MINIMIZATION PROBLEM WITHOUT CALCULUS

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> "Could you explain how to do this problem without using calculus?
> Thank you"

This week I received this interesting question from a student, in reference to a homework problem. We have just begun our section on minimization problems, and every problem in the section can be solved using either the closed interval method or the open interval method. For many of the problems, calculus is the only way to solve them. For every problem, calculus is the easiest way to solve them. However, a few problems also have solutions which involve no calculus whatsoever. Here we will present a method for solving one of these problems, completely calculus free. We will need a famous result from algebra, which we will now establish.

1. The Algebraic / Geometric Mean Inequality

This inequality states that, for any two positive numbers a and b, the following holds:

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

So for example, if $a = 2$ and $b = 18$, the inequality tells us that $\sqrt{2 \times 18} \leq \frac{2+18}{2}$ or $6 \leq 10$. The proof of this inequality comes from the fact that the square of any number is non-negative. So, we start with the number $a - b$.

$$\begin{aligned} 0 &\leq (a-b)^2 \\ 0 &\leq a^2 - 2ab + b^2 \\ 4ab &\leq a^2 + 2ab + b^2 \\ ab &\leq \frac{(a+b)^2}{4} \\ \sqrt{ab} &\leq \frac{a+b}{2} \end{aligned}$$

The proof was fairly quick, but the result can be used to prove some impressive and non-obvious results.

2. The Problem

> "Find the equation of the line through the point $(3, 5)$ that cuts off
> the least area from the first quadrant."

The straightforward, calculus-based way to solve this problem involves writing the line as $y - 5 = m(x - 3)$ and writing the area bounded by the line as a function of m. Once that is done, the open interval method can be used; since we know the slope is negative, m is restricted to the interval $(-\infty, 0)$. That is the easy way to solve this problem. We will do the problem differently.

We know a lot about the line already. For instance, its slope should be negative. The x-intercept is to the right of 3; that is, it occurs at $(3+s,0)$, where s is some positive number. Also, the y-intercept occurs above 5, so the y-intercept is at $(0,5+t)$ where t is a positive number. No mater what angle we put the line at, it will still have a rectangle below and to the left of it, with one corner at the origin and the opposite corner at $(3,5)$. We can split up the area into three chunks:

- The rectangle, which is fixed, and has an area of $3\times 5=15$.
- A lower-right triangle, with an area of $\frac{1}{2}s\times 5$.
- An upper-left triangle, with an area of $\frac{1}{2}3\times t$.

Then the total area is given by $A=15+\frac{5s+3t}{2}$. The lower-left and upper-right triangles are similar, so $\frac{s}{5}=\frac{3}{t}$, and so $s=\frac{15}{t}$. Substituting into our area formula, we find

$$A=15+\frac{5^2\times 3\frac{1}{t}+3t}{2}.$$

Here is our chance to apply the arithmetic / geometric mean inequality. Take $a=5^2\times 3\frac{1}{t}$ and $b=3t$. Then the inequality tells us

$$\begin{aligned}A&\geq 15+\sqrt{ab}\\&=15+\sqrt{5^2\times 3\frac{1}{t}\times 3t}\\&=15+15\\&=30\end{aligned}$$

This tells us that the smallest area we can possibly have is 30. All we need to do is find a value of t where $A=30$, and we will know we have found the best one. In fact, if we choose $t=5$, then we get the area to be exactly 30. Then the y-intercept of our line must be $(0,10)$ and so since the line also passes through $(3,5)$, we find

$$y=-\frac{5}{3}x+10.$$