THE BANACH MEASURE PROBLEM

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ABSTRACT. A conjecture of Banach states that a bounded (countably additive, signed) measure defined on all subsets of a continuum is absolutely continuous with respect to counting measure. Banach and Kuratowski [1] prove the conjecture on the hypothesis that a continuum has the least cardinality of all uncountable sets. Paul Cohen [2] shows that the continuum hypothesis is not a consequence of the axiom of choice in Zermelo-Fraenkel set theory. A proof of the Banach conjecture is now given without hypothesis other than the axiom of choice in Zermelo-Fraenkel set theory. The argument is motivated by a proof [3] of the Stone-Weierstrass theorem from the Krein-Milman theorem [4].

A bounded measure σ on the subsets of a set S which is absolutely continuous with respect to counting measure has a Radon-Nikodym derivative. The derivative is a real-valued function $\sigma'(\xi)$ of ξ in S such that the sum

$$\sum |\sigma'(\xi)| < \infty$$

taken over the elements ξ of S converges. The value

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$$\sigma(E) = \sum \sigma'(\xi)$$

of the measure on a set E is a sum over the elements ξ of E.

A bounded measure on the subsets of the set Z of nonnegative integers is absolutely continuous with respect to counting measure.

A bounded measure on the subsets of a set S is a bounded real-valued function $\sigma(E)$ of subsets E of S such that for every mapping π of S into Z a bounded measure $\pi\sigma$ on the subsets of Z is defined whose value

$$\pi\sigma(E) = \sigma(\pi^{-1}E)$$

on every subset E of Z is the value of σ on the set $\pi^{-1}E$ of elements ξ of S such that $\pi\xi$ belongs to E.

A continuum is a set which has the same cardinality as the class S of all subsets of Z.

A subset C of Z inherits a well-ordering from Z. The inequality $a < b$ for elements a and b

of C means that the inequality holds for the elements a and b of Z. An n-th element of C is defined for an element n of Z as an element of which there are exactly n smaller elements of C, when such an element exists. An n-th element of C exists if, and only if, C contains more than n elements.

A projection of S onto a subset of S is a mapping π of S onto a subset πS of S which leaves every element of πS fixed.

An equivalence relation is defined on the subsets of Z for every positive integer r. Equivalence of subsets A and B of Z means that, for every element n of Z not greater than r, an n-th element of A exists if, and only if, an n-th element of B exists, and then these elements are equal.

A projection π_r of S onto a countable subset $\pi_r S$ of S is defined by taking a subset of Z into the equivalent subset of Z which contains no more than r elements. A subset of Z which contains no more than r elements is an element of $\pi_r S$.

A Banach space $\mathcal{C}(S)$ of real-valued functions defined on S is defined as the closure in the topology of uniform convergence on S of functions which vanish outside of some finite set. A function $f(\xi)$ of ξ in S with real values belongs to $\mathcal{C}(S)$ if, and only if, for every $\epsilon > 0$ the inequality

$$|f(\xi)| < \epsilon$$

holds outside of some finite set. The norm of an element f of $\mathcal{C}(S)$ is a maximum

$$\|f\| = \max |f(\xi)|$$

taken over the elements ξ of S.

The dual Banach space $\mathcal{C}^*(S)$ is the set of linear functionals on $\mathcal{C}(S)$ which are continuous in the metric topology defined by the norm. The norm

$$\|L\| = \sup |Lf|$$

of an element L of $\mathcal{C}^*(S)$ is a least upper bound taken over the elements f of $\mathcal{C}(S)$ of norm at most one.

The Krein-Milman topology of $\mathcal{C}^*(S)$ is the weak topology of $\mathcal{C}^*(S)$ as the dual space of $\mathcal{C}^*(S)$. An element f of $\mathcal{C}(S)$ defines a linear functional

L into Lf on $\mathcal{C}^*(S)$ which is continuous for the Krein-Milman topology.

A regular convex subset of $\mathcal{C}^*(S)$ is a convex subset which is compact for the Krein-Milman topology. A linear functional on $\mathcal{C}^*(S)$ which is continuous for the Krein-Milman topology maps a regular convex subset of $\mathcal{C}^*(S)$ onto a compact set of real numbers.

A subset of $\mathcal{C}^*(S)$ is said to be bounded if every element of $\mathcal{C}(S)$ defines a linear functional on $\mathcal{C}^*(S)$ which maps the subset into a bounded subset of real numbers.

A regular convex subset of $\mathcal{C}^*(S)$ is bounded. A bounded convex subset of $\mathcal{C}^*(S)$ is regular if it is closed for the Krein-Milman topology.

The Krein-Milman theorem states that a regular convex subset of $\mathcal{C}^*(S)$ is the closed convex span of its extreme points.

An extreme point of a convex set

is an element of the convex set which is not a proper convex combination of distinct elements a and b of the convex set. The convex combination is proper if equality does not hold in either of the inequalities

$0 \leq t \leq 1.$

A computation of extreme points appears in the proof of the Stone-Weierstrass theorem. The regular convex set is the set of elements of $\mathcal{C}^*(S)$ of norm at most one.

An element h of $\mathcal{C}(S)$ defines a continuous linear transformation of $\mathcal{C}^*(S)$ into itself which takes L into hL defined by

$$(hL)f = L(hf)$$

for every element f of $\mathcal{C}(S)$.

An element L of $\mathcal{C}^*(S)$ of norm one is not an extreme point of the convex set if the elements

$$hL \quad \text{and} \quad L - hL$$

of the convex set are nonzero for some element h of $\mathcal{C}(S)$ whose values are taken in the interval $[0,1]$. For then a proper convex combination

$$L = (1-t)L_+ + tL_-$$

of elements L_+ and L_- of the convex set of norm one defined by

$$(1-b)L_+ = L - hL$$

and

$$b L_- = hL.$$

Since an extreme point of the convex set has norm one, an extreme point annihilates every element h of $\mathcal{C}(S)$, whose values are taken in the interval $[0, 1]$, such that hL is not equal to L.

Such functions, whose values are zero or one, are defined for every positive integer r, to have equal values at elements of S which are equivalent with respect to r. An extreme point chooses a member of the equivalence defined by r by annihilating functions which vanish on the equivalence class. An extreme point chooses by the arbitrariness of r an element of S by annihilating functions which vanish at that element. An extreme point is plus or minus the linear functional which assigns function values at an element of S

A subalgebra of $\mathcal{C}(S)$ is spanned by the functions whose values are zero or one and which for some positive integer r have equal values at elements of S which are equivalent with respect to r. The subalgebra is dense in $\mathcal{C}(S)$ by the Stone-Weierstrass theorem.

The set of bounded measures on the subsets of S which are absolutely continuous with respect to counting measure is a Banach space $\mathcal{M}(S)$ in the norm

$$\|\sigma\| = \sum |\sigma'(\xi)|$$

with summation over the elements ξ of S.

An element σ of $\mathcal{M}(S)$ defines a linear functional on $\mathcal{C}(S)$ which takes a function f with vanishes outside of a finite set into the sum

$$\int f \, d\sigma = \sum f(\xi)\sigma'(\xi)$$

taken over the elements ξ of S. The integral is otherwise defined by continuity in the metric topology of $\mathcal{C}(S)$ defined by the norm. Continuity is a consequence of the inequality

$$\left| \int f \, d\sigma \right| \leq \|f\| \, \|\sigma\|$$

with the norm of f taken in $\mathcal{C}(S)$ and the norm of σ taken in $\mathcal{M}(S)$.

The inequality states that integration is a contractive transformation of $\mathcal{M}(S)$ into $\mathcal{C}^*(S)$. Integration is shown to be an isometric transformation of $\mathcal{M}(S)$ into $\mathcal{C}^*(S)$ by constructing for every $\epsilon > 0$ a function f in $\mathcal{C}(S)$ of norm at most one such that

$$\left| \int f \, d\sigma \right| \geq \|\sigma\| - \epsilon.$$

A finite subset E of S is chosen so that

$$\|\sigma\| - \epsilon \leq \sum |\sigma'(\xi)|$$

with summation over the elements ξ of E. The function is chosen to vanish outside of E and is defined in E so that

$$f'(\xi)\sigma'(\xi) = |\sigma'(\xi)|.$$

Integration is an isometric transformation of $M(S)$ onto $\mathcal{C}^*(S)$. The surjective property is shown by constructing for a given element L of $\mathcal{C}^*(S)$ an element σ of $M(S)$ such that the condition

$$Lf = \int f\, d\sigma$$

is satisfied by every element f of $\mathcal{C}(S)$. It is sufficient by continuity to satisfy the condition for functions which vanish outside of a finite set. It is sufficient by linearity to satisfy the condition for functions which vanish outside of a set containing only one element. The element ξ defines the Radon–Nikodym derivative $\sigma'(\xi)$ so that the condition is satisfied.

The Krein-Milman topology is applied in a proof of the Banach conjecture.

THEOREM. A bounded measure defined on all subsets of a continuum is absolutely continuous with respect to counting measure.

Proof. A bounded measure defined on all subsets of a set is the difference of two bounded nonnegative measures defined on all subsets of the set. A bounded nonnegative measure defined on all subsets of a set is the sum of two bounded nonnegative measures defined on all subsets of the set, of which one is absolutely continuous with respect to counting measure, and of which one vanishes on all countable subsets.

It needs to be shown that a bounded nonnegative measure defined on all subsets of S vanishes on all subsets if it vanishes on all countable subsets.

For every positive integer r a measure

$$\sigma_r = \pi_r \sigma$$

is defined by

$$\sigma_r(E) = \sigma(\pi_r^{-1} E)$$

for every subset E of S. The measure is absolutely continuous with respect to counting measure since π_r maps S onto the countable subset $\pi_r S$ of S whose elements are the subsets of Z containing no more than r elements. The measure σ_r is nonnegative and has the same value on S as σ.

When $r < n$,

$$\sigma_r = \pi_r \sigma_n$$

is the measure whose value on a subset E of S is the value of σ_n on the set of elements ξ of S such that $\pi_r \xi$ belongs to E.

Integration maps the set of all measures σ_r onto a bounded subset of $\mathcal{C}^*(S)$ whose closed convex span in the Krein-Milman topology is a regular convex subset Δ of $\mathcal{C}^*(S)$.

Integration maps an element μ of $\mathcal{M}(S)$ into Δ if, and only if, for every element f of $\mathcal{C}(S)$ the integral

$$\int f d\mu$$

belongs to the closed convex span of integrals

$$\int f d\sigma_r$$

taken over all positive integers r. The condition is satisfied for all elements f of $\mathcal{C}(S)$ if it is satisfied for all elements whose values are zero or one and which for some positive integer r have equal values at elements which are equivalent with respect to r.

Integration maps a unique element σ_∞ into Δ such that

$$\pi_r \sigma_\infty = \sigma_r = \pi_r \sigma$$

for every positive integer r.

The measure σ is annihilated by annihilating the measure σ_∞. The measure σ_∞ is annihilated by annihilating its Radon-Nikodym derivative at every element of S.

Argue by contradiction assuming that

$$\sigma_\infty'(\eta) \neq 0$$

for some element η of S. A nonnegative measure μ is defined on the subsets of S which agrees with σ on subsets of the set of elements ξ such that

$$\pi_r \xi = \pi_r \eta$$

for every positive integer r, and which annihilates every subset of the set of elements ξ such that

$$\pi_r \xi \neq \pi_r \eta$$

for some positive integer r.

These conditions imply that the measure μ annihilates every set which does not contain η. A contradiction is obtained since σ does not annihilate the set whose only element is η.

This completes the proof of the theorem.

The proof of the Banach conjecture has implications which require a specialist in the branch of set theory which issues from to Cohen proof.

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