $f(x)$ between $x=a$ and $x=b$

ex: $\int_0^2 x+1\, dx = 4$



The fundamental theorem of calculus gives us a way to compute definite integrals using antiderivatives:

if $F'(x) = f(x)$ ($F(x)$ is an antiderivative of $f(x)$) then $\int_a^b f(x)dx = F(b) - F(a)$.

ex/

$$\int_1^{e^2} \frac{3}{x}\,dx = 3\int_1^{e^2} \frac{1}{x}\,dx$$
$$= 3\ln(|x|)\Big|_1^{e^2}$$
$$= 3\ln(e^2) - 3\ln(1)$$
$$= 3\cdot 2 - 3\cdot 0$$
$$= 6$$

$$\int_0^{\pi/4} \tan^2\theta \csc^2\theta\,d\theta = \int_0^{\pi/4} \left(\frac{\sin\theta}{\cos\theta}\right)^2 \left(\frac{1}{\sin\theta}\right)^2 d\theta$$

$$= \int_0^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta} \frac{1}{\sin^2\theta}\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta\,d\theta$$

$$= \tan\theta \Big|_0^{\pi/4}$$

$$= \tan(\pi/4) - \tan(0)$$
$$= 1 - 0$$
$$= 1$$

we can also use the FTC to find the total change in some quantity if we know the derivative of that quantity.

ex The velocity of my car in feet per second is $v(t) = \frac{t+5}{0.1\sqrt{t}}$. How much does my position change between $t=4$ and $t=16$?

If position is $s(t)$, then $s'(t) = v(t)$

Hence,

$$s(16) - s(4) = \int_4^{16} v(t)\,dt$$

$$= \int_4^{16} \frac{t+5}{0.1\sqrt{t}}\,dt$$

$$= \frac{1}{0.1}\int_4^{16} \frac{t+5}{\sqrt{t}}\,dt$$

$$= 10\int_4^{16} \left(\frac{t}{\sqrt{t}} + \frac{5}{\sqrt{t}}\right)dt$$

$$= 10\int_4^{16} \left(t^{\frac{1}{2}} + 5t^{-\frac{1}{2}}\right)dt$$

$$= 10\left(\frac{2}{3}t^{3/2} + 5\cdot 2t^{\frac{1}{2}}\right)\Big|_4^{16}$$

$$= 10\left(\frac{2}{3}(16)^{3/2} + 10(16)^{\frac{1}{2}}\right) - 10\left(\frac{2}{3}(4)^{3/2} + 10(4)^{\frac{1}{2}}\right)$$

$$= 10\left(\frac{2}{3}4^3 + 10(4)\right) - 10\left(\frac{2}{3}2^3 + 10(2)\right)$$

$$= 10\left(\frac{2}{3}64 + 40\right) - 10\left(\frac{2}{3}8 + 20\right)$$

$$= 10\cdot\frac{2}{3}\cdot 64 + 400 - 10\cdot\frac{2}{3}\cdot 8 - 200$$

$$= 10\cdot\frac{2}{3}(64-8) + 400 - 200$$

$$= 10\cdot\frac{2}{3}(56) + 200$$

$$= \frac{1720}{3}$$

$$= 573 + \frac{1}{3} \quad \text{feet}$$

N

please fill out this get-to-know-you card:

In all caps: | FULL NAME
fill in: | what I should call you
| major/field of interest
| something you could talk about for hours and not get bored
copy this expression exactly: | $\int_{3x^2}^{12} 5\sin(\pi y)e^y\,dy + \frac{d}{dx}\sqrt{x-4}$

Integration by substitution

first challenge: $\int e^{x^3} 3x^2\,dx$

Detective mode: it looks like someone used the chain rule to get here!

$$\frac{d}{dx} e^{x^3} = e^{x^3}\frac{d}{dx}x^3 = e^{x^3}3x^2$$

therefore $e^{x^3}+C = \int e^{x^3}3x^2\,dx$

Technique for doing integration by substitution

$$\int e^{x^3} 3x^2\,dx$$

① identify what looks like the "inside function" and call it "u"

$$u = x^3$$

② find $\frac{du}{dx}$ and solve for "dx"

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx$$

$$\frac{du}{3x^2} = dx$$

③ use step ② to replace "dX" in the integral, and simplify

$$\int e^{x^3} 3x^2 \, dX$$
$$= \int e^{x^3} 3x^2 \left(\frac{du}{3x^2}\right)$$
$$= \int e^{x^3} du$$

④ use step ① to replace all the "x" with "u"

$$= \int e^u du$$

⑤ do the integral!

$$\int e^u du = e^u + C$$

⑥ use step ① to write your answer in terms of X.

$$e^u + C = e^{x^3} + C$$

example $\int x^3 \cos(x^4+2)\, dX$

① $u = x^4 + 2$

② $\frac{du}{dX} = 4x^3$

$du = 4x^3 dX$

$dX = \frac{1}{4x^3} du$

③ $\int x^3 \cos(4x^4+2) \frac{1}{4x^3} du$

$= \int \cos(4x^4+2) \frac{1}{4} du$

④ $= \int \cos(u) \frac{1}{4} du$

⑤ $= \sin(u) \frac{1}{4} + C$

⑥ $= \sin(x^4+2) \frac{1}{4} + C$

example $\int e^{5x} dX$

① $u = 5x$

② $\frac{du}{dx} = 5 \rightarrow dx = \frac{1}{5} du$

③ $\int e^{5x} \frac{1}{5} du$

④ $\int e^u \frac{1}{5} du$

⑤ $\frac{1}{5} e^u + C$

⑥ $\frac{1}{5} e^{5x} + C$

Activity! (worked in pairs)

1. write down 2 simple functions $f(x)$ and $g(x)$ (e.g. $f(x) = x^3$, $g(x) = \sin(x)$)
2. write down $f(g(x))$ and take the derivative
   $$\frac{d}{dx}(\sin(x))^3 = 3(\sin(x))^2 \cos(x)$$
3. give the function to a classmate. without telling them $f(x)$ or $g(x)$, see if they can find $\int \left(\frac{d}{dx} f(g(x))\right) dx$.
4. Talk about the process with them. Could you tell what they chose for $f(x)$ and $g(x)$?

What about definite integrals?

$$\int_1^3 2x\sqrt{x^2-1}\,dx$$

Two ways to do this.

First way: do all 6 steps as above, then worry about the limits.

$$u = x^2 - 1$$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{1}{2x}\,du$$

$$\int_1^3 2x\sqrt{x^2-1}\,dx = \int_{x=1}^{3} 2x\sqrt{x^2-1}\,\frac{1}{2x}\,du$$

$$= \int_{x=1}^{3} \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big|_{x=1}^{3}$$

$$= \frac{2}{3}(x^2-1)^{3/2}\Big|_1^3$$

$$= \frac{2}{3}(8)^{3/2} - \frac{2}{3}(0)^{3/2}$$

second way: use step ① to change the bounds when you replace dx

if $x=3$ then $u=3^2-1=8$

if $x=1$ then $u=1^2-1=0$

$$\int_1^3 2x\sqrt{x^2-1}\,dx = \int_0^8 2x\sqrt{x^2-1}\,\frac{1}{2x}\,du$$
$$= \int_0^8 \sqrt{u}\,du$$
$$= \frac{2}{3}u^{3/2}\Big|_0^8$$
$$= \frac{2}{3}8^{3/2} - \frac{2}{3}(0)^{3/2}$$

example

The function $f(t)=\frac{1}{2}\sin(2\pi t/5)$, where t is in seconds and $f(t)$ is in liters/second, is a good model for the rate of air flow into the lungs. How much air has flowed into the lungs between $t=0$ and $t=5$?

Lesson 2
more integration
by substitution
Jan 12

example

$$\int x^3 \sqrt{x^2+1}\, dx = \int x^3 \sqrt{x^2+1}\, \frac{1}{2x}\, du$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$dx = \frac{1}{2x}\, du$

$$= \frac{1}{2}\int x^2 \sqrt{x^2+1}\, du$$
$$= \frac{1}{2}\int (u-1)\sqrt{u}\, du$$
$$= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right) du$$
$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$
$$= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C$$

example $\int \frac{x}{\sqrt[4]{x+2}}\, dx$

$u = x+2$

$du = dx$

$$= \int \frac{u-2}{\sqrt[4]{u}}\, du$$
$$= \int u^{3/4} - 2u^{-1/4}\, du$$
$$= \frac{4}{7}u^{7/4} - 2\,\frac{4}{3}u^{3/4} + C$$
$$= \frac{4}{7}u^{7/4} - \frac{8}{3}u^{3/4} + C$$

example

Set up an integral to find the area under the curve $y = 3x\sin(x^2)$ over $\frac{1}{2} \le x \le \frac{3}{2}$.

$$\int_{1/2}^{3/2} 3x\sin(x^2)\, dx$$

Definition The average value of $f(x)$ over the interval $a < x < b$ is

$$\frac{1}{b-a}\int_a^b f(x)\, dx$$

example Set up an integral to find the average value of $\frac{\sin(\ln(x))}{x}$ over $1 < x < 3$, then find it.

example

A certain savings account pays dividends at a rate of $15e^{0.03t}$ dollars/month, where t is in months. How much money does it pay over the course of the 4th month? Round your answer to 4 decimal places.

example

According to a study from 2013 out of the university of Antioquia in medellín colombia, a yorkshire terrier grows at a rate of

$$Y'(t) = \frac{0.21 e^{\frac{1}{10}(15-t)}}{10\left(e^{\frac{1}{10}(15-t)}+1\right)^2}$$

where $Y'(t)$ is in kg/week & t is in weeks. If it weighs 0.4 kg when it is born, find the average weight of the dog over the course of its first 8 weeks.

Note : $\frac{2.1}{e^{1.5}+1} = 0.38$

$$\frac{21}{8} \ln\left(\frac{1+e^{-0.7}}{1+e^{-1.5}}\right) = 0.53$$

$0.53 + 0.02$

$= 0.55$ kg

$= 1.21$ lbs

Lesson 3
Jan 17

Integration using the natural log

Important rules:

$$\ln(AB) = \ln(A) + \ln(B)$$
$$\ln(A/B) = \ln(A) - \ln(B)$$
$$\ln(A^n) = n\ln(A)$$
$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$
$$\int \frac{1}{x}\,dx = \ln(|x|) + C$$

note: we will find $\int \ln(|x|)dx$ on friday!

__example__ $\frac{d}{dx}\ln(3x^2 + \cancel{5x} + 3) = \frac{1}{3x^2 + \cancel{5x} + 3}\frac{d}{dx}(3x^2 + \cancel{5x} + 3)$

$= \frac{1}{3x^2+5x+3}(6x \cancel{+5})$

$= \frac{6x \cancel{+5}}{3x^2 \cancel{+5x} + 3}$

Notice this means $\int \frac{6x \cancel{+5}}{3x^2 \cancel{+5x} + 3}dx = \ln(|3x^2 \cancel{+5x} + 3|) + C$

The moral of the story: don't freak if logarithms show up!

__example__ $\int \frac{6x^2+2x}{6x^3+3x^2+4}dx$

notice $\frac{d}{dx}6x^3+3x^2+4$
$= 18x^2+6x$
$= 3(6x^2+2x)$

$u = 6x^3+3x^2+4$

$\frac{du}{dx} = 18x^2+6x$
$= 3(6x^2+2x)$

$\frac{1}{3(6x^2+2x)}du = dx$

$\int \frac{6x^2+2x}{6x^3+3x^2+4}\frac{1}{3(6x^2+2x)}du = \frac{1}{3}\int\frac{1}{u}du$

$= \frac{1}{3}\ln(|u|) + C$

$= \frac{1}{3}\ln(|6x^3+3x^2+4|) + C$

example $\int 2x^{-1}\cos(\ln(5x))\,dx \quad = \int 2\cancel{x}^{-1}\cos(u)\cancel{x}\,du$

$u = \ln(5x)$

$\frac{du}{dx} = \frac{1}{5x}5 = \frac{1}{x}$

$x\,du = dx$

$= \int 2\cos(u)\,du$

$= 2\sin(u) + C$

$= 2\sin(\ln(5x)) + C$

example $\int_3^8 \frac{5}{x\ln(x)}\,dx$, round your answer to 3 decimal places.

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$x\,du = dx$

$\int_{\ln(3)}^{\ln(8)} \frac{5}{\cancel{x}\ln(x)}\cancel{x}\,du$

$= \int_{\ln(3)}^{\ln(8)} \frac{5}{u}\,du$

$= 5\ln(|u|)\Big|_{\ln(3)}^{\ln(8)}$

$= 5\ln(\ln(8)) - 5\ln(\ln(3))$

$\boxed{= 3.190}$

example

Three months ago, ten porgs arrived on a new island on Ah-TO. The population of porgs is modeled well by

$$P(t) = \frac{200e^t}{1+e^t},$$ where

t is in months from now. What will be the average number of porgs on the island over the course of the next month? Round to the nearest porg.

example $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$ choose a good u!

Integration by parts

Remember the product rule:

$$\frac{d}{dx} uv = \frac{du}{dx} v + u \frac{dv}{dx}$$

Integrate:

$$\int \frac{d}{dx} uv \, dx = \int v \frac{du}{dx} dx + \int u \frac{dv}{dx} dx$$

$$uv = \int v \, du + \int u \, dv$$

$$\boxed{uv - \int v \, du = \int u \, dv}$$

<u>example</u>

$$\int x \sin(\pi x) \, dx$$

choose $u = x$ which gets simpler when you take the derivative

$$\frac{du}{dx} = 1 \quad \text{so} \quad du = dx$$

That means $dv = \sin(\pi x) dx$

$$\int dv = \int \sin(\pi x) dx$$

$$v = -\frac{1}{\pi} \cos(\pi x) + C \quad \text{(choose } C=0\text{)}$$

$$v = -\frac{1}{\pi} \cos(\pi x)$$

integration by parts rule:

$$\int x \sin(\pi x) = -\frac{x}{\pi} \cos(\pi x) - \int -\frac{1}{\pi} \cos(\pi x) dx$$

$$= -\frac{x}{\pi} \cos(\pi x) + \frac{1}{\pi} \int \cos(\pi x) dx$$

$$= -\frac{x}{\pi} \cos(\pi x) + \left(\frac{1}{\pi}\right)^2 \sin(\pi x) + C$$

Check our work:

$$\frac{d}{dx}\left[-\frac{x}{\pi} \cos(\pi x) + \left(\frac{1}{\pi}\right)^2 \sin(\pi x) + C\right]$$

$$-\frac{1}{\pi} \cos(\pi x) + x \sin(\pi x) + \frac{1}{\pi} \cos(\pi x)$$

$$= x \sin(\pi x)$$

ex $\int \ln(x)\,dx$

$u = \ln(x)$ $\qquad dv = dx$

$\frac{du}{dx} = \frac{1}{x}$ $\qquad v = x$

$du = \frac{1}{x}\,dx$

$$\int \ln(x) = \int u\,dv = uv - \int v\,du$$
$$= x\ln x - \int x \tfrac{1}{x}\,dx$$
$$= x\ln x - \int 1\,dx$$
$$= x\ln x - x + C$$

ex $\int x^5 \ln(x^4)\,dx$

$u = \ln(x^4)$ $\qquad dv = x^5\,dx$

$= 4\ln(x)$ $\qquad v = \frac{x^6}{6}$

$\frac{du}{dx} = \frac{4}{x}$

$du = \frac{4}{x}\,dx$

$$\int (\ln(x^4))(x^5\,dx) = \int u\,dv = uv - \int v\,du$$
$$= \frac{x^6}{6}\ln(x^4) - \int \frac{x^6}{6}\frac{4}{x}\,dx$$
$$= \frac{x^6}{6}\ln(x^4) - \frac{2}{3}\int x^5\,dx$$
$$= \frac{x^6}{6}\ln(x^4) - \frac{2}{3}\frac{x^6}{6} + C$$
$$= \frac{x^6}{6}\ln(x^4) - \frac{x^6}{9} + C$$

Definite integral rule for integration by parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

ex $\int_0^{10} x(3x-1)^7\,dx$

$u = x$ $\qquad dv = (3x-2)^7 dx$

$\frac{du}{dx} = 1$ $\qquad v = \frac{1}{3}\frac{1}{8}(3x-2)^8$

$du = dx$

$$\int_0^{10} u\,dv = uv\Big|_0^{10} - \int_0^{10} v\,du$$

$$= x\left(\frac{1}{3}\frac{1}{8}(3x-2)^8\right)\Big|_0^{10} - \int_0^{10} \frac{1}{3}\frac{1}{8}(3x-2)^8 dx$$

$$= \frac{x}{3\cdot 8}(3x-2)^8\Big|_0^{10} - \frac{1}{3\cdot 8}\frac{1}{3}\frac{1}{9}(3x-2)^9\Big|_0^{10}$$

$$= \frac{1}{3\cdot 8}(3x-2)^8\left[x - \frac{1}{3\cdot 9}(3x-2)\right]\Big|_0^{10}$$

$$= \frac{1}{3\cdot 8}(28)^8\left(10 - \frac{1}{3\cdot 9}(28)\right) - \frac{1}{3\cdot 8}(-2)^8\left(0 - \frac{1}{3\cdot 9}(-2)\right)$$

ex $\int_0^{\sqrt{5}} \frac{3x^3}{\sqrt{4+x^2}}\,dx$

What method do we use?

by parts:

$u = 3x^2 \qquad dv = x(4+x^2)^{-\frac{1}{2}}\,dx$

$\frac{du}{dx} = 6x \qquad v = (4+x^2)^{1/2}$

$du = 6x\,dx$

$$\int_0^{\sqrt{5}} u\,dv = uv\Big|_0^{\sqrt{5}} - \int_0^{\sqrt{5}} v\,du$$

$$= 3x^2(4+x^2)^{\frac{1}{2}}\Big|_0^{\sqrt{5}} - \int_0^{\sqrt{5}} 6x(4+x^2)^{\frac{1}{2}}\,dx$$

ex $\int (4z^2+2)\,e^{2z}\,dz$

$u = 4z^2+2 \qquad dv = e^{2z}\,dz$

$du = 8z\,dz \qquad v = \frac{1}{2}e^{2z}$

$$\int u\,dv = uv - \int v\,du$$

$$= (4z^2+2)\left(\tfrac{1}{2}e^{2z}\right) - \int 8z\ \tfrac{1}{2}e^{2z}\,dz$$

$$= (2z^2+1)e^{2z} - \int 4ze^{2z}\,dz$$

Now we need $\int 4ze^{2z}\,dz$. A good time to use integration by parts!

$u = 4z \qquad dv = e^{2z}\,dz$

$du = 4\,dz \qquad v = \frac{1}{2}e^{2z}$

$$\int u\,dv = uv - \int v\,du$$

$$= (4z)\left(\tfrac{1}{2}e^{2z}\right) - \int 4\left(\tfrac{1}{2}e^{2z}\right)dz$$

$$= 2z\,e^{2z} - e^{2z} + C$$

SO: $\int (4z^2+2)e^{2z}\,dz = (2z^2+1)e^{2z} - \left(2ze^{2z} - e^{2z}\right) + C$

$= (2z^2+1)e^{2z} - 2ze^{2z} + e^{2z} + C$

$= \left(2z^2+1-2z+1\right)e^{2z} + C$

$= \left(2z^2-2z+2\right)e^{2z} + C$

Example

$$\int 13x^2 \cos(-6x)\,dx$$

$u = 13x^2 \qquad dv = \cos(-6x)\,dx$

$du = 26x\,dx \qquad v = \frac{1}{-6}\sin(-6x)$

$$13x^2 \frac{1}{-6}\sin(-6x) - \int \frac{1}{-6}\sin(-6x)\,26x\,dx$$

$$= -\frac{13}{6}x^2\sin(-6x) + \frac{13}{3}\int x\sin(-6x)\,dx$$

$$\int x\sin(-6x)\,dx$$

$u = x \qquad dv = \sin(-6x)\,dx$

$du = dx \qquad v = \frac{1}{6}\cos(-6x)$

$$\frac{x}{6}\cos(-6x) - \int \frac{1}{6}\cos(-6x)\,dx$$

$$\frac{x}{6}\cos(-6x) + \frac{1}{36}\sin(-6x) + C$$

$$\int 13x^2\cos(-6x)\,dx = -\frac{13}{6}x^2\sin(-6x) + \frac{13}{3}\left[\frac{x}{6}\cos(-6x) + \frac{1}{36}\sin(-6x) + C\right]$$

$$= -\frac{13}{6}x^2\sin(-6x) + \frac{13}{18}\cos(-6x) + \frac{13}{108}\sin(-6x) + C$$

Choosing your method wisely

"find the area of the region under the curve $x(x-8)^7$ over the interval $8 \le x \le 9$ by integration by parts"

$$\int_8^9 x\,(x-8)^7\,dx$$

$u = x \qquad dv = (x-8)^7 dx$

$du = dx \qquad v = \frac{1}{8}(x-8)^8$

$$\frac{x}{8}(x-8)^8 \Big|_8^9 - \int_8^9 \frac{1}{8}(x-8)^8\,dx$$

$$\frac{x}{8}(x-8)^8 \Big|_8^9 - \frac{1}{8\cdot 9}(x-8)^9 \Big|_8^9$$

$$\left[\frac{9}{8}(1)^8 - \frac{8}{8}(0)^8\right] - \frac{1}{8\cdot 9}\left[1^9 - 0^9\right]$$

$$\frac{9}{8} - \frac{1}{8\cdot 9}$$

$$\frac{81}{8\cdot 9} - \frac{1}{8\cdot 9} = \frac{80}{8\cdot 9} = 10/9$$

"find $\int_8^9 x(x-8)^7 dx$ by any method"

Try substitution:

$u = x - 8$

$du = dx$

$$\int_0^1 (u+8)\,u^7\,du$$

$$\int_0^1 u^8 + 8u^7\,du$$

$$\frac{u^9}{9} + u^8 \Big|_0^1 = \frac{1}{9} + 1 = 10/9$$

example find $\int \frac{4\ln(\sqrt{t+1})}{(t+1)^2}\,dt$

idea use substitution to make it easier,
then integrate by parts.

example $\int x^3\sqrt{1+x^2}\,dx$

Differential Equations

Example

$$\frac{dy}{dx} = \frac{x^3}{y}. \quad \text{Find } y(x)$$

$$y\,dy = x^3\,dx$$

$$\frac{y^2}{2} = \frac{x^4}{4} + C_1$$

$$y^2 = \frac{x^4}{2} + C_2$$

$$y = \pm\sqrt{\frac{x^4}{2} + C_2}$$

Example

$$y'(t) = -5y \quad , \quad y(0) = 13.$$

Example

$$y'(x) = 5x\,e^{y - x^2} \quad , \quad y = 9 \text{ when } x = 1.$$

Newton's Law of Cooling:

If an object has temperature $y(t)$ and its environment has constant temperature T

Then $$\frac{dy}{dt} = k(y - T)$$

where k is some constant.

Example

When I take a pie out of the oven (350°) and put it on the counter (70°), it takes an hour and a half to reach 90°. What is k? How much longer will it be until it reaches 80°?

$$\frac{90 \ln\left(\frac{1}{28}\right)}{\ln\left(25/280\right)} - 90$$

Example

$y' = -3t^n$ with $y(0) = 19$, n is constant, $n \geq 0$.

Example.

The growth rate of a tree is jointly proportional to the height of the tree and its distance from the canopy (100 ft up). Write a differential equation to describe $h(t)$.

Example

A radioactive sample decays at a rate proportional to the size of the sample. Write a differential equation describing the size of the sample.

Separation of Variables

This is the tool to use whenever you can write your problem as

$$y'(t) = f(y)g(t)$$

like:

$$y' = y^2(t+4)$$

$$p' = \left(\frac{1}{p^2}\right)t^3$$

$$\frac{dL}{dx} = e^L \sin(x)$$

What if the problem doesn't match that pattern? often: make it fit.

ex Find the general ~~sol^n to~~

$$10x^2y' = y' + 3xe^{-y}$$

$$(10x^2-1)y' = 3xe^{-y}$$

$$y' = \frac{3x}{10x^2-1}e^y$$

ex Find the particular sol^n to

$$\frac{dy}{dx} - 5x^2(1+y) = 0$$

when $y(0) = 4$

$$\frac{1}{1+y}dy = 5x^2dx$$

$$\ln(1+y) = \frac{5}{3}x^3 + C$$

$$e^{\ln(1+y)} = e^{\frac{5}{3}x^3+C}$$

$$1+y = Ae^{5/3x^3}$$

$$y = Ae^{5/3x^3} - 1$$

ex A sauce on the stove reduces at a rate proportional to its volume. If it takes 10 minutes to reach 75% of its initial volume, how long does it take to reach 50% of its initial volume?

ex The rate at which I grade papers is inversely proportional to the number of papers I have left to grade. If it takes me 1 hour to grade 40 papers, how much longer would it take me to grade 45 papers?

$$\frac{dP}{dt} = \frac{k}{P}$$

$$P\,dp = k\,dt$$

$$\tfrac{1}{2}P^2 = kt + C$$

$$P = \sqrt{2kt + C}$$

know: $P(0) = 40$, $P(1) = 0$

$\downarrow$ $\quad\quad\quad\quad$ $\downarrow$

$40 = \sqrt{C}$ $\quad\quad$ $0 = \sqrt{2k + 1600}$

$1600 = C$ $\quad\quad$ $0 = 2k + 1600$

$\boxed{k = -800}$

what if $P(0) = 45$, but $k = -800$?

$$P = \sqrt{-1600t + C}$$

ex $\frac{dy}{dt} + y\sin t = 0$, $\quad y(\pi) = 12$.

A 500-gallon tank initially contains 260 gallons of pure water, and an agitator which keeps the tank well-mixed. Think of it as a giant blender.

Brine containing 2 pounds of salt per gallon flows in at a rate of 5 gallons per minute, and the mixture drains out of the tank at 3 gallons per minute. How much salt is in the tank after 6 minutes? Do we have the tools to solve this?

$$\frac{dy}{dx} = \frac{\cos(5x)}{e^{5y}}$$

Substance A is converted to substance B at a rate proportional to the square of the amount of substance A. Initially there are 70 grams of substance A, and after 1 hour there are only 15 grams. How much will there be after 12 hours?

The rate of change in number of miles of road cleared by a snow plow is inversely proportional to the depth of the snow. We know 24 mph are cleared when the snow is 2.2 inches thick, and 13 mph are cleared when the snow is 7 inches thick. How many miles per hour are cleared when the snow is 13 inches thick?

Linear Differential Equations & Integrating Factors

Lesson 9

Standard Form: $\frac{dy}{dx} + P(x)\,y = Q(x)$

Integrating factor: $u(x) = e^{\int P(x)\,dx}$

General solution: $y(x) = \frac{1}{u(x)} \int u(x)Q(x)\,dx$

Important notes:
- To use this method, the equation must be in standard form.
- any multiple of $u(x)$ will work just as well.
- Don't forget $+C$ on the last integral

example

$$\frac{dy}{dx} + \frac{1}{x}y = (x-2)^2, \text{ where } x>0.$$

here $P(x) = \frac{1}{x}$, $Q(x) = (x-2)^2$

$$u = e^{\int \frac{1}{x}dx} = e^{\ln|x|+c} = e^c e^{\ln(|x|)}$$

$$= e^c|x| = e^c x$$

Then
$$y = \frac{1}{e^c x}\int e^c x (x-2)^2\,dx$$

$$= \frac{1}{x}\int x(x^2-4x+4)\,dx$$

$$= \frac{1}{x}\int x^3-4x^2+4x\,dx$$

$$= \frac{1}{x}\left(\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 + C\right)$$

$$= \frac{x^3}{4} - \frac{4x^2}{3} + 2x + \frac{C}{x}$$

Note:
- keep the $+C$.
- The previous one cancelled.

why does it work?

$$\frac{dy}{dx} + \frac{1}{x} y = (x-2)^2$$

$$x\frac{dy}{dx} + y = x(x-2)^2$$

$$\frac{d}{dx}(xy) = x(x-2)^2$$

$$xy = \int x(x-2)^2 dx$$

example

$$\frac{dy}{dx} + \tan x \, y = 6\cos x$$

example $t^3y' + \ y = 5t^4e^{\frac{1}{2t^2}}$ $y(1)=0$

$$y' + \frac{1}{t^3}y = 5te^{\frac{1}{2t^2}}$$

$$\int P(t)dt = \int \frac{1}{t} dt$$
$$= \int t^{-3} dt$$
$$= -\frac{1}{2}t^{-2} + C$$
$$u(t) = e^{-\frac{1}{2}t^{-2}+C}$$
$$= e^C e^{-\frac{1}{2}t^{-2}}$$

$$y = \frac{1}{e^C e^{-\frac{1}{2}t^{-2}}} \int e^C e^{-\frac{1}{2t^2}} 5te^{\frac{1}{2t^2}} dt$$
$$= \frac{1}{e^{-\frac{1}{2t^2}}} \int 5t e^{-\frac{1}{2t^2} + \frac{1}{2t^2}} dt$$
$$= e^{\frac{1}{2t^2}} \int 5t \, dt$$
$$= e^{\frac{1}{2t^2}} \left(\frac{5}{2}t^2 + C\right)$$

$\Rightarrow C = -\frac{5}{2}$ $\qquad y = \frac{5}{2}(t^2-1)e^{\frac{1}{2t^2}}$

example

$$4\frac{dy}{dt} - y = t$$

example $(y-238)\sin(x)\,dx - dy = 0$

example $\dfrac{dy}{dx} + 3x^2 y = 6x^2$

Solving Real Problems with Differential Equations

1. Identify variables
2. write a differential equation
3. Identify the type of differential equation
4. Solve the differential equation by the method from (3).
5. use your solution to solve the problem

example

A person weighs 60 kg and eats 1600 calories/day. 850 calories are used per day by her base metabolism, and she uses 15 cal/day per kg of body weight. If she stores energy as fat (which has 10,000 calories/kg), How much will she weigh after a week on the diet? after a month?

W = weight (kg)

t = time (days)

$$\frac{dW}{dt} = \frac{1\text{ kg}}{10{,}000}\left(\frac{1600\text{ cal}}{\text{day}} - \frac{85\text{ cal}}{\text{day}} - \frac{15W\text{ cal}}{\text{day}}\right)$$

$$\frac{dW}{dt} = \frac{1515}{10000} - \frac{15}{10000}W$$

Example

a 500 gallon tank initially contains 260 gallons of pure water. brine flows in at 5 gal/minute, carrying 2 lbs salt/gallon. Mixture drains out of the tank at 3 gal/min. How much salt is in the tank when full?

$$\frac{dS}{dt} = 10 - \frac{3S}{260+2t}$$

$$\frac{dS}{dt} + \frac{3}{260+2t} S = 10$$

$$P = \frac{3}{260+2t} \qquad Q = 10$$

$$\int P\,dt = \int \frac{3}{260+2t}\,dt$$

$$= \frac{3}{2} \ln|260+2t|$$

$$u = e^{\int P dt} = e^{\frac{3}{2}\ln|260+2t|}$$

$$= e^{\ln((260+2t)^{3/2})}$$

$$= (260+2t)^{3/2}$$

$$y = \frac{1}{u} \int u Q\,dt$$

$$= (260+2t)^{-3/2} \int 10\,(260+2t)^{3/2}\,dt$$

$$= (260+2t)^{-3/2}\, 10 \left(\frac{2}{5}(260+2t)^{5/2} \cdot \frac{1}{2} + C \right)$$

$$= 2(260+2t) + \frac{C}{(260+2t)^{3/2}}$$

$$y(0)=0 \Rightarrow 2(260) + \frac{C}{260^{3/2}} = 0$$

$$-2(260)^{5/2} = C$$

$$y = 2(260+2t) - \frac{2(260)^{5/2}}{(260+2t)^{3/2}}$$

$$y(120) = 2(500) - \frac{2(260)^{5/2}}{(500)^{3/2}}$$

$$= 1000 - 2 \cdot 260 \cdot \left(\frac{260}{500}\right)^{3/2}$$

$$\approx 805$$

How to study linear differential equations

choose $f(x)$ and $g(x)$ simple — $f(x) = x^2$, $g(x) = e^x$

find $f'(x)$, $g'(x)$ — $f'(x) = 2x$, $g'(x) = e^x$

write $y'(x) f(x) + f'(x) y = g'(x)$

$$\frac{dy}{dx} + \frac{f'(x)}{f(x)} y = \frac{g'(x)}{f(x)} \qquad \frac{dy}{dx} + \frac{2}{x} y = \frac{e^x}{x^2}$$

give it to a friend to solve!

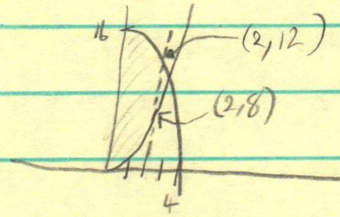
$$u = f(x)$$

$$y = \frac{g(x) + C}{f(x)}$$

Lesson 11

The Area Between Curves

ex Find the area between $y = 16 - x^2$ and $y = x^3$ over the interval $0 \le x \le 2$.



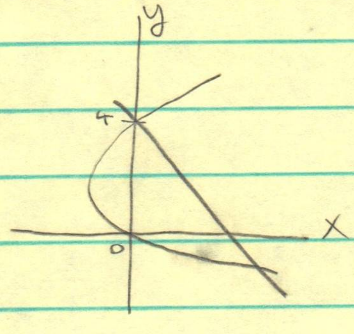
In general: $\text{Area} = \int_a^b (\text{upper function}) - (\text{lower function})\, dx$

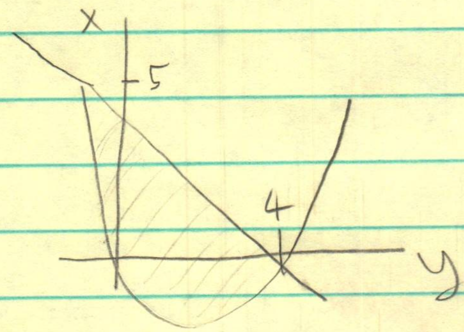
$(x+2)(x-3) = x^2 - x - 6$

ex Find the area bounded by $y = x^2 - 1$ and $y = x + 5$

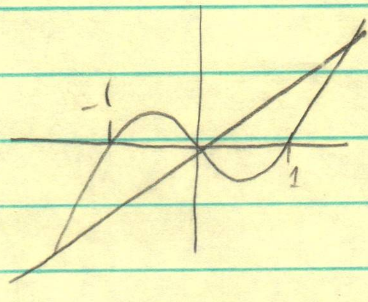
$(y+1)(y-4) = y^2 - 3y - 4$

ex Find the area bounded by $x = y^2 - 4y$ and $x + y = 4$





ex Find the area of the region bounded by $y = x^3 - x$ and $y = 3x$



$x^3 - x = 3x$

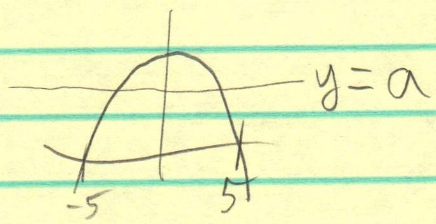
$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x+2)(x-2) = 0$

ex Find the equation of the horizontal line that divides the region in half.

$y = 25 - x^2$, $y = 0$



ex Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = 3$.

Solids of Revolution

Example

My foam cup has a radius of 2.4 cm at its base, and 4.1 cm at its top. The cup is 4.4 cm tall. What is its volume?

(c.f. 241.4 mL)

Example

Find the volume of the solid generated by revolving the region enclosed by $y = \frac{1}{8}\sec(x)$, $y=0$, $x=0$, and $x=\frac{\pi}{5}$ about the x-axis

Example

Find the volume of a sphere with radius 1.

Example

Find the volume of the solid generated by revolving the region enclosed by $y=x^2$, $x=0$, and $y=8$ about the y-axis

example

Find the volume of the solid generated by revolving the region enclosed by $y=\ln(x)$, $y=4$, and the axes about the y-axis.

Solids of Revolution with Cavities

example

find the volume of the solid generated by rotating the region bounded by $x=0$, $x=1$, $y=\sqrt{x}$, and $y=e^x$ about the x-axis

general rule:

if the region is bounded above by $f(x)$ and below by $g(x)$ then $V=\int_a^b \pi(f(x))^2-\pi(g(x))^2\,dx$

NOT $V=\int_a^b \pi(f(x)-g(x))^2\,dx$

example

The region bounded by $y=-x^2+8x+30$ and $y=30-x$ is rotated about the x-axis. find the volume.

example

the region bounded by $y=0$, $x=e^3$, and $y=\frac{1}{6}\ln(x)$ is rotated about the y-axis.

example

The region inside $x^2+y^2=25$ and above $y=3$ is rotated about the x-axis

ex

the region bdd by $y=2x$, $x=1$, $x=5$, and $y=0$ is rotated abt. the y-axis.

volumes of rotations about lines other than axes.

ex R bounded by $y = x^2$ and $y = 9$,
rotate abt. line $y = -4$

ex R bounded by $y = \frac{1}{2}x^2$, $y = 2$, and $x = 0$
rotate abt. line $x = 5$

ex R bounded by $y = -x^2 + 3x + 3$
$y = 3 - x$
rotate abt. line $y = 10$

ex R bounded by $y = \frac{3}{2}\sqrt{x}$, $y = 0$, $x = 16$
rotate abt. line $x = 16$

Propane Tank Problem

Propane tank is generated by rotating $x^2 + 81y^2 = 100$ abt the y-axis. How deep when it's $\frac{1}{4}$ full?

MA 16020
Lesson 15
Improper Integrals

Sometimes, it is useful to consider the area under a curve from some point onwards. (This is often useful in probability, diff eqs, etc.) We consider integrals of the form $\int_a^\infty f(x)\,dx$, which are improper integrals with an infinite bound.

By definition, $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$, if the limit exists. If the limit does not exist, we say the integral $\int_a^\infty f(x)dx$ diverges.

Ex 1. Find the integral, if it converges. $\int_1^\infty \frac{1}{3x-2}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{1}{3x-2}\,dx = \lim_{b\to\infty} \frac{1}{3}\ln|3x-2|\Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{3}\ln(3b-2) - \frac{1}{3}\ln(1)\right)$

$= \lim_{b\to\infty} \frac{1}{3}\ln(3b-2) = \infty$ so the integral $\boxed{\text{diverges}}$

Ex 2. Find the integral, if it converges $\int_0^\infty \frac{5x}{e^{2x}}\,dx$

$= \lim_{b\to\infty} \int_0^b 5xe^{-2x}\,dx$ $\quad u = 5x \quad v = -\frac{1}{2}e^{-2x}$ $\quad du = 5\,dx \quad dv = e^{-2x}\,dx$

$\int 5xe^{-2x}\,dx = -\frac{5}{2}xe^{-2x} - \int -\frac{5}{2}e^{-2x}\,dx = -\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x}$

so $\lim_{b\to\infty}\left[-\frac{5}{2}xe^{-2x} - \frac{5}{4}e^{-2x}\right]_0^b = \lim_{b\to\infty}\left[-\frac{5}{2}be^{-2b} - \frac{5}{4}e^{-2b} + 0 + \frac{5}{4}\right]$

By L'Hôpital's Rule, $\lim_{b\to\infty} \frac{-5b}{2e^{2b}} = \lim_{b\to\infty} \frac{-5}{4e^{2b}} = 0$

Also $\lim_{b\to\infty} -\frac{5}{4}e^{-2b} = 0$

So we get $0 - 0 + 0 + \frac{5}{4} = \frac{5}{4}$

Hence, $\int_0^\infty \frac{5x}{e^{2x}}\,dx = \boxed{\frac{5}{4}}$

Ex 3. Find the integral, if it converges $\int_1^\infty \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx$

$= \lim_{b\to\infty} \int_1^b \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,dx \qquad u = -2\sqrt{x} = -2x^{1/2},\ du = -x^{-1/2}\,dx$

So $-du = \frac{1}{\sqrt{x}}\,dx$

$\int \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-2\sqrt{x}}$

Thus, $\lim_{b\to\infty}\left[-\frac{1}{2}e^{-2\sqrt{x}}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2\sqrt{b}} + \frac{1}{2}e^{-2}\right] = 0 + \frac{1}{2}e^{-2}$

So $\int_1^\infty \frac{e^{-2\sqrt{x}}}{2\sqrt{x}}\,dx = \boxed{\frac{1}{2e^2}}$

There are also <u>improper integrals</u> <u>with an infinite discontinuity</u>. Such integrals are of the form $\int_a^b f(x)\,dx$ where $f(x)$ has an infinite discontinuity on $a \le x \le b$.

If $f(x)$ has an infinite discontinuity at a,

$$\int_a^b f(x)\,dx = \lim_{s\to a^+}\int_s^b f(x)\,dx, \text{ if it exists}$$

If $f(x)$ has an infinite discontinuity at b,

$$\int_a^b f(x)\,dx = \lim_{s\to b^-}\int_a^s f(x)\,dx, \text{ if it exists.}$$

Ex 4. Find the integral, if it converges $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$\frac{1}{\sqrt{x}}$ has an infinite discontinuity at $x=0$

$\lim_{s\to 0^+}\int_s^1 x^{-1/2}\,dx = \lim_{s\to 0^+}\left[2x^{1/2}\right]_s^1 = \lim_{s\to 0^+}\left[2s^{1/2} - 2\right]$

$= 0 - 2 = 2$

Thus, $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \boxed{2}$

Ex 5. Find the integral if it converges $\int_0^\pi 6\tan(\frac{\theta}{2})\,d\theta$

$\tan(\frac{\theta}{2})$ has an infinite discontinuity when $\cos(\frac{\theta}{2}) = 0$

i.e., when $\frac{\theta}{2} = \frac{\pi}{2} + n\pi \Rightarrow \theta = \pi + 2n\pi$

issue is at π

$$\lim_{s\to\pi^-} \int_0^s 6\tan(\tfrac{\theta}{2})\,d\theta = \lim_{s\to\pi^-} \left[-12\ln|\cos(\tfrac{\theta}{2})|\right]_0^s$$

$$= \lim_{s\to\pi^-} \left[-12\ln|\cos(\tfrac{s}{2})| + 0\right] = \lim_{t\to 0^+} -12\ln t = -\infty$$

So $\int_0^\pi 6\tan(\frac{\theta}{2})\,d\theta$ **diverges**

A fun example. Find $\int_{-1}^1 \frac{1}{x}\,dx$, if it exists

$\frac{1}{x}$ has an infinite discontinuity at $x=0$

so $\int_{-1}^1 \frac{1}{x}\,dx = \int_{-1}^0 \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$

$$= \lim_{s\to 0^-} \int_{-1}^s \frac{1}{x}\,dx + \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx$$

$$= \lim_{s\to 0^-} \ln|x|\Big|_{-1}^s + \lim_{t\to 0^+} \ln|x|\Big|_t^1$$

$$= \underbrace{\lim_{s\to 0^-}\left(\ln|s| - \ln|-1|\right)}_{\substack{-\infty - 0 \\ \text{diverges!}}} + \underbrace{\lim_{t\to 0^+}\left(\ln|1| - \ln|t|\right)}_{\substack{0 + \infty \\ \text{diverges!}}}$$

So $\int_{-1}^1 \frac{1}{x}\,dx$ diverges.

If you're not careful, you would do

$\int_{-1}^1 \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^1 = \ln(1) - \ln|-1| = 0 - 0 = 0$

which is wrong

Geometric Series

Recall: $\sum_{n=3}^{5} n^2\cos(n\pi) = 3^2\cos(3\pi) + 4^2\cos(4\pi) + 5^2\cos(5\pi)$

example: write in summation notation:

$$\frac{1}{5} - \frac{4}{10} + \frac{9}{15} - \frac{16}{20} + \frac{25}{25} = \sum_{n=1}^{5} \frac{n^2}{5n}(-1)^{n+1}$$

New concept: Infinite sums

$$f(1)+f(2)+f(3)+\dots = \sum_{n=1}^{\infty} f(n) = \lim_{b\to\infty} \sum_{n=1}^{b} f(n)$$

mention partial sums → par

example: write in summation notation:

$$3.\overline{8} = 3.888\ldots = 3 + \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \dots$$

$$= 3 + \sum_{n=1}^{\infty} \frac{8}{(10)^n}$$

example: compute $2+2+2+\dots$

$$2+2+2+\dots = \sum_{n=1}^{\infty} 2 = \lim_{b\to\infty} \sum_{n=1}^{b} 2$$

$$= \lim_{b\to\infty} \underbrace{2+2+\dots+2}_{b \text{ times}}$$

$$= \lim_{b\to\infty} 2b = \infty$$

New concept: geometric series

any series of the form $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots$

example: write $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ in summation notation, & guess what the limit converges to.

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

The trick

$$\sum_{n=0}^{b} ar^n = a + ar + ar^2 + \dots + ar^{b-1} + ar^b$$

$$(1-r)\sum_{n=0}^{b} ar^n = a(1-r) + ar(1-r) + \dots + ar^{b-1}(1-r) + ar^b(1-r)$$

$$= a - ar + ar - ar^2 + \dots + ar^{b-1} - ar^b + ar^b - ar^{b+1}$$

$$(1-r)\sum_{n=0}^{b} ar^n = a - ar^{b+1}$$

$$\sum_{n=0}^{b} ar^n = \frac{a - ar^{b+1}}{1-r}$$

$$\sum_{n=0}^{\infty} ar^n = \lim_{b\to\infty} \frac{a - ar^{b+1}}{1-r}$$

$$= \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

example find $\sum_{n=0}^{\infty} 3\left(-\frac{1}{4}\right)^n$

ans: $a = 3$, $r = -\frac{1}{4}$

$$\sum_{n=0}^{\infty} 3\left(-\frac{1}{4}\right)^n = \frac{3}{1-\left(-\frac{1}{4}\right)} = \frac{3}{1+\frac{1}{4}} = \frac{3}{(5/4)} = \frac{12}{5}$$

example find $\sum_{n=1}^{\infty} 7(-1)^n \left(\frac{3}{4}\right)^{n+1}$

example find $\sum_{n=0}^{\infty} 4\left(\frac{8}{3}\right)^n$

Lesson 17
Feb 23

Applied Geometric Series

Amoxicillin has a bioavailability of 95%, and a biological halflife of 61.3 mins ≈ 1 hour. My doctor told me to take a 500 mg pill every six hours. If I continued the treatment, how much amoxicillin would be in my body just before I take a pill?

$$A(t) = (0.95)(500)\left(\tfrac{1}{2}\right)^t$$

$$S = (0.95)(500)\left(\tfrac{1}{2}\right)^6 + (0.95)(500)\left(\tfrac{1}{2}\right)^{12} + (0.95)(500)\left(\tfrac{1}{2}\right)^{18} + \dots$$

$$= (0.95)(500)\left(\left(\tfrac{1}{2}\right)^6\right)^1 + (0.95)(500)\left(\left(\tfrac{1}{2}\right)^6\right)^2 + (0.95)(500)\left(\left(\tfrac{1}{2}\right)^6\right)^3 + \dots$$

$$= \sum_{n=1}^{\infty} (0.95)(500)\left(\left(\tfrac{1}{2}\right)^6\right)^n$$

$m = n-1$

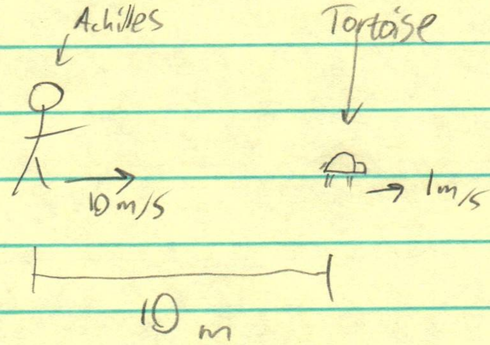
$$= \sum_{m=0}^{\infty} (0.95)(500)\left(\tfrac{1}{2}\right)^6\left(\left(\tfrac{1}{2}\right)^6\right)^m$$

$a = (0.95)(500)\left(\tfrac{1}{2}\right)^6$, $r = \left(\tfrac{1}{2}\right)^6$, $|r| < 1$.

$$S = \frac{(0.95)(500)\left(\tfrac{1}{2}\right)^6}{1 - \left(\tfrac{1}{2}\right)^6} = 7.54 \text{ mg}$$

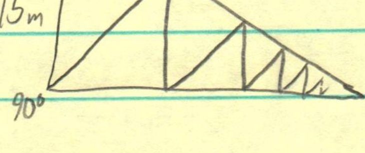
Suppose 30% of money people receive is saved, 70% is spent. The government gives out 80 billion dollars as a tax rebate, to stimulate the economy. How much spending does the stimulus generate, assuming people spend their rebate?

Zeno
(of Elea, 450 BC)



$0.99\overline{9}$

Ye Triangle



Mention: Investment problem, on Lon-Capa

Lesson 16
Feb 26

Functions of Several Variables

In 1686, Newton gave us his law of universal gravitation:

$$F(m, M, r) = G\frac{mM}{r^2}$$

which is a function in 3 variables. We know now that $G = 6.674 \times 10^{-11}\ N(m^2/kg^2)$. We can use the law to figure out the force of gravity on any object. If I am 65 kg, the moon is 7.34×10^{22} kg, and there are 3.6×10^8 meters between us, then

$$F(65, 7.34 \times 10^{22}, 3.6 \times 10^8) = G\frac{(65)(7.3 \times 10^{22})}{(3.6 \times 10^8)^2} = 0.0024N$$

→ 3 grams of sand

At the time, nobody knew the value of G.

<u>The Schiehallion experiment</u> (1774 - 1776

Find the mass of a mountain, then measure its force on a known mass (a pendulum).

They needed the mass of the mountain!

They invented:

Contour Lines

to describe a new function of 2 variables:

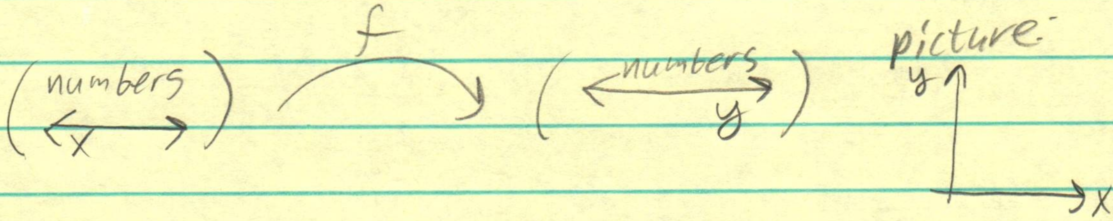
$h(x, y)$

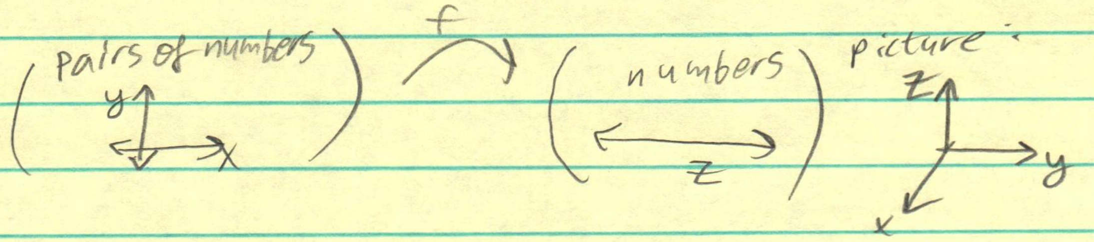
the height at a given longitude & lattitude.

Then used this to find the mountain's mass (err < 20%)

later, 1786, henry cavendish found it better (err < 1%)

Picturing functions of 2 variables using contour lines

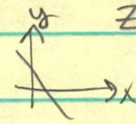




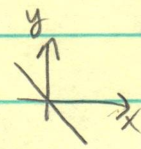
example

Sketch the function $f(x,y) = 1-x-y$ using level curves. $z = 1-x-y$

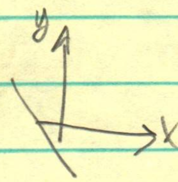
$z=0$: $1-x-y=0$
$1-x=y$



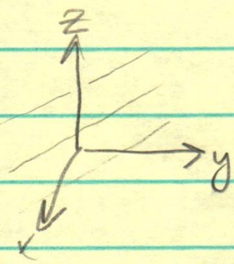
$z=1$: $1-x-y=1$
$-x=y$



$z=2$: $1-x-y=2$
$-x-1=y$



answer:



example: find the shape of the level curves (contour lines) of

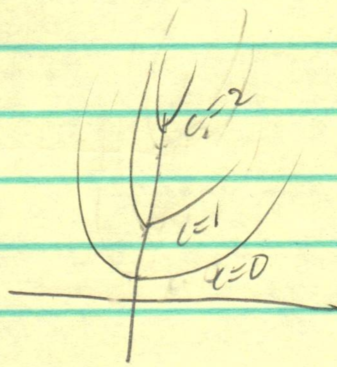
$$f(x,y) = \ln(y - 4x^2)$$

$$c = \ln(y - 4x^2)$$

$$e^c = y - 4x^2$$

$$y = 4x^2 + e^c$$

parabola, y-intercept of e^c

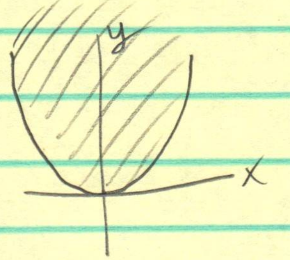


Q: what is the domain of $f(x,y)$ above?

$$y - 4x^2 > 0$$

$$y > 4x^2$$

$$D = \{(x,y) \mid y > 4x^2\}$$



example

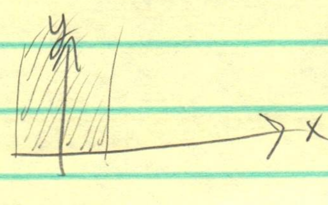
what is the domain of $f(x,y) = \dfrac{\ln(y)}{\sqrt{1-x^2}}$?

we need: $y > 0$ $\quad$ $1 - x^2 > 0$

$$1 > x^2$$

$$1 > \sqrt{x^2} = |x|$$

ans: $\{(x,y) \mid y > 0 \text{ and } 1 > |x|\}$



example find the domain of $\frac{1}{x-y} + \sqrt{x}$

we need $x-y \neq 0$ and $x \geq 0$

$x \neq y$

ans: $\{ (x,y) \mid x-y \neq 0, \ x \geq 0 \}$

example

describe the level curves of

$$f(x,y) = e^{(x-4)^2 + (y+1)^2}$$

Lesson 19
Feb 28

Partial Derivatives

Say $f(x,y) = 3xy + y^2$

Remember how to take the derivative w.r.t. x:

$$\frac{d}{dx} f(x,y) = \frac{d}{dx}(3xy + y^2)$$
$$= \frac{d}{dx}(3xy) + \frac{d}{dx}(y^2)$$
$$= 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

This is OMG! TMI! Too much information!
when x and y don't depend upon eachother, use the "Partial derivative" $\frac{\partial}{\partial x}$ treating y as a constant.

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(3xy + y^2)$$
$$= \frac{\partial}{\partial x}(3xy) + \frac{\partial}{\partial x}(y^2)$$
$$= 3y + 0$$
$$= 3y$$

we can also find $\frac{\partial}{\partial y} f(x,y)$:

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(3xy + y^2)$$
$$= \frac{\partial}{\partial y}(3xy) + \frac{\partial}{\partial y}(y^2)$$
$$= 3x + 2y$$

Notice $f'(x,y)$ makes no friggin' sense. what's the variable of differentiation? Instead, write:

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y)$$

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y)$$

Remember: partial derivatives are easier than total / full / normal derivatives. That's why we use them.

example find $f_x(0,1)$ and $f_y(0,1)$ if
$f(x,y) = 3y\sec(\pi x)$

example say $z = \sqrt{3xy^2+1}$
find $\frac{\partial z}{\partial y}$ at $(1,2)$.

example A certain shop sells pens and pencils. These are substitute goods, meaning if people buy more of one they will buy less of the other. If the shop arranges to sell x pens and y pencils, their revenue is given by

$$R(x,y) = 200x + 100y - 3x^2 - 8xy - 4y^2$$

where R is in dollars.

Find the marginal revenue with respect to the number of pens & pencils sold.

A JOKE what do you get when you add a rabbit, a half a rabbit, a quarter rabit, and so on? Technically 2 rabbits, but it's only possible if you split hares.

example find $\frac{\partial z}{\partial x}$ if $z = \frac{(4x+8y)^3}{\ln(y+1)}$

example find f_y if $f(x,y) = \frac{3xy^3}{x^2y+4}$

Second Partial derivatives.

example if $f(x,y) = x^2 e^y + y\ln(x)$

find $\frac{\partial}{\partial x} f_x(x,y)$

New rule: $\frac{\partial}{\partial x} f_x(x,y) = (f_x)_x(x,y) = f_{xx}(x,y)$

Now find $\frac{\partial}{\partial y} f_x(x,y) = (f_x)_y(x,y) = f_{xy}(x,y)$

Now find $\frac{\partial}{\partial y} f_y(x,y)$ and $\frac{\partial}{\partial x} f_x(x,y)$.

Notice: $f_{xy}(x,y) = f_{yx}(x,y)$. This always happens!

we only need f_{xx}, f_{xy}, and f_{yy}.

example find all the second partial derivatives of $f(x,y) = ye^{x^2 y}$.

example find $\frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(x^2 y + \frac{3\sin(y)e^y}{\sqrt{y^4+3y+2\ln(y)}}\right)$

The trick: may as well do $\frac{\partial}{\partial y}\frac{\partial}{\partial x}\left(x^2 y + \frac{3\sin(y)e^y}{\sqrt{y^4+3y+2\ln(y)}}\right)$

Notation remember $\frac{d}{dx}\frac{d}{dx} f(x) = \frac{d^2}{(dx)^2} f(x) = \frac{d^2 f}{dx^2}$

same idea: $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y)$

example Find all the second partial derivatives of

$$f(u,v) = \frac{u\ln(8uv)}{3v}$$

example Find all the second partial derivatives of $f(s,t) = 3te^{\cos(4s-8)}$

tell a joke!

Differentials of Multivariable Functions Lesson 21

Remember, in one variable:

$$\frac{\Delta f}{\Delta x} \approx \frac{df}{dx}$$

and so we said $\Delta f \approx \frac{df}{dx} \Delta x$

we used this to estimate functions:

eg estimate $\sqrt{4.01} - 2$ using differentials

solution: take $f(x) = \sqrt{x}$, $x = 4$, and $\Delta x = 0.01$

Then $\sqrt{4.01} - 2 = f(4+0.01) - f(4) = \Delta f$

here $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

we find $\Delta f \approx \frac{1}{4}(0.01) = 0.0025 \approx 0.002498$

New idea: apply this concept to multivariable functions

$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

example say $f(x,y) = 4x^2 + 3xy - 3y^2$.

Estimate $f(1.02, 1.99) - f(1,2)$ using differentials

here $(x,y) = (1,2)$ and $\Delta x = 0.02$, $\Delta y = -0.01$

we need $f_x(x,y) = 3y + 8x$ so $f_x(1,2) = 8 + 6 = 14$

$f_y(x,y) = 3x - 6y$ so $f_y(1,2) = 3 - 12 = -9$

Then

$$\Delta f \approx (14)(0.02) + (-9)(-0.01)$$
$$= 0.28 + 0.09$$
$$= 0.37$$

actual value: $f(1.02, 1.99) - f(1,2) = 0.3707$

$PV = nk_B T$
$\underbrace{nk_B}_{0.36\ J/K}$

Example: A piston in an engine holds 1 liter of air, and is held at 293 kelvin. If the piston is compressed to 0.8 liters and the temperature increased to 320 kelvin, how much will the pressure change?

Recall ideal gas law: $PV = nRT$

$PV = nk_B T$

$nk_B \approx 0.357$ Joules/kelvin

All About Error

The absolute error is the worst-case difference from the nominal (intended) value. example:

the tolerances in dimensions of the bricks are given in millimeters. For example: the width of a given cinder block is 100 plus 3 or -5.

$$95 < W < 103$$

$$w = 99 \pm 4$$

the width has an absolute error of 4 mm

The other chosen values here are length = 440, height = 215

The relative errror is the absolute error divided by the nominal value. example:

the tolerance in density is given as $\pm 10\%$.

If the density is $1000\ kg/m^3$, then this means $900 < \text{density} < 1100$

Relative error is more useful.

example If a brick measures $440 \times 190 \times 215$ mm^3 millimeters ± 4 mm and has a density of $1200 \frac{kg}{m^3} \pm 10\%$, then what is the mass of the block? what is the absolute error? what is the relative error?

$$M = \frac{1}{(1000)^3} D w^3$$

$$= \frac{1}{(1000)^3} 1200 \times 400^3 = 11.35 \text{ kg}$$

$$= 102.2 \text{ kg}$$

Absolute error:

$$\Delta M = \frac{\partial M}{\partial D}\Delta D + \frac{\partial M}{\partial l}\Delta l + \frac{\partial M}{\partial w}\Delta w + \frac{\partial M}{\partial h}\Delta h$$

$$= \frac{1}{(1000)^3}\left[lwh\,\Delta D + Dwh\,\Delta l + Dlh\,\Delta w + Dlw\,\Delta h\right]$$

$$= \frac{1}{(1000)^3}\left[4\right.$$

$$\Delta M = \frac{\partial M}{\partial D}\Delta D + \frac{\partial M}{\partial w}\Delta w$$

$$= \frac{w^3}{1000^3}\Delta D + \frac{3Dw^2}{1000^3}\Delta w$$

$$= \frac{440^3}{1000^3}(\pm 120) + \frac{3(1200)(440)^2}{1000^3}(\pm 4)$$

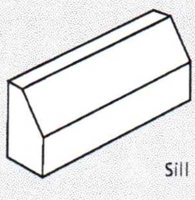
$$= 13.01$$

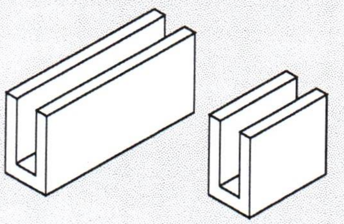
relative error: $\frac{13}{102} = 12.7\%$

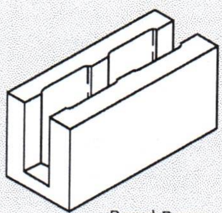
CONCRETE BLOCK ASSOCIATION

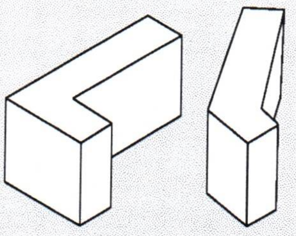
A Guide to Selection & Specification

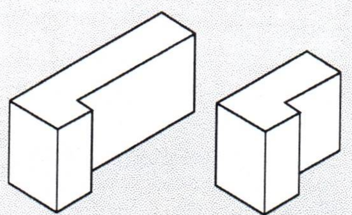
Special Blocks
Blocks produced to form an extensive 'kit-of-parts' which gives the designer-specifier greater flexibility.

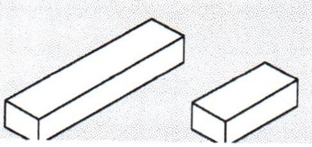

Sill


Lintel


Bond Beam


Quoin/Return


Cavity closers



Dimensional tolerances

Tolerances permitted by BS EN 771-3 are:

Tolerance Category	D1	D2
Length	+3	+1
	-5	-3
Width	+3	+1
	-5	-3
Height	+3	+2
	-5	-2

Limiting deviations in millimetres.

Blocks will generally be supplied to D1 tolerance category unless otherwise specified.

Manufacturers are permitted to supply to tighter tolerances on any dimension within a tolerance category.

Block Strengths

Blocks are available in compressive strengths from 2.9N/mm² to 40N/mm² (Solid) and 2.9N/mm² to 22.5N/mm² (cellular and hollow). Common strengths are 3.6N/mm² and 7.3N/mm².

Density

Aggregate concrete blocks are available in the net dry density range of 650 – 2400kg/m³ with a tolerance of ± 10%.

The full range of densities will not necessarily be available from all manufacturers.

Gross dry densities are typically used for cellular and hollow units and for the same products will be lower than net dry densities.

Configuration

Units to BS EN 771-3: Aggregate concrete masonry units will fall within one of the 4 groups specified in BS EN 1996-1-1: Eurocode 6 – Design of masonry structures.

Group 1 < 25% formed voids
Group 2 > 25% < 60% formed vertical voids
Group 3 > 25% < 70% formed vertical voids
Group 4 > 25% < 50% formed horizontal voids

Generally units will fall within Group 1 and Group 2 configurations.

Additional Details (when relevant)

- Block description
- Density or unit weights
- Flatness of surface (only applicable to facing units)
- Thermal resistance
- Durability
- Water absorption (not required for specification in the UK)
- Moisture movement (not required for specification in the UK)
- Water vapour permeability (not required for specification in the UK)
- Reaction to fire (designated non-combustible by UK Building Regulations)
- Shear bond strength (not required for specification in the UK)
- Flexural bond strength (not required for specification in the UK)

Visit www.cba-blocks.org.uk for the latest information, news and views from the CBA.
CBA Technical Helpline 0116 222 1507

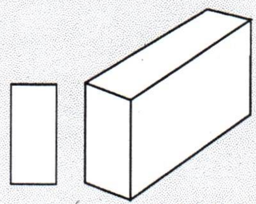
THB BRITISH PRECAST modern masonry

CONCRETE BLOCK ASSOCIATION

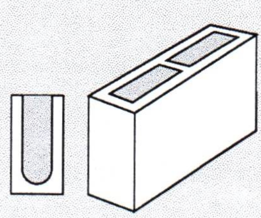
CI/SfB Ff2 (Ajv)

Aggregate Concrete Blocks
A Guide to Selection & Specification

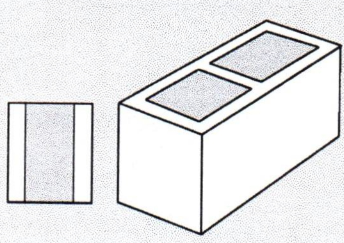
Block Types



Solid blocks
Blocks which contain no formed voids. (Group 1)



Cellular blocks
Blocks which contain one or more formed voids which do not fully penetrate the block. (Group 1 or Group 2 according to void percentage.)



Hollow blocks
Blocks which contain one or more formed voids which fully penetrate the block. (Group 1 or Group 2 according to void percentage.)

Blocks are available in various strengths and surface textures.

Concrete Block Association
60 Charles Street, Leicester LE1 1FB
Tel: 0116 253 6161 Fax: 0116 251 4568
Email: enquiries@cba-blocks.org.uk

Introduction

This guide has been produced to assist designers and specifiers with the selection and specification of building blocks, which are manufactured in accordance with BS EN 771-3. The appropriate CBA Data Sheets should be consulted for more detailed information, including the selection of the UHB (Universal Housing Block) for housing projects. BS EN 771-3 contains many more properties than its predecessor BS 6073-1 but not all these are required for use in the UK. Therefore only those properties specifically needed for an application should be specified.

Care should be taken to ensure that the mix of properties specified are mutually compatible and therefore available in a single block type.

Block Specification

For convenience blocks may be categorized using the following block details for all specifications:

- Block description (e.g. standard common blocks)
- Dimensions (e.g. 440mm x 100mm x 215mm)
- Tolerance category (e.g. D1)
- Strength (e.g. 7.3 N/mm²)
- Net dry density (e.g. 2000kg/m³)
- Configuration (e.g. solid/Group 1)

Within these categories there are three configurations, solid, cellular and hollow: see opposite.

Block Description

Block types are available in various ranges produced by CBA manufacturers, which may be generally described as follows:

Standard common blocks

Blocks suitable for general building work, offering excellent all round performance and normally available in 440 x 215mm face size. In addition to their loadbearing capabilities, they provide an excellent background for plastering and rendering as well as for fixings. This type of block is not normally intended for use in facing applications as variations in colour and texture may occur.

Close textured/Paint grade common blocks

Blocks manufactured with a close texture and suitable for direct painting.

Standard facing blocks

Blocks manufactured for applications where shape and texture consistency are of prime importance. Slight variations in colour may be discernible.

Architectural masonry facing blocks

Blocks manufactured to high standards of dimensional accuracy and consistency of colour and texture. The blocks are intended for use in situations where the visual appearance of the wall is of primary concern. The blocks are available in a range of colours, textures, finishes and shapes.

See back for Special Block shapes.

Dimensions

Face sizes/co-ordinating dimensions

Aggregate concrete blocks are typically available in two standard face sizes (length x height) of 440 x 215mm and 390 x 190mm. Other face sizes are available to aid manual handling. To obtain the co-ordinating dimensions add the specified joint thickness (normally 10mm) to the height and length of the block.

Block dimensions should be specified in the order length x width x height.

Intermediate widths may be available from some manufacturers.

Dimensions of commonly available blocks are:

	Face size (mm)	
Width	440 x 215	390 x 190
	75	-
	90	90
	100	100
	140	140
	190	190
	215	-

Example: A speaker pumps out 100 watts, $\pm 20\%$. The amount of that energy impacting a persons ear is given by $S = \frac{E}{(1000)D^2}$, where D is how far away they are standing. If $D = 10m \pm 1m$, what is the absolute error in the energy impacting them?

The Chain Rule in Multiple variables

Remember the rule for differentials:

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

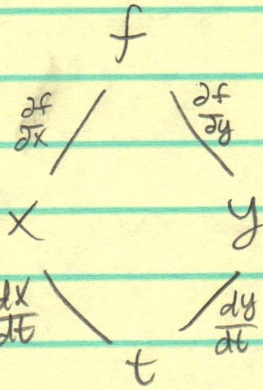
what if x and y depend on a third variable, t?

write: $$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}$$

If we take $\lim_{\Delta t \to 0}$ we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

This is the chain rule for multiple variables. Here's a diagram which may help:



we can use this to make complicated derivatives simple. example:

$$\text{Say } f(x,y) = x^4 y^3$$
$$x = \sin(\pi t)$$
$$y = e^t$$

find $\frac{df}{dt}$ when $t = 1/2$.

then $x = \sin(\pi/2) = 1$

$y = e^{1/2} = \sqrt{e}$

$\frac{\partial f}{\partial x} = 4x^3y^3 = 4(1)^3(\sqrt{e})^3 = 4e^{3/2}$

$\frac{\partial f}{\partial y} = 3x^4y^2 = 3(1)^4(\sqrt{e})^2 = 3e$

$\frac{dx}{dt} = \pi\cos(\pi t) = \pi\cos(\pi/2) = 0$

$\frac{dy}{dt} = e^t = e^{1/2} = \sqrt{e}$

$\frac{df}{dt} = 4e^{3/2}(0) + 3e(\sqrt{e}) = 3e^{3/2}$

ex a closed box with a square base is getting narrower and taller. The width is decreasing at 2 cm/min and the height is increasing at 3 cm/min. How fast is the surface area changing when the box is 8 cm wide and 10 cm tall?

ex Remember the ideal gas law: $PV = nRT$. If the pressure is decreasing at 0.8 Pa/min and the volume is increasing at 1.3 L/min, how fast is the temperature changing?

Extrema in Two Variables

Look at the model of $z = 4x e^{-x^2-y^2}$

It obviously has one maximum and one minimum. Where are they? Before we found critical points by setting the derivative to zero. Here we have 2 derivatives.

<u>Definition</u> (a,b) is a critical point of $f(x,y)$ if $f_x(a,b)=0$ <u>and</u> $f_y(a,b)=0$.

<u>example</u>

$$f(x,y) = 4x e^{-x^2-y^2}$$

$$f_x(x,y) = 4e^{-x^2-y^2} + 4x(-2x)e^{-x^2-y^2}$$

$$= 4e^{-x^2-y^2}(1-2x^2)$$

$$f_y(x,y) = 4x(-2y)e^{-x^2-y^2}$$

$$= -8xy\, e^{-x^2-y^2}$$

so the critical points are where

$$1-2x^2=0 \quad \text{AND} \quad xy=0$$

$$\tfrac{1}{2}=x^2 \quad \text{AND} \quad (x=0 \text{ or } y=0)$$

$$x=\pm\sqrt{1/2} \quad \text{AND} \quad (x=0 \text{ or } y=0)$$

($x=0$ impossible)

critical points are at $(\sqrt{1/2},0)$ and $(-\sqrt{1/2},0)$

Next question: how do we classify the critical points?

Are they maxima? minima? neither?

Answer: use the second derivative test!

The second derivative test for 2 variables

Say you want to check whether a critical point (a,b) is a maximum, minimum, or neither. Then find the discriminant:

$$D = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

Then if $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a minimum

$D > 0$ $\quad f_{xx}(a,b) < 0$, maximum

$D < 0$ Then (a,b) is a saddle point

$D = 0$ the test fails

Example use the second derivative test on $f(x,y) = x^2 - y^2$.

First find the critical points:

$$f_x(x,y) = 2x = 0 \implies x = 0$$

$$f_y(x,y) = -2y = 0 \implies y = 0$$

the only critical point is at $(0,0)$

Now find the discriminant:

$$f_{xx}(x,y) = 2$$

$$f_{yy}(x,y) = -2$$

$$f_{xy}(x,y) = 0$$

$$D = (2)(-2) - (0)^2$$

$$= -4$$

$D < 0$ so $(0,0)$ is a saddle point.

Example use the second derivative test on the "monkey saddle" $f(x,y) = x^3 - 3xy^2$

First find critical points: $f_x(x,y) = 3x^2 - 3y^2 = 0$

$$x^2 = y^2$$
$$x = \pm y$$

$$f_y(x,y) = -6xy = 0 \Rightarrow x=0 \text{ or } y=0$$

$(x=0 \text{ or } y=0)$ AND $x = \pm y$

$\Rightarrow$ $(0,0)$ is the only critical point

Now find the discriminant: $f_{xx}(x,y) = 6x$

$$f_{yy}(x,y) = -6x$$
$$f_{xy}(x,y) = -6y$$

$$D = f_{xx}(0,0)\, f_{yy}(0,0) - (f_{xy}(0,0))^2$$
$$= (0)(0) - (0)^2$$
$$= 0$$

The test fails! (this happens some times)

example Find and classify the critical points of

$f(x,y) = (\frac{1}{2}y^2 - 1)(\frac{1}{3}x^3 - x)$

First find critical points:

$$f_x(x,y) = (\tfrac{1}{2}y^2 - 1)(x^2 - 1) = 0$$

$\frac{1}{2}y^2 - 1 = 0$ OR $x^2 - 1 = 0$

$y = \pm\sqrt{2}$ OR $x = \pm 1$

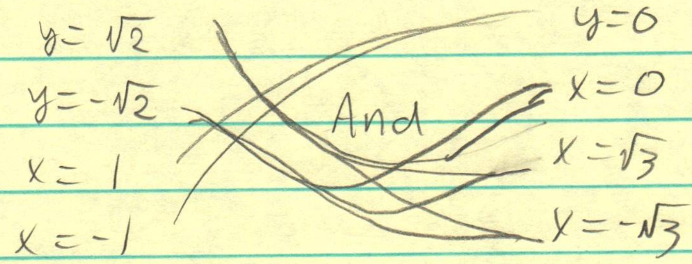
$$f_y(x,y) = y\,(\tfrac{1}{3}x^3 - x) = 0$$

$y = 0$ OR $\frac{1}{3}x^3 - x = 0$

$y = 0$ OR $x(\frac{1}{3}x^2 - 1) = 0$

$y = 0$ OR $x = 0$ OR $x = \pm\sqrt{3}$

organize the possibilities



Critical points are:

$(1,0)$, $(-1,0)$, $(0,\sqrt{2})$, $(0,-\sqrt{2})$

$(\sqrt{3},\sqrt{2})$, $(\sqrt{3},-\sqrt{2})$, $(-\sqrt{3},\sqrt{2})$, $(-\sqrt{3},-\sqrt{2})$

More compactly: $(\pm 1, 0)$, $(0, \pm\sqrt{2})$, $(\sqrt{3}, \pm\sqrt{2})$, $(-\sqrt{3}, \pm\sqrt{2})$

Now we can find the discriminant:

$$f_{xx}(x,y) = (\tfrac{1}{2}y^2-1)(2x)$$

$$f_{yy}(x,y) = (\tfrac{1}{3}x^3-x)$$

$$f_{xy}(x,y) = y(x^2-1)$$

$$D = (\tfrac{1}{2}y^2-1)(2x)(\tfrac{1}{3}x^3-x) - (y(x^2-1))^2$$

$$D(1,0) = (-1)(2)(\tfrac{1}{3}-1) - 0$$

$$= (-1)(2)(-\tfrac{2}{3})$$

$$= 4/3 > 0 \Rightarrow (1,0) \text{ is a max or min}$$

$$f_{xx}(1,0) = (-1)(2) < 0 \Rightarrow (1,0) \text{ is a maximum}$$

$$D(-1,0) = (-1)(-2)(-\tfrac{1}{3}+1) - 0$$

$$= 4/3 > 0 \Rightarrow (-1,0) \text{ is a max or a min}$$

$$f_{xx}(-1,0) = (-1)(-2) > 0 \Rightarrow (-1,0) \text{ is a minimum}$$

Optimization Problems from Life

example a certain shop buys cheese wholesale at \$2/pound, in swiss and pepperjack. Market research shows if the price of swiss is set at x \$/lb and the price of pepperjack is set at y \$/lb, then people will buy $20 - x - 2y$ pounds of swiss and $1 + 4x - 8y$ pounds of pepper jack. How should the manager price the cheese to maximize profit?

$$\text{profit} = \text{revenue} - \text{cost}$$

$$\text{cost} = 2(20-x-2y) + 2(1+4x-8y)$$

$$\text{revenue} = x(20-x-2y) + y(1+4x-8y)$$

$$\text{profit} = (x-2)(20-x-2y) + (y-2)(1+4x-8y)$$

$$P_x = (20-x-2y) - (x-2) + 4(y-2)$$

$$= 20 - x - 2y - x + 2 + 4y - 8$$

$$= 14 - 2x + 2y = 0$$

$$P_y = -2(x-2) + (1+4x-8y) - 8(y-2)$$

$$= -2x + 4 + 1 + 4x - 8y - 8y + 16$$

$$= 21 + 2x - 16y = 0$$

$$35 - 14y = 0$$

$$y = \frac{35}{14} = 2.5$$

$$x = 7 + y = 9.5$$

ex a biologist is preparing a bacteria culture. The bacteria need a medium with 10% salinity to grow. The percentage salinity is given by $S = 0.01\, x^2y^2z$, where x, y, and z are the amounts in liters of 3 nutrients which are combined to make the medium. The costs of the solutions are 5, 1, and 2 $/L respectively. What is the minimum possible cost of the medium?

$C = 5x + y + 2z$

$10 = 0.01\, x^2y^2z$

$1000 = x^2y^2z$

$z = \frac{1000}{x^2y^2}$

$C = 5x + y + \frac{2000}{x^2y^2}$

$C_x = 5 - 2\frac{2000}{x^3y^2} = 0$

$C_y = 1 - 2\frac{2000}{x^2y^3} = 0$

$5x^3y^2 = 4000$

$y^2 = \frac{800}{x^3}$

$y = \sqrt[2]{800/x^3}$

$x^2y^3 = 4000$

$x^2\left(\sqrt{800/x^3}\right)^3 = 4000$

$x^2\left(\frac{800}{x^3}\right)^{3/2} = 4000$

$5x^3y^2 = x^2y^3$

$5x = y$

$x^2(5x)^3 = 4000$

$125x^5 = 4000$

$x^5 = 32$

$x = 2$

$y = 5x = 10$

$z = \frac{1000}{2^2\, 10^2} = \frac{10}{4} = 2.5$

$C = 5(2) + 10 + 2(2.5)$

$= 10 + 10 + 5$

$= 25$

ex A new product costs \$40 to manufacture and sells for \$60. If x thousand dollars are spent on development and y thousand dollars on promotion, customers will buy approximately

$$\frac{450y}{y+1} + \frac{200x}{x+1}$$

units. How much should the manufacturer spend on development and on promotion?

$$\text{profit} = P = 20\left(\frac{450y}{y+1} + \frac{200x}{x+1}\right) - 1000(x+y)$$

$$P_x = 20\,\frac{200(x+1) - 200x}{(x+1)^2} - 1000 = 0$$

$$4000 = 1000(x+1)^2$$

$$4 = (x+1)^2$$

$$2 = x+1$$

$$1 = x$$

$$P_y = 20\,\frac{450(y+1) - 450y}{(y+1)^2} - 1000 = 0$$

$$9000 = 1000(y+1)^2$$

$$3 = y+1$$

$$2 = y$$

\$1000 on development,
\$2000 on promotion.

Lesson 25

Lagrange Multipliers

<u>example</u> minimize $f(x,y) = x^2 + y^2$

subject to $2x + 8y = 6$.

The old way: solve the constraint for one variable, substitute, and minimize:

$$y = \frac{6-2x}{8}$$

$$f(x) = x^2 + \left(\frac{6-2x}{8}\right)^2$$

Then differentiate.

The new way: write $g(x,y) = 2x + 8y = 6 = c$

Find $f_x = 2x \qquad g_x = 2$

$f_y = 2y \qquad g_y = 8$

Solve Lagrange Equations:

$f_x = \lambda g_x \quad , \quad f_y = \lambda g_y \quad , \quad g = c$

$2x = \lambda 2 \qquad 2y = \lambda 4 \qquad 2x + 8y = 6$

$x = \lambda \qquad y = 2x \qquad 2x + 16x = 6$

$x = \frac{6}{18} = \frac{2}{9} \qquad y = \frac{4}{9}$

~~$\frac{4}{9} + 8y = 6$~~

~~$y = \frac{6 - \frac{4}{9}}{8}$~~

<u>example</u> maximize $15x^{3/2}y$ subject to $x = 55 - y$

$f(x,y) = 15x^{3/2}y \qquad g(x,y) = x + y = 55 = c$

$f_x = \frac{45}{2}x^{\frac{1}{2}}y \qquad g_x = 1$

$f_y = 15x^{3/2} \qquad g_y = 1$

$\frac{45}{2}x^{\frac{1}{2}}y = \lambda \qquad 15x^{3/2} = \lambda$

$\frac{45}{2}x^{\frac{1}{2}}y = 15x^{3/2} \Rightarrow \frac{3}{2}x^{\frac{1}{2}}y = x^{3/2} \Rightarrow \frac{3}{2}y = x$

$\frac{3}{2}y + y = 55$

$\frac{5}{2}y = 55 \quad y = 22$

$x = 33$

example maximize $\ln(3xy^2)$ subject to $8x^2+7y^2=4$

$f(x,y) = \ln(3xy^2)$

$= \ln(3)+\ln(x)+2\ln(y)$

$g(x,y)=8x^2+7y^2=4=C$

$f_x = \frac{1}{x}$ $\quad g_x = 16x$

$f_y = \frac{2}{y}$ $\quad g_y = 14y$

$\frac{1}{x}=16\lambda x$ $\quad \frac{2}{y}=14\lambda y$

$\lambda = \frac{1}{16x^2}$ $\quad \frac{2}{y}=\frac{14}{16x^2}y$

$16x^2 = 7y^2$

$8x^2+16x^2=4$

$24x^2=4$

$x^2=\frac{1}{6}$

$x=\pm\sqrt{\frac{1}{6}}$

$y=\pm\sqrt{\frac{16\frac{1}{6}}{7}}$

$=\pm\sqrt{\frac{16}{42}}=\pm\sqrt{\frac{8}{21}}$

$f(x,y)=\ln\left(3\sqrt{\frac{1}{6}}\ \frac{8}{21}\right) = -0.7623$

example find the minimum and maximum values of $e^{-xy/5}$ subject to $5x^2+19y^2 \le 13$

Idea: find interior critical points, then the extrema on the boundary.

example Find the minimum value of $f(x,y)=x^2e^{y^2}$ subject to $5y^2+2x=18$

Applied Optimization with Constraints

lesson 26

A certain cat likes to sit at the top of a round tower and watch people pass by. The tower has a radius of four feet. The temperature at a point x feet to the east of the center and y feet to the north of the center is given by $f(x,y) = y^2 - x^2 + 70$ degrees farenheight. what is the warmest the cat can get, given that it prefers to sit on the ledge?

Objective: $y^2 - x^2 + 70$ constraint: $x^2 + y^2 = 4^2$

$f_x = -2x$ $g_x = 2x$

$f_y = 2y$ $g_y = 2y$

System:
$$-2x = \lambda 2x$$
$$2y = \lambda 2y$$
$$x^2 + y^2 = 16$$

WRONG WAY: since $-2x = \lambda 2x$, λ must be -1

since $2y = \lambda 2y$, λ must be 1

hence, the problem is broken.

Better way: since $-2x = \lambda 2x$, $-2xy = \lambda 2xy$

since $2y = \lambda 2y$, $2xy = \lambda 2xy$

hence,
$$-2xy = 2xy$$
$$0 = 4xy$$
$$0 = xy$$

Therefore $x = 0$ and $y = 0$

what? but $x^2 + y^2 = 16$! The problem is broken!

no, we should have said $x = 0$ OR $y = 0$

if $x = 0$ then $y = \pm 4$, if $y = 0$ then $x = \pm 4$.

My final exam is in 24 hours, and I haven't studied. If I take it now, I will get 100/500 points, a failing grade. If I read my notes for x hours, I can improve my grade by $x(46-x)$ points. If I work practice problems for y hours, I can improve my grade by $y(50-y)$ points. However, I will lose $(x+y)^2$ points from fatigue. How much should I read, how much should I practice?

---

objective: $f(x,y) = 100 + x(46-x) + y(50-y) - (x+y)^2$

$= 100 + 46x - x^2 + 50y - y^2 - (x^2+2xy+y^2)$

$= 100 + 46x - 2x^2 + 50y - 2y^2 - 2xy$

constraint: $x+y \leq 24$

First, try Lagrange multipliers: assume $x+y=24$

$f_x = 46 - 4x - 2y \qquad g_x = 1$

$f_y = 50 - 2x - 4y \qquad g_y = 1$

$46-4x-2y = \lambda$

$50-2x-4y = \lambda$

$46-4x-2y = 50-2x-4y$

$2y-2x = 4$

$y-x = 2$

$y+x = 24$

$2y = 26$

$y = 13$

$x = 11$

$f(x,y) = 390$

$\frac{390}{500} = 72\%$, a low C.

Now try to find critical points:

$$46 - 4x - 2y = 0$$
$$50 - 2x - 4y = 0$$
$$25 - x - 2y = 0$$
$$21 - 3x = 0$$
$$x = 7$$
$$y = \frac{25-7}{2} = 9$$
$$f(7,9) = 486 \qquad \frac{486}{500} = 97.2\%, \text{ an A+}$$

sleep the remaining 8 hours.

Find the minimum cost of producing 50000 units of a product, where x is the number of units of labor at \$45/unit & y is the number of units of capital expended at \$45/unit. The famous Cobb-Douglas function says the production is given by $P(x,y) = 100x^{0.6}y^{0.4}$.

ans: $x = 588$, $y = 392$, $C = 980 \times 45 = 44{,}100$

Double integrals

We like to do integrals to solve problems. Sometimes, what we want to integrate is itself an integral. Ex:

$$\int_1^3 \left(\int_0^2 x^2\, dx \right) dy = \int_1^3 \int_0^2 x^2\, dx\, dy$$

This is called a double integral. We work them from the inside to the outside.

$$\int_1^3 \int_0^2 x^2\, dx\, dy = \int_1^3 \frac{x^3}{3}\Big|_{x=0}^{2}\, dy$$

$$= \int_1^3 \frac{4}{3}\, dy$$

$$= \frac{4}{3} y \Big|_1^3$$

$$= 8/3$$

Often the inside integral will depend on the variable from the outside integral:

$$\int_0^3 \int_1^4 2x + y\, dy\, dx$$

$$= \int_0^3 2xy + \frac{y^2}{2}\Big|_1^4\, dx$$

$$= \int_0^3 8x + 8 - 2x - \tfrac{1}{2}\, dx$$

$$= \int_0^3 6x + 7.5\, dx$$

$$= 3x^2 + 7.5x\Big|_0^3 = 27 + 22.5 = 49.5$$

There are lots of tricks for simplifying double integrals. we can use the constant multiple rule:

$$\int_1^2 \int_0^{\sqrt{5}} 4x^2 y^3 \, dy \, dx$$

$$= 4 \int_1^2 \int_0^{\sqrt{5}} x^2 y^3 \, dy \, dx$$

$$= 4 \int_1^2 x^2 \left(\int_0^{\sqrt{5}} y^3 \, dy \right) dx$$

$$= 4 \int_0^{\sqrt{5}} y^3 \, dy \int_1^2 x^2 \, dx$$

$$= 4 \left. \frac{y^4}{4} \right|_0^{\sqrt{5}} \left. \frac{x^3}{3} \right|_1^2$$

$$= \left(\sqrt{5}^4 \right) \left(\frac{8}{3} - \frac{1}{3} \right)$$

$$= 25 \cdot 7/3 = 175/3$$

We can use this rule to solve double integrals very fast, if they can be split up:

$$\int_0^1 \int_0^{\pi/2} 6y \cos(x) \, dx \, dy$$

$$= 6 \int_0^1 y \, dy \int_0^{\pi/2} \cos(x) \, dx$$

$$= 6 \left(\left. \tfrac{1}{2} y^2 \right|_0^1 \right) \left(\left. \sin(x) \right|_0^{\pi/2} \right)$$

$$= 6 \left(\tfrac{1}{2} \right) (1) = 3$$

we can't always use that trick:

$$\int_{-3}^{3}\int_{0}^{4} (2x+4y)\, dy\, dx$$

we have other tricks, like the sum rule:

$$\int_{-3}^{3}\int_{0}^{4} 2x\, dy\, dx + \int_{-3}^{3}\int_{0}^{4} 4y\, dy\, dx$$

$$= 2\int_{0}^{4} dy \int_{-3}^{3} x\, dx + 4\int_{-3}^{3} dx \int_{0}^{4} y\, dy$$

$$= 2 \cdot 4 \cdot \left(\frac{x^2}{2}\Big|_{-3}^{3}\right) + 4\,(6)\, \frac{y^2}{2}\Big|_{0}^{4}$$

$$= 0 + 4 \cdot 6 \cdot \tfrac{1}{2} \cdot 16$$

$$= 12 \cdot 16 = 192$$

often, the bounds of the inside integral has a variable of the outside integral.

$$\int_{0}^{3}\int_{0}^{x^2} 3yx\, dy\, dx$$

$$= 3\int_{0}^{3} x \int_{0}^{x^2} y\, dy\, dx$$

$$= 3\int_{0}^{3} x \left(\frac{y^2}{2}\Big|_{0}^{x^2}\right) dx$$

$$= \frac{3}{2}\int_{0}^{3} x\, (x^2)^2\, dx$$

$$= \frac{3}{2}\int_{0}^{3} x^5\, dx = \frac{3}{2}\,\frac{1}{6}\, x^6\Big|_{0}^{3} = \frac{1}{4}\, 3^6 = \frac{729}{4}$$

examples

$$\int_{3\pi}^{4\pi} \int_{y}^{\frac{\pi}{2}} -12 \sec(y) \sin(x) \, dx \, dy$$

$$\int_{1}^{e^2} \int_{0}^{4\ln(x)} 3x \, dy \, dx$$

$$\int_{0}^{\sqrt{\pi/4}} \int_{x^2}^{\pi/2} 4x \sin y \, dy \, dx$$

$$\int_{0}^{\pi} \int_{0}^{1} 13y^8 \cos(x) \, dy \, dx$$

Double Integrals as Volumes

Consider my loaf of bread. It measures 4" x 11" at its base. It is well-modeled as filling the region below $z = 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}$ and above the rectangle $0 \le x \le 4$, $0 \le y \le 11$.

What is its volume?

I can find it by getting the volume of each slice, then adding them up.

Find the area of one slice, at a given y-value:

$$A(y) = \int_0^4 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}\, dx$$

Add up the areas, if each slice has a width "dy":

$$V = \int_0^{11} A(y)\, dy$$

$$= \int_0^{11}\int_0^4 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}\, dx\, dy \approx 146.5$$

It's a double integral!

$11 \times 4 \times 4 = 176$

The general rule is, the volume above the region R and below $z = f(x,y)$ is given by

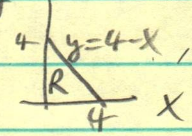
$$\iint_R f(x,y)\, dA$$

This means "figure out the limits yourself" (pointing to R)

This means "dy dx or dx dy" (pointing to dA)

What if I cut the loaf along the plane $y = 4 - x$? What is the volume of the triangular part of the loaf?

here R is the triangle:



so the volume is

$$\iint_R 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}\, dA$$

How do we figure out the limits? first look at our slice of y:

$$A(y) = \int_0^{4-y} 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}\, dx$$

Then add up the slices:

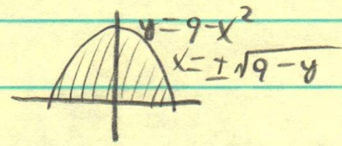
$$V = \int_0^4 \int_0^{4-y} 4 - \frac{(y-5.5)^2}{30} - \frac{(x-2)^2}{4}\, dx\, dy$$

<u>Example</u>

Find the volume below $z = 7+3y$ above the region $-3 \le x \le 3$, $0 \le y \le 9-x^2$.

we want $\iint_R 7+3y\, dA$

where R is



There are 2 ways to find it: slice in x or in y.

$$A(x) = \int_0^{9-x^2} 7+3y\, dy$$

$$V = \int_{-3}^{3} A(x)\, dx$$

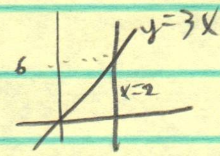
$$= \int_{-3}^{3}\int_0^{9-x^2} 7+3y\, dy\, dx$$

$$A(y) = \int_{-\sqrt{9-y}}^{\sqrt{9-y}} 7+3y\, dx$$

$$V = \int_0^9 \int_{-\sqrt{9-y}}^{\sqrt{9-y}} 7+3y\, dx\, dy$$

sometimes one way is much easier. eg:

$$\iint_R \frac{1}{x^2+6}\, dA$$ where R is bdd by $y=3x$, x-axis, $x=2$



we could y on the outside:

$$\int_0^6 \int_{y/3}^2 \frac{1}{x^2+6}\, dx\, dy$$

(This way needs trig substitution)

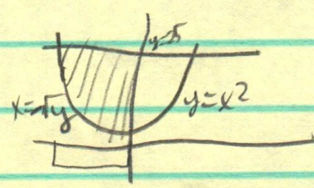
or x on the outside:

$$\int_0^2 \int_0^{3x} \frac{1}{x^2+6}\, dy\, dx$$

(this way does not)

Therefore, it makes sense to change the order of integration sometimes.

eg switch the order of $\int_{-5}^0 \int_{x^2}^{25} f(x,y)\, dy\, dx$



$$\int_0^{25} \int_{-\sqrt{y}}^0 f(x,y)\, dx\, dy$$

example: Evaluate $\int_0^4 \int_{x^2}^{16} -4x\sqrt{1+y^2}\, dy\, dx$

$$= -4\int_0^4 x \int_{x^2}^{16} \sqrt{1+y^2}\, dy\, dx$$

Now we are stuck!

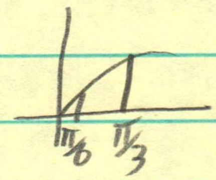
Try changing the order.

$$\int_0^{16} \int_0^{\sqrt{y}} -4x\sqrt{1+y^2}\, dx\, dy$$

extra time problem:

G bdd by x-axis, $y = 5\sin x$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{3}$

find $\iint_G 7\sec^2 x \, dy \, dx$



$$\int_{\pi/6}^{\pi/3} \int_0^{5\sin x} 7\sec^2 x \, dy \, dx$$

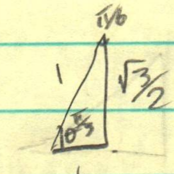
$$7 \int_{\pi/6}^{\pi/3} \sec^2 x \int_0^{5\sin x} dy \, dx$$

$$7 \cdot 5 \int_{\pi/6}^{\pi/3} \sec^2 x \sin x \, dx$$

$$7 \cdot 5 \int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos^2 x} \, dx$$

$u = \cos x$

$du = -\sin x \, dx$



$\sec = \frac{1}{\cos} = \frac{\text{hyp}}{\text{adj}}$

$$7 \cdot 5 \int_{\pi/6}^{\pi/3} \sec x \tan x \, dx$$

$$7 \cdot 5 \sec x \Big|_{\pi/6}^{\pi/3}$$

$$7 \cdot 5 \left(\sec \frac{\pi}{3} - \sec \frac{\pi}{6} \right)$$

$$7 \cdot 5 \cdot \left(2 - \frac{2}{\sqrt{3}} \right)$$

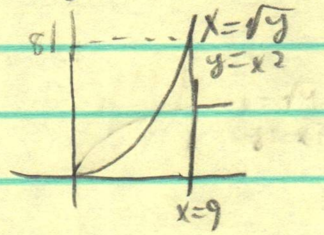
$$70 \left(1 - \frac{1}{\sqrt{3}} \right) = 29.586$$

Advanced Double Integrals and Average Value (lesson 29)

ex compute

$$\int_0^{81}\int_{\sqrt{y}}^{9} 9\sqrt{x^3+1}\,dx\,dy$$

- The inside integral does not match the pattern for any integration rule we have.
- Try changing the order:



$$\int_0^{9}\int_0^{x^2} 9\sqrt{x^3+1}\,dy\,dx$$

$$= 9\int_0^9 \sqrt{x^3+1}\; y\Big|_0^{x^2}\,dx$$

$$= 9\int_0^9 x^2\sqrt{x^3+1}\,dx$$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$

$9^3+1 = 730$

$$= 3\int_1^{730}\sqrt{u}\,du$$

$$= 3\left(\tfrac{2}{3}\right)u^{\frac{3}{2}}\Big|_1^{730}$$

$$= 2\left(730^{3/2}-1\right) \approx 39445.03$$

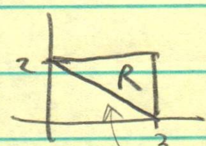
New Idea

The average value of $f(x,y)$ over a region R is

$$\frac{1}{\text{area of } R} \iint_R f(x,y)\, dA$$

example

Find the average value of x^2 over the triangle with vertices $(2,0)$, $(2,3)$, and $(0,3)$.



$y = -\frac{2}{3}x + 2$

$x = -\frac{3}{2}y + 3$

$$\int_0^3 \int_{2-\frac{2}{3}x}^{2} x^2 \, dy \, dx$$

$$= \int_0^3 x^2 \int_{2-\frac{2}{3}x}^{2} 1 \, dy \, dx$$

$$= \int_0^3 x^2 \left(2 - \left(2 - \frac{2}{3}x\right)\right) dx$$

$$= \int_0^3 x^2 \left(\frac{2}{3}x\right) dx$$

$$= \frac{2}{3} \int_0^3 x^3 \, dx$$

$$= \frac{2}{3} \left. \frac{x^4}{4} \right|_0^3$$

$$= \frac{2}{3} \cdot \frac{81}{4} = \frac{27}{2} = 13.5$$

area $= \frac{1}{2}bh = 3$

avg value $= \frac{13.5}{3} = \frac{9}{2} = 4.5$

ex Find the average value of $e^y\sqrt{x+e^y}$
on the rectangle with vertices (0,0), (8,0), (8,4), and (0,4)

ex evaluate $\iint -8x^2\,dA$ over the region in the first quadrant bounded by the hyperbola $xy=36$ and the lines $y=x$, $y=0$, and $x=12$

Intro Linear Algebra

Lesson 30

Let's solve the system of equations

$$\begin{cases} 3x - 6y = 9 \\ -2x + 9y = 4 \end{cases}$$ (Row 1) (Row 2)

using substitution: solve R_1 for y:

$$3x - 9 = 6y$$

$$\frac{3x-9}{6} = y = \frac{1}{2}x - \frac{3}{2}$$

Substitute into R_2:

$$-2x + 9\left(\frac{3x-9}{6}\right) = 4$$

$$-2x + \frac{9}{2}x - \frac{27}{2} = 4$$ } more work

$$x = 7$$ → put this in Row 1 $y = 2$

A better way, use elimination:

$$3x - 6y = 9$$
$$-2x + 9y = 4$$

Replace Row 1 with (Row 1)/3:

$$x - 2y = 9$$
$$-2x + 9y = 4$$

add 2(Row 1) to Row 2:

$$x - 2y = 3$$
$$0 + 5y = 10$$

divide Row 2 by 2:

$$x - 2y = 3$$
$$y = 2$$

now substitute $y=2$ into row 1, $x - 4 = 3$
$x = 7$

Much faster, less work!

An even better way, use a matrix to do elimination.

$$\left[\begin{array}{cc|c} 3 & -6 & 9 \\ -2 & 9 & 4 \end{array}\right]$$

This is an "augmented" matrix

Now proceed in the same way:

$\frac{1}{3}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ -2 & 9 & 4 \end{array}\right]$

$2R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 5 & 10 \end{array}\right]$

$\frac{1}{5}R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 2 \end{array}\right]$

This gets you there with fewer distractions.

We will use matrices from now on.

ex

solve the system
$$x + 7y + z = -11$$
$$y - z = -4$$
$$z = 2$$

This one is easiest to solve by substitution:

$z = 2$ so R_2 gives us $y = -4 + z = -2$

$x = -11 - 7y - z = -11 + 14 - 2 = 1$

The solution is $(1, -2, 2)$.

As a matrix, the system is
$$\left[\begin{array}{ccc|c} 1 & 7 & 1 & -11 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

This is called "echelon form" which is french for "staircase form". It has 1s on the diagonal, with 0s beneath the 1s.

Big Idea: To solve a system of linear equations, use elimination to put it in Echelon form, then use substitution to find the answer. This is called "Gaussian Elimination" and it is the fastest way anybody on the planet knows how to solve a system of equations.

ex. solve

$$7y + 5z = 7$$
$$2y + z = 2$$
$$x + 3y - z = 5$$

$$\left[\begin{array}{ccc|c} 0 & 7 & 5 & 7 \\ 0 & 2 & 1 & 2 \\ 1 & 3 & -1 & 5 \end{array}\right]$$

$R_1 \to R_3$
$R_3 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 2 & 1 & 2 \\ 0 & 7 & 5 & 7 \end{array}\right]$$

$\frac{1}{2}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 7 & 5 & 7 \end{array}\right]$$

$-7R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 3/2 & 0 \end{array}\right]$

$5 - \frac{7}{2}$

$\frac{10}{2} - \frac{7}{2} = \frac{3}{2}$

$\frac{2}{3}R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$R_3: \; z = 0$

$R_2: \; y + \frac{1}{2}z = 1$

$y = 1$

$R_1: \; x + 3y - z = 5$

$x + 3 = 5$

$x = 2$

soln is $(2, 1, 0)$.

We used 3 types of row operations:

- swap two rows
- multiply a row by a nonzero constant
- add a multiple of 1 row to another

These are the only types you need.

ex. compare the sol^ns of

$x+y=3$
$x-y=1$

$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right]$

$R_2 - R_1 \to R_2$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right]$

$-\frac{1}{2}R_2 \to R2$ $\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right]$

$y=1$

$x = 3-1$
$= 2$

$(2,1)$

"consistent Independent"

$2x+y=5$
$4x+2y=10$

$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 4 & 2 & 10 \end{array}\right]$

$\frac{1}{2}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 4 & 2 & 10 \end{array}\right]$

$-4R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 \end{array}\right]$

$x + \frac{1}{2}y = \frac{5}{2}$
$x = \frac{5}{2} - \frac{1}{2}y$

has soln $y = t$
$x = \frac{5}{2} - \frac{1}{2}t$

$\left(\frac{5}{2} - \frac{1}{2}t, t\right)$

"consistent dependent"

$x-y=3$
$3x-3y=4$

$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 3 & -3 & 4 \end{array}\right]$

$-3R_1 + R_2 \to R_2$

$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 0 & -5 \end{array}\right]$

This means

$x-y=3$
and
$0=-5$

"inconsistent"

ex solve

$$2x + 3y + 3z = 8$$
$$x + 2y + z = 4$$
$$x + 3y = 4$$

as a matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 8 \\ 1 & 2 & 1 & 4 \\ 1 & 3 & 0 & 4 \end{array}\right]$$

$R_1 \to R_2$, $R_2 \to R_1$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 3 & 3 & 8 \\ 1 & 3 & 0 & 4 \end{array}\right]$$

$R_2 - 2R_1 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 0 & 4 \end{array}\right]$$

$R_3 - R_1 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$-R_2 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 - R_2 \to R_3$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + 2y + z = 4$$
$$y - z = 0$$

set $t = z$
$y = z = t$

$x = 4 - 2y - z$
$= 4 - 3t$

answer is

$(4-3t, t, t)$

consistent, dependent.

a ball is thrown so That its vertical position is $s(t) = \frac{1}{2}at^2 + vt + h$. We measure its position at 3 times: $s(1) = 47$, $s(2) = 75$, $s(3) = 47$. Find a, v, and h.

$$s(1) = \frac{1}{2}a(1)^2 + v(1) + h = 47$$

$$\frac{1}{2}a + v + h = 47$$

$$s(2) = \frac{1}{2}a(2)^2 + v(2) + h = 75$$

$$2a + 2v + h = 75$$

$$s(3) = \frac{1}{2}a(3)^2 + v(3) + h = 47$$

$$\frac{9}{2}a + 3v + h = 47$$

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 1 & 47 \\ 2 & 2 & 1 & 75 \\ \frac{9}{2} & 3 & 1 & 47 \end{array}\right]$$

Gauss-Jordan Elimination and Reduced Echelon Form

Check out this matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

which represents the system

$x=5, \quad y=7, \quad z=-3.$

The solution is right there! no work at all!

This matrix is in <u>reduced echelon form</u>: every row starts with a 1, and each 1 has only zeros above it.

<u>Big idea</u>: get the matrix in reduced echelon form, and we won't even need to do substitution to get the answer.

This is called "Gauss-Jordan Elimination".

It is usually a bit slower than Gaussian elimination, but it has other advantages.

<u>example</u> solve the system with Gauss-Jordan Elimination.

$$4x + 2y = 10$$
$$3x + 7y = 13$$

$$\left[\begin{array}{cc|c} 4 & 2 & 10 \\ 3 & 7 & 13 \end{array}\right]$$

$\frac{1}{4}R_1 \to R_1$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 3 & 7 & 13 \end{array}\right]$

$7 - \frac{3}{2} = \frac{14}{2} - \frac{3}{2} = \frac{11}{2}$

$13 - \frac{15}{2} = \frac{26}{2} - \frac{15}{2} = \frac{11}{2}$

$-3R_1 + R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & \frac{11}{2} & \frac{11}{2} \end{array}\right]$

$\frac{2}{11}R_2 \to R_2$ $\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 1 \end{array}\right]$

$R_1 - \frac{1}{2}R_2 \to R_1$ $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$ $\frac{4}{2} = 2$

sol$^{\underline{n}}$ is $x = 2$, $y = 1$.

The system is consistent & independent

<u>example</u> solve
$$2x + 3y + 3z = 8$$
$$x + 2y + z = 4$$
$$x + 3y = 4$$
using Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|c} 2 & 3 & 3 & 8 \\ 1 & 2 & 1 & 4 \\ 1 & 3 & 0 & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 3 & 3 & 8 \\ 1 & 3 & 0 & 4 \end{array}\right]$

$R_2 - 2R_1 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 1 & 3 & 0 & 4 \end{array}\right]$

$R_3 - R_1 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$-R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$

$R_3 - R_2 \to R_3$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Now we have Echelon form, we see The system is consistent dependent.

$R_1 - 2R_2 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Now we have reduced echelon form.

choose $t = z$. Then: $y - z = 0 \Rightarrow y = z = t$

$x + 3z = 4 \Rightarrow x = 4 - 3z = 4 - 3t$

solution is $(4-3t, t, t)$, for any number t.

Example Solve

$$\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \\ 9x + 10y + 11z = 12 \end{cases} \quad ①$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

$-20+8=-16$

$R_2 - 5R_1 \to R_2$ $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -16 \\ 9 & 10 & 11 & 12 \end{array}\right]$$

$R_3 - 9R_1 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -16 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

$10-18=-8$
$11-27=-16$
$12-36=-24$

$-\frac{1}{4}R_2 \to R_2$ $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

$R_3 + 8R_2 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

$4 \times 8 = 32$
$32-24=6$

$\frac{1}{6}R_3 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

we have Echelon form.
system is inconsistent,
no point in continuing.

Matrix Operations

This is the last topic covered on Exam 5.

First, a word about abstraction.

For thousands of years, people computed formulas by memorizing procedures, e.g. to find the area of a circle, square its radius and then multiply your result by π. Then in the 1600s, some folks got the idea of writing things as functions: $A(r) = \pi r^2$. This made it easier to do simplifications: $f(x) = 3(x+2) - 6 = 3x + 6 - 6 = 3x$ and it let them do more clever things to the functions, like take derivatives.

Up to now we have used matrices only as a tool for solving systems of equations. This is like using functions, but only ever substituting values into them. Now we will transition into looking at matrices themselves.

Matrices come in all sizes:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ -3 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -1 & 7 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$$

here A is a 3×2 matrix, B is a 1×3 matrix, C is a 2×2 matrix. Only C is a square matrix.

<u>Always</u> read row x column.

We can multiply matrices by scalar numbers by multiplying into every term: $3A = \begin{bmatrix} 3 & 6 \\ 12 & 24 \\ -9 & 0 \end{bmatrix}$

Let's say $D = \begin{bmatrix} -1 & 3 \\ -2 & -6 \\ 9 & 1 \end{bmatrix}$. What are the dimensions of D?   Ans: 3x2. NOT 2x3.

If 2 matrices have the same dimensions, we can add them term by term:

$$A + D = \begin{bmatrix} 1 & 2 \\ 4 & 8 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 3 \\ -2 & -6 \\ 9 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 2 & 2 \\ 6 & 1 \end{bmatrix}$$

---

example if $E = \begin{bmatrix} 1/2 & 1 \\ 7 & 1/6 \end{bmatrix}$, find $6E - 2C$.

$$6E = \begin{bmatrix} 3 & 6 \\ 42 & 1 \end{bmatrix} \qquad 2C = \begin{bmatrix} 2 & 14 \\ 0 & 2 \end{bmatrix}$$

$$6E - 2C = \begin{bmatrix} 1 & -8 \\ 42 & -1 \end{bmatrix}$$

---

example what is $A + C$?

Ans: The question is bogus.

New concept: Matrix Multiplication

This is where all these matrix rules become useful.
The definition is a bit odd, though.
First, look at the 2x1 matrix $\begin{bmatrix} x \\ y \end{bmatrix}$.
Multiplication works like this:

$$\begin{bmatrix} 1 & 2 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 6x - y \end{bmatrix}$$

That is, a 2x2 matrix times a 2x1 matrix gives a 2x1 matrix. This lets us write systems of linear equations problems without using augmented matrices.

Now, what if we want

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

we could work it from the right:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3x+0y \\ 0x+y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 3x+2y \\ 0(3x)+y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ y \end{bmatrix}$$

But with the right way of looking at multiplication, we can work it from the left.

BIG IDEA: MATRIX MULTIPLICATION

to find the entry in the i^{th} row and the j^{th} column of AB, combine the i^{th} row of A with the j^{th} column of B.

Example:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3+2\cdot 0 & 1\cdot 0+2\cdot 1 \\ 0\cdot 3+1\cdot 0 & 0\cdot 0+1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

Sure enough, $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ y \end{bmatrix}$

Important in order to multiply 2 matrices AB, The number of columns of A must be the same as the number of rows of B.

Example

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 2\cdot(-3) & 1\cdot 3 + 2\cdot 1 \\ -1\cdot 0 + 1\cdot(-3) & -1\cdot 3 + 1\cdot 1 \\ 0\cdot 0 + 4\cdot(-3) & 0\cdot 3 + 4\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 5 \\ -3 & -2 \\ -12 & 4 \end{bmatrix}$$

Notice if A is $n \times m$ and B is $p \times q$, Then AB is $n \times q$ (BUT ONLY IF $m = p$)

example

what is $\begin{bmatrix} 0 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 4 \end{bmatrix}$?

Answer: gibberish. you cant multiply 2×2 by 3×2.

example

find the dimensions of

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 7 \\ 5 & 4 & 3 \end{bmatrix}$$

It must be 1×3

$$\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$$

Notice: we can find A^2 by taking AA. But only if A is square! otherwise A^2 is gibberish.

Important In general, $AB \neq BA$

example

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 7 \\ 5 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$$

but $\begin{bmatrix} 6 & 2 & 7 \\ 5 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix}$ is nonsense.

example

say $A = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}$

Find AB and find BA.

$$AB = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 3 & 2 \end{bmatrix}$$

example say $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Find CD and DC

C has a special name: it is called the 3x3 identity matrix.

example say $E = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $F = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$

find EF

spend the rest of the time multiplying matrices.

Matrix Inverses

We have addition, subtraction, multiplication for matrices. What about division?

Here is the idea of division. $\frac{1}{5}$ is the only number that, if you multiply by 5, you get 1. Since we know this, we can treat division like multiplication: $20 \div 5 = (20)(5^{-1}) = 4$

We want matrix division to work the same way, so we need an idea of "1" as a matrix.

eg compute: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Notice when we multiplied by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we got the same matrix back. Just like multiplying by 1! $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is called the 3x3 identity matrix. In general, the nxn identity matrix has 1s on the diagonal, zeroes everywhere else. We write

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(I is for identity, since it preserves the identity of whatever it multiplies).

The Big IDea: if $AB = I$, Then A is the inverse of B, written as $A = B^{-1}$. Also, $B = A^{-1}$.

example

if $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}$

check whether $A = B^{-1}$.

Ans: if $A = B^{-1}$ then $AB = I$. Let's find AB:

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3/2 & -2 \\ -1/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $A = B^{-1}$. Yay!

How to find A^{-1}, if you know A?

Answer: Gauss-Jordan Elimination.

use row reduction on $[A \mid I]$.

This will give you $[I \mid A^{-1}]$.

Example

Find A^{-1} if $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$R_2 - R_1 \to R_2$ $$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -4 & -1 & 1 \end{array}\right]$$

$-\frac{1}{4}R_2 \to R_2$ $$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/4 & -1/4 \end{array}\right]$$

$R_1 - 3R_2 \to R_1$ $$\left[\begin{array}{cc|cc} 1 & 0 & 1/4 & 3/4 \\ 0 & 1 & 1/4 & -1/4 \end{array}\right]$$

This means $A^{-1} = \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{bmatrix}$.

Check our work:

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why do we need matrix inverses? Let's say you had a simple equation, $3x = 9$. How would you solve it? multiply on both sides by $\frac{1}{3}$:

$$\left(\tfrac{1}{3}\right) 3x = \left(\tfrac{1}{3}\right) 9 \rightarrow 1x = 3 \rightarrow x = 3$$

we can do the same with matrices.

<u>example</u>

use the fact that $\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{bmatrix}$

to solve the system

$$\begin{cases} x + 3y = 12 \\ x - y = 8 \end{cases}.$$

The trick:

$$\begin{bmatrix} x + 3y \\ x - y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

so we really just need to solve

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

for $\begin{bmatrix} x \\ y \end{bmatrix}$. Just multiply by the inverse:

$$\begin{bmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 1/4 & -1/4 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$ $x = 9, y = 1$

example

If $A = \begin{bmatrix} -\frac{1}{11} & \frac{5}{11} & \frac{1}{11} \\ -\frac{8}{11} & \frac{18}{11} & -\frac{3}{11} \\ -\frac{10}{11} & \frac{17}{11} & -\frac{1}{11} \end{bmatrix}$. then $A^{-1} = \begin{bmatrix} 3 & 2 & -3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$.

use this to solve

$$\begin{cases} -\frac{1}{11}x + \frac{5}{11}y + \frac{1}{11}z = 2 \\ -\frac{8}{11}x + \frac{18}{11}y - \frac{3}{11}z = -1 \\ -\frac{10}{11}x + \frac{17}{11}y - \frac{1}{11}z = 1 \end{cases}.$$

solution:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$A^{-1}A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$I\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 13 \end{bmatrix}$$

example If $A = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$, find A^{-1} and use it to solve $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 4 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_3 - 2R_1 \to R_3$ $\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right]$

$\frac{1}{2} R_1 \to R_1$ $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{array}\right]$

$R_2 \leftrightarrow R_3$ $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 1 & 0 \end{array}\right]$

$R_3 - 2R_2 \to R_3$ $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 3 & 5 & 1 & -2 \end{array}\right]$

$\frac{1}{3} R_3 \to R_3$ $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 5/3 & 1/3 & -2/3 \end{array}\right]$

$R_2 + R_3 \to R_2$ $\left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 5/3 & 1/3 & -2/3 \end{array}\right]$

$$R_1 - \tfrac{1}{2}R_2 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 2/3 & -1/6 & -1/6 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 5/3 & 1/3 & -2/3 \end{array}\right]$$

$$\tfrac{2}{3} - \tfrac{5}{6} = \tfrac{4}{6} - \tfrac{5}{6}$$

$$R_1 - \tfrac{1}{2}R_3 \to R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/6 & -1/3 & 1/6 \\ 0 & 1 & 0 & -1/3 & 1/3 & 1/3 \\ 0 & 0 & 1 & 5/3 & 1/3 & -2/3 \end{array}\right]$$

$$-\tfrac{1}{6} - \tfrac{1}{6} = -\tfrac{1}{3}$$

$$-\tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{6}$$

$$A^{-1} = \begin{bmatrix} -1/6 & -1/3 & 1/6 \\ -1/3 & 1/3 & 1/3 \\ 5/3 & 1/3 & -2/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & 2 \\ 10 & 2 & -4 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & 2 \\ 10 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -2 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & 2 \\ 10 & 2 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -6 \\ 0 \\ 30 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Determinants

Example Find the inverse of $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$ uh-oh! there is no way to get the matrix into the form $[I \mid A^{-1}]$.

The matrix is not invertible!

Facts:
- Most matrices have inverses (are invertible).
- Some matrices are not invertible.
- if a matrix does not have an inverse, it is called non-invertible or it is called singular (because it is so special).
- To save work, we have a test invented in ancient China to see if a matrix is invertible.

definition when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant of A, written as det(A) or as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is the number $ad - bc$.
if $\det(A) \neq 0$ than A is invertible, and if $\det A = 0$ then A is singular.

example check if each matrix is singular.

$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 4 & 6 \end{bmatrix}$

$\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2\cdot 2 - 4\cdot 1 = 0$

A is singular

$\begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} = -1\cdot 3 - (-1)2 = -3 + 2 = -1$

B is non-singular

$\begin{vmatrix} 0 & 1 \\ 4 & 6 \end{vmatrix} = 0\cdot 6 - 1\cdot 4 = -4$

C is invertible.

Question: how do we find the determinant of a 3x3 matrix?

Answer: "cofactor expansion"

we can write the determinant in terms of minors.

<u>definition</u> M_{ij} is the <u>minor</u> for the i^{th} row and j^{th} column of a matrix. It is the determinant of everything outside the i^{th} row and j^{th} column.

<u>example</u> say $A = \begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & 3 \\ -1 & -2 & 2 \end{bmatrix}$

then

$$M_{11} = \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = 2+6 = 8$$

$$M_{23} = \begin{vmatrix} 4 & 0 \\ -1 & -2 \end{vmatrix} = -8-0 = -8$$

$$M_{31} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 0-2 = -2$$

<u>definition</u> C_{ij} is the <u>cofactor</u> for the i^{th} row and j^{th} column. $C_{ij} = (-1)^{i+j} M_{ij}$

<u>example</u>

$$C_{11} = (-1)^2 M_{11} = 8$$

$$C_{23} = (-1)^5 M_{23} = 8$$

$$C_{31} = (-1)^4 M_{31} = -2$$

Method for computing determinants

to find a determinant of a 3x3 matrix, pick a row or column which has easy entries. Then, multiply each entry by its cofactor, and add them up.

example find $\det(A)$ if $A = \begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & 3 \\ -1 & -2 & 2 \end{bmatrix}$

choose the second column. then

$$\det(A) = 0(-1)^3 \begin{vmatrix} 3 & 3 \\ -1 & 2 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 4 & 2 \\ -1 & 2 \end{vmatrix} + (-2)(-1)^5 \begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 2 \\ -1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix}$$

$$= (8+2) + 2(12-6)$$

$$= 10 + 12$$

$$= 22$$

example find $\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & -1 \\ 4 & 4 & 3 \end{vmatrix}$

expand along the first column.

$M_{11} = \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 6+4 = 10$

$M_{21} = \begin{vmatrix} 2 & 4 \\ 4 & 3 \end{vmatrix} = 6-16 = -10$

$M_{31} = \begin{vmatrix} 2 & 4 \\ 2 & -1 \end{vmatrix} = -2-8 = -10$

$C_{11} = (-1)^2\, 10 = 10$

$C_{21} = (-1)^3 (-10) = 10$

$C_{31} = (-1)^4 (-10) = -10$

$$\begin{vmatrix} 1 & 2 & 4 \\ 3 & 2 & -1 \\ 4 & 4 & 3 \end{vmatrix} = 1(10) + 3(10) + 4(-10) = 0$$ (the matrix is singular)

example For what values of x is $A = \begin{bmatrix} x-2 & -4 \\ 1 & x+3 \end{bmatrix}$ singular?

Answer: $\det(A) = \begin{vmatrix} x-2 & -4 \\ 1 & x+3 \end{vmatrix} = (x-2)(x+3) - (1)(-4)$

$= x^2 + x - 6 + 4$

$= x^2 + x - 2$

$= (x+2)(x-1)$

so $\det(A) = 0$ when $x = 1$ or $x = -2$.

example solve $0 = \begin{vmatrix} x+7 & 4 & -3 \\ -4 & x-10 & 12 \\ -10 & -4 & x \end{vmatrix} = (x+6)(x-6)(x-3)$

Eigenvalues

Take the matrix $A=\begin{bmatrix}4 & -6\\ 1 & -1\end{bmatrix}$ and multiply by the vectors $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

$$\begin{bmatrix}4 & -6\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix} \quad \begin{bmatrix}4 & -6\\ 1 & -1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \quad \begin{bmatrix}4 & -6\\ 1 & -1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}$$

notice: $\begin{bmatrix}1\\1\end{bmatrix}$ is not special, but $A\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$ and $A\begin{bmatrix}3\\1\end{bmatrix}=2\begin{bmatrix}3\\1\end{bmatrix}$

That makes $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ special because multiplying them by A is the same as multiplying them by a number. $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}3\\1\end{bmatrix}$ are eigenvectors of A. $\begin{bmatrix}2\\1\end{bmatrix}$ is associated to the eigenvalue 1 and $\begin{bmatrix}3\\1\end{bmatrix}$ is associated to the eigenvalue 2.

example which of these vectors are eigenvectors of $B=\begin{bmatrix}15 & 12\\ -16 & -13\end{bmatrix}$?

$$\vec{x}=\begin{bmatrix}1\\-1\end{bmatrix} \quad \vec{y}=\begin{bmatrix}2\\-2\end{bmatrix} \quad \vec{z}=\begin{bmatrix}-3\\4\end{bmatrix} \quad \vec{u}=\begin{bmatrix}1\\0\end{bmatrix} \quad \vec{v}=\begin{bmatrix}1\\1\end{bmatrix}$$

$B\vec{u}=\begin{bmatrix}15\\-16\end{bmatrix}$ (no) $\qquad B\vec{v}=\begin{bmatrix}17\\-29\end{bmatrix}$ (no)

$B\vec{x}=\begin{bmatrix}3\\-3\end{bmatrix}=3\vec{x}$ (yes, eigenvalue 3)

$B\vec{y}=\begin{bmatrix}6\\-6\end{bmatrix}=3\vec{y}$ (yes, eigenvalue 3)

$B\vec{z}=\begin{bmatrix}-3\\4\end{bmatrix}=\begin{bmatrix}3\\-4\end{bmatrix}=-\vec{z}$ (yes, eigenvalue -1)

Question: how do we find the eigenvalues of a matrix?

Answer: we are looking for a value of the variable λ which gives a solution to $A\vec{x}=\lambda\vec{x}$, where $\vec{x}$ is not just the zero vector.

Question: how would we solve $9x=yx$?

answer: move everything to the right: $0=yx-9x$

factor: $0=(y-9)x$

and use the zero multiple rule, $y-9=0$ or $x=0$.

Same Idea

take $A\vec{x}=\lambda\vec{x}$

move to right: $0=\lambda\vec{x}-A\vec{x}$

I want to factor, but A is a matrix and λ is a number, so I can't take $(\lambda-A)\vec{x}$. Instead, we know $I\vec{x}=\vec{x}$ so I can write:

$$0=\lambda I\vec{x}-A\vec{x}$$

$$0=(\lambda I-A)\vec{x}$$

$$(\lambda I-A)\vec{x}=0$$

we know $\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$ solves this. we want another solution! If $(\lambda I-A)$ is invertible, this is not possible.

Big idea: λ is an eigenvalue of A exactly when $\lambda I-A$ is not invertible, when $\lambda I-A$ is singular, when $\det(\lambda I-A)=0$.

example Find the eigenvalues of $A = \begin{bmatrix} 7 & -3 \\ 4 & 0 \end{bmatrix}$

solution set $\det[\lambda I - A) = 0$

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 7 & -3 \\ 4 & 0 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} -7 & 3 \\ -4 & 0 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{bmatrix} \lambda-7 & 3 \\ -4 & \lambda \end{bmatrix} \right) = 0$$

$$(\lambda - 7)\lambda + 12 = 0$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0$$

the eigenvalues of A are 3 and 4

example find the eigenvalues of $\begin{bmatrix} 19 & -30 \\ 12 & -19 \end{bmatrix} = B$

solution $\lambda I - B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 19 & -30 \\ 12 & -19 \end{bmatrix}$

$$= \begin{bmatrix} \lambda-19 & 30 \\ -12 & \lambda+19 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-19 & 30 \\ -12 & \lambda+19 \end{vmatrix} = (\lambda-19)(\lambda+19) + (12)(30) = 0$$

$$\lambda^2 - 19^2 + 360 = 0$$

$$\lambda^2 - 361 + 360 = 0$$

$$\lambda^2 - 1 = 0$$

eigenvalues are 1 and

$$(\lambda+1)(\lambda-1) = 0$$

How do we find eigenvectors?

Once we know λ, we need to find a (nonzero) $\vec{x}$ which solves $A\vec{x} = \lambda\vec{x}$ or equivalently, $(\lambda I - A)\vec{x} = 0$.

We can use Gaussian elimination!

<u>example</u> for $A = \begin{bmatrix} 7 & -3 \\ 4 & 0 \end{bmatrix}$ we found $\lambda = 3$ or $\lambda = 4$.

Find an eigenvector for each eigenvalue.

First choose $\lambda = 3$. Then $\lambda I - A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 7 & -3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ -4 & 3 \end{bmatrix}$

so we need all the solutions to $\begin{bmatrix} -4 & 3 \\ -4 & 3 \end{bmatrix}\vec{x} = 0$

$$\left[\begin{array}{cc|c} -4 & 3 & 0 \\ -4 & 3 & 0 \end{array}\right] \xrightarrow{R_2 - R_1 \to R_2} \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

~~Choose $x_2 = t$ so $-4x_1 + 3t = 0$~~

~~$\frac{3t}{4} = x_1$~~

~~We can choose any t we like~~

if $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$ this means $-4a + 3b = 0$

choose $b = t$, so $-4a + 3t = 0$ $\quad a = \frac{3t}{4}$

So the solution is $\vec{x} = \begin{bmatrix} \frac{3t}{4} \\ t \end{bmatrix}$. We can choose any t we like (not zero), so why not choose $t = 4$?

$\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

check our work: $A\vec{x} = \begin{bmatrix} 7 & -3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 21-12 \\ 12 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

we got it!

3x3 eigenvalues & eigenvectors

Find the eigenvalues & eigenvectors:

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 10 & -6 & -16 \\ -4 & 2 & 6 \end{bmatrix}$$

ans: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, 0 \quad \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, 2 \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, -1$

$$\lambda I - A = \begin{bmatrix} \lambda-1 & 1 & 2 \\ -10 & \lambda+6 & 16 \\ 4 & -2 & \lambda-6 \end{bmatrix}$$

$$\det(\lambda I - A) = (\lambda-1)\begin{vmatrix} \lambda+6 & 16 \\ -2 & \lambda-6 \end{vmatrix} - (1)\begin{vmatrix} -10 & 16 \\ 4 & \lambda-6 \end{vmatrix} + (2)\begin{vmatrix} -10 & \lambda+6 \\ 4 & -2 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda+6)(\lambda-6)-(-2)(16)\right] - \left[(-10)(\lambda-6)-(4)(16)\right] + 2\left(20-4(\lambda+6)\right)$$

$$= (\lambda-1)\left[\lambda^2-36+32\right] - \left[-10\lambda+60-64\right] + 2\left[20-4\lambda-24\right]$$

$$= (\lambda-1)\left[\lambda^2-4\right] - \left[-10\lambda-4\right] + 2\left[-4\lambda-4\right]$$

$$= \lambda^3 - \lambda^2 - 4\lambda + 4 + 10\lambda + 4 - 8\lambda - 8$$

$$= \lambda^3 - \lambda^2 - 2\lambda$$

$$= \lambda(\lambda^2 - \lambda - 2)$$

$$= \lambda(\lambda-2)(\lambda+1)$$

alternate method:

$$\begin{vmatrix} \lambda-1 & 1 & 2 \\ -10 & \lambda+6 & 16 \\ 4 & -2 & \lambda-6 \end{vmatrix} \begin{matrix} \lambda-1 & 1 \\ -10 & \lambda+6 \\ 4 & -2 \end{matrix}$$

$(\lambda-1)(\lambda+6)(\lambda-6) + (1)(16)(4) + (2)(-10)(-2) - (4)(\lambda+6)(2) - (-2)(16)(\lambda-1)$
$- (\lambda-6)(-10)(1)$

$(\lambda-1)(\lambda^2-36) + 64 \quad + 40 \quad - 8\lambda - 48 + 32\lambda - 32 + 10\lambda - 60$

$\lambda^3 - \lambda^2 - 36\lambda + 36 + 104 \quad - 8\lambda - 48 \quad + 32\lambda - 32 + 10\lambda - 60$

$\lambda^3 - \lambda^2 - 2\lambda$

in any case, $\lambda = 0, 2,$ or -1.

now find solutions ~~x~~ to $(\lambda I - A)\vec{x} = \vec{0}$.

choose a λ: $\lambda = 0$. $\lambda I - A = -A = \begin{bmatrix} -1 & 1 & 2 \\ -10 & 6 & 16 \\ 4 & -2 & -6 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ -10 & 6 & 16 & 0 \\ 4 & -2 & -6 & 0 \end{array}\right]$$

$-R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -10 & 6 & 16 & 0 \\ 4 & -2 & -6 & 0 \end{array}\right]$$

$R_2 + 10R_1 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -4 & -4 & 0 \\ 4 & -2 & -6 & 0 \end{array}\right]$$

$R_3 - 4R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$-\frac{1}{4}R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right]$$

$R_3 - 2R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 + R_2 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

now say $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then $a - c = 0$, $b + c = 0$

let $c = t$. Then $a = t$, $b = -t$.

$\vec{x} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$ take $t = 1$, $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

check our work:

$$A\vec{x} = \begin{bmatrix} 1 & -1 & -2 \\ 10 & -6 & -16 \\ -4 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

good ✓

$$B = \begin{bmatrix} 4 & 12 & 10 \\ 6 & 34 & 26 \\ -9 & -48 & -37 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} (-2) \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} (2) \quad \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix} (1)$$