Hilbert Spaces of Entire Functions

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Preface

Anyone approaching Hilbert spaces of entire functions for the first time will see the theory as an application of the classical theory of entire functions. The main tools are drawn from classical analysis. These are the Phragmén-Lindelöf principle, the Poisson representation of positive harmonic functions, the factorization theorem for functions of Pólya class, Nevanlinna's theory of functions of bounded type, and the Titchmarsh-Valiron theorem relating growth and zeros of entire functions of exponential type. Chapter 1 is an expository account of these fundamental principles as they are used in my applications.

The origins of Hilbert spaces of entire functions are found in a theorem of Paley and Wiener that characterizes finite Fourier transforms as entire functions of exponential type which are square integrable on the real axis. This result has a striking consequence which is meaningful without any knowledge of Fourier analysis. The identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = (\pi/a) \sum_{-\infty}^{+\infty} |F(n\pi/a)|^2$$

holds for any entire function $F(z)$ of exponential type at most a which is square integrable on the real line. The formula is ordinarily derived from a Fourier series expansion of the Fourier transform of $F(z)$. In the fall of 1958, I discovered an essentially different proof which requires nothing more than a knowledge of Cauchy's formula and basic properties of orthogonal sets. The identity is a special case of a general formula which relates mean squares of entire functions on the whole real axis to mean squares on a sequence of real points. Certain Hilbert spaces, whose elements are entire functions, enter into the proof of the general identity.

Since the identity has its origins in Fourier analysis, I conjectured that a generalization of Fourier analysis was associated with these spaces. I spent the years 1958–1961 verifying this conjecture. The outlines of the theory are best seen by using the invariant subspace concept. A fundamental problem is to determine the invariant subspaces of any bounded linear transformation in Hilbert space and to write the transformation as an integral in terms of invariant subspaces. A similar problem can be stated for an unbounded or partially defined transformation once the invariant subspace concept is clarified. For those who have had no previous acquaintance with Hilbert spaces of entire functions, it may help to say that they are the invariant subspaces appropriate for a certain kind of transformation. The theory of Hilbert spaces of entire functions is the best behaved of all invariant subspace theories. Nontrivial invariant subspaces always exist for nontrivial transformations. Invariant subspaces are totally ordered by inclusion. The transformation admits an integral representation in terms of its invariant subspaces. This representation is stated as a generalization of the Paley-Wiener theorem and Fourier transformation. Chapter 2 is devoted to the theory of the spaces. The rest of the book is concerned with examples and applications.

The known examples of Hilbert spaces of entire functions belong to the theory of special functions, a subject which is very old in relation to most of modern analysis. The foundations of the theory were laid by Leonard Euler (1707–1783) in the century following the discovery of the calculus. I do not take the historical approach to the subject, which is already so well represented by Whittaker and Watson. I have tried instead to find characteristic features of known eigenfunction expansions which can be used for a systematic derivation of their properties. For example, the Fourier expansion is characterized by its relation to translations. I determine all Hilbert spaces of entire functions which admit a two-sided isometric shift. The Hankel transformation is characterized by its relation to homogeneous substitutions. I determine all Hilbert spaces of entire functions which admit homogeneous substitutions as isometries. The eigenfunction expansions associated with Jacobi polynomials are of great interest as they include the Legendre expansion, which occurs in the quantum mechanical theory of angular momentum. The corresponding Hilbert spaces of entire functions are like the spaces of Fourier analysis in that they admit a two-sided shift. But the shift is not isometric. The isometric property is replaced by an inner product identity. The Gauss and Kummer expansions are closely related to the Jacobi expansion. The corresponding Hilbert spaces of entire functions are characterized by a similar identity. An interesting application of these expansions is given in M. Rosenblum's theory of the Hilbert matrix.

Hilbert spaces of entire functions also have other applications. An obvious area is approximation by polynomials or entire functions of exponential type. Since it was through such problems that I discovered the spaces, it was easy enough for me to include material for readers with these interests.

I give an account of my thesis results on the problem of local operators on Fourier transforms. This is equivalent to an approximation problem for entire functions of exponential type. The results on the problem include a simple proof and a generalization of Levinson's theorem on nonvanishing Fourier transforms. I also include a proof of the remarkable theorem of Beurling and Malliavin on the domain of local operators. The theorem can also be stated as an existence theorem for certain kinds of subspaces of a Hilbert space of entire functions. In this form it has several applications. One of these is the construction of measures of finite total variation which are supported in a given set and whose Fourier transforms vanish in a given interval. The extreme points of the convex set of such measures of total variation at most one have interesting special properties. Although it is easy to construct entire functions with given zeros, it is quite difficult to estimate the functions so obtained. I use the extreme point method to construct nontrivial entire functions whose zeros lie in a given set and whose reciprocals admit absolutely convergent partial fraction decompositions. A classical problem is to estimate an entire function of exponential type in the complex plane from estimates on a given sequence of points. I construct Hilbert spaces of entire functions of exponential type with norm determined by what happens on a given sequence of real points.

Some additional examples of Hilbert spaces of entire functions were not completed in time for inclusion in the book and will be published elsewhere. Several unexplained items in my list of references are material required for this unpublished work.

The book presumes a knowledge of Hilbert space and analytic function theory as presented in my book with James Rovnyak on *Square Summable Power Series* (Holt, Rinehart and Winston, 1966), referred to here as SSPS. I expect to supplement this eventually with publication of my lecture notes on *Quantum Power Series*.

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Contents

CHAPTER 1

Entire Functions

1. PHRAGMÉN-LINDELÖF PRINCIPLE

In this chapter we review the analytic function theory which is required as background knowledge for the theory of Hilbert spaces of entire functions presented in Chapter 2. The notation $z = x + iy$ is used for a complex variable. We are concerned mainly with properties of functions analytic in the upper half-plane, $y > 0$.

If a function $f(z)$ is analytic in the unit disk, $|z| < 1$, and has a continuous extension to the closed disk, then $|f(z)|$ must attain a maximum value in the closed disk. By the maximum principle, the maximum does not occur in the interior of the disk when $f(z)$ is not a constant. It follows that $f(z)$ is bounded by 1 in the disk if it is bounded by 1 on the boundary of the disk. The situation is different in the upper half-plane because maxima may not exist in the closure of an unbounded region.

Consider a function $f(z)$ which is analytic in the upper half-plane, which is continuous in the closed half-plane, and which is bounded by 1 on the real axis. We would like to conclude that $f(z)$ is bounded by 1 in the half-plane. The example $f(z) = e^{-iz}$ shows that some hypothesis is necessary. The Phragmén-Lindelöf principle states that this conclusion is valid if $f(z)$ is bounded in the half-plane, or if it satisfies a weaker hypothesis of the same nature. The notation $\log^+ x$ is used for max $(0, \log x)$ when $x \geq 0$.

THEOREM 1. Assume that $f(z)$ is analytic in the upper half-plane, that $|f(z)|$ has a continuous extension to the closed half-plane, and that

$$\liminf_{a\to\infty} a^{-1} \int_0^\pi \log^+ |f(ae^{i\theta})| \sin\theta \, d\theta = 0.$$

If $|f(z)|$ is bounded by 1 on the real axis, then it is bounded by 1 in the upper half-plane.

Proof of Theorem 1. We start with an estimate of $|f(z)|$ in the upper half-disk, $|z| < 1$ and $y > 0$, from a knowledge that $|f(z)|$ is bounded by 1 on the real part of the boundary of the half-disk. If $h(\theta)$ is any continuous, real valued function of real θ which is periodic of period 2π, there exists a function $g(z)$, which is analytic for $|z| < 1$, such that $\operatorname{Re} g(z)$ has a continuous extension to the closed half-disk and $h(\theta) = \operatorname{Re} g(e^{i\theta})$ for all real θ. It is given by

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, h(\theta)d\theta$$

for $|z| < 1$. We use this construction with $h(\theta) = \log^+|f(e^{i\theta})|$ for $0 \le \theta \le \pi$. Extend $h(\theta)$ so as to be an odd periodic function of real θ which is periodic of period 2π. This is possible because $|f(z)|$ is bounded by 1 on the real axis. Since

$$g(z) = \frac{1}{2\pi}\int_0^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, h(\theta)d\theta - \frac{1}{2\pi}\int_0^{\pi} \frac{e^{-i\theta}+z}{e^{-i\theta}-z}\, h(\theta)d\theta$$

for $|z| < 1$, we obtain

$$\begin{aligned}\operatorname{Re} g(z) &= \frac{1-|z|^2}{2\pi}\int_0^{\pi} \frac{h(\theta)d\theta}{|e^{i\theta}-z|^2} - \frac{1-|z^2|}{2\pi}\int_0^{\pi} \frac{h(\theta)d\theta}{|e^{-i\theta}-z|^2}\\ &= \frac{1-|z|^2}{2\pi}\int_0^{\pi} \frac{4yh(\theta)\sin\theta\, d\theta}{|e^{i\theta}-z|^2\,|e^{-i\theta}-z|^2}.\end{aligned}$$

Note that $\operatorname{Re} g(z)$ vanishes on the real part of the unit disk. Since we assume that $|f(z)| \le 1$ on the real axis, $f(z)/\exp g(z)$ is bounded by 1 on the boundary of the upper half-disk. By the maximum principle, the function is bounded by 1 in the interior of the half-disk. Explicitly, we have

$$\log|f(z)| \le \frac{1-|z|^2}{2\pi}\int_0^{\pi} \frac{4y\log^+|f(e^{i\theta})|\sin\theta\, d\theta}{|e^{i\theta}-z|^2\,|e^{-i\theta}-z|^2}$$

for $|z| < 1$ and $y > 0$. The same argument applies with $f(z)$ replaced by $f(az)$ when $a > 0$. If z is replaced by z/a in the resulting inequality, it reads

$$\log|f(z)| \le \frac{a^2-|z|^2}{2\pi}\int_0^{\pi} \frac{4ay\log^+|f(ae^{i\theta})|\sin\theta\, d\theta}{|ae^{i\theta}-z|^2\,|ae^{-i\theta}-z|^2}$$

for $|z| < a$ and $y > 0$. If $|z| < \epsilon a$ where $\epsilon < 1$, then

$$|ae^{i\theta}-z| \ge a(1-|z/a|) \ge a(1-\epsilon).$$

For each fixed z in the upper half-plane, we obtain

$$\log|f(z)| \le (2y/\pi)(1-\epsilon)^{-4}\liminf_{a\to\infty} a^{-1}\int_0^{\pi} \log^+|f(ae^{i\theta})|\sin\theta\, d\theta.$$

Since ϵ is arbitrary,

$$\log |f(z)| \leq (2y/\pi) \liminf_{a\to\infty} a^{-1} \int_0^\pi \log^+ |f(ae^{i\theta})| \sin\theta \, d\theta$$

and the theorem follows.

2. ANALYTIC FUNCTIONS WITH GIVEN MODULUS ON THE REAL AXIS

In applying the Phragmén-Lindelöf principle, we will need to construct functions which are analytic in the upper half-plane and which have a given modulus at points on the real axis.

THEOREM 2. Let $h(x)$ be a continuous function of real x such that $h(x) \geq 1$ for all real x and

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log h(t)dt < \infty.$$

Then the formula

$$\log f(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1+z^2}{1+t^2}\frac{\log h(t)dt}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{+\infty} \frac{\log h(t)dt}{1+t^2}$$

defines a function $f(z)$, which is analytic in the upper half-plane, such that $|f(z)| \geq 1$, $|f(z)|$ has a continuous extension to the closed half-plane, and $|f(x)| = h(x)$ for all real x.

Proof of Theorem 2. It is clear that $\log f(z)$ is a well-defined function which is analytic in the upper half-plane. Since

$$\frac{1+z^2}{i(t-z)} - \frac{1+\bar z^2}{i(t-\bar z)} + \frac{z-\bar z}{i} = 2y\frac{1+t^2}{|t-z|^2},$$

we obtain

$$\log |f(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\log h(t)dt}{(t-x)^2+y^2}$$

for $y > 0$. Since we assume that $h(x) \geq 1$ for all real x, it follows that $|f(z)| \geq 1$ for $y > 0$. The main problem is to show that $|f(z)|$ is continuous in the closed half-plane and has $h(x)$ as boundary value function. Let u be any fixed real number. We must show that $\log h(u) = \lim \operatorname{Re} \log f(z)$ as $z \to u$ through nonreal values. Explicitly the problem is to show that

$$\log h(u) = \lim_{z\to u} \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\log h(t)dt}{(t-x)^2+y^2}.$$

Since

$$1 = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{dt}{(t-x)^2+y^2},$$

it is sufficient to show that

$$0 = \lim_{z \to u} \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|\log h(t) - \log h(u)| dt}{(t-x)^2 + y^2}.$$

If $\epsilon > 0$ is given, choose $\delta > 0$ so that $|\log h(t) - \log h(u)| \leq \frac{1}{2}\epsilon$ whenever $|t - u| \leq \delta$. If $|u - x| \leq \frac{1}{2}\delta$, then

$$\frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|\log h(t) - \log h(u)| dt}{(t-x)^2 + y^2} \leq \tfrac{1}{2}\epsilon + \frac{y}{\pi} \int_{-\infty}^{u-\delta} \frac{|\log h(t) - \log h(u)| dt}{(t - u + \frac{1}{2}\delta)^2}$$

$$+ \frac{y}{\pi} \int_{u+\delta}^{+\infty} \frac{|\log h(t) - \log h(u)| dt}{(t - u - \frac{1}{2}\delta)^2} \leq \epsilon$$

when y is sufficiently small.

PROBLEM 1. Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that $\operatorname{Re} f(z)$ has a continuous extension to the closed half-plane and that $h(x)$ is a bounded, continuous function of real x such that $0 \leq h(x) \leq \operatorname{Re} f(x)$ for all real x. Show that

$$\operatorname{Re} f(x + iy) \geq \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{h(t) dt}{(t-x)^2 + y^2}$$

for $y > 0$. *Hint:* Apply the Phragmén-Lindelöf principle to an appropriate function which is analytic and bounded in the upper half-plane.

PROBLEM 2. Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. If $\operatorname{Re} f(z)$ has a continuous extension to the closed half-plane, show that there exists a function $g(z)$, which is analytic and has a nonnegative real part in the upper half-plane, such that

$$\operatorname{Re} f(x + iy) = \operatorname{Re} g(x + iy) + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(t) dt}{(t-x)^2 + y^2}$$

for $y > 0$. Show that $\operatorname{Re} g(z)$ is continuous in the closed half-plane and that $\operatorname{Re} g(x) = 0$ for all real x.

PROBLEM 3. Let $g(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that $\operatorname{Re} g(z)$ is continuous in the closed half-plane, and that $\operatorname{Re} g(x) = 0$ for all real x. Show that $\operatorname{Re} g(x + iy) = py$ where p is a constant. *Hint:* Show that

$$\operatorname{Re} g(z) = \frac{a^2 - |z|^2}{2\pi} \int_0^{\pi} \frac{4ay \operatorname{Re} g(ae^{i\theta}) \sin \theta \, d\theta}{|ae^{i\theta} - z|^2 \, |ae^{-i\theta} - z|^2}$$

when $a > 0$, $|z| < a$, and $y > 0$.

Th 3

PROBLEM 4. In Problems 2 and 3, show that

$$p = \lim_{y\to\infty} \operatorname{Re} f(iy)/y.$$

3. STIELTJES INVERSION FORMULA

A more general treatment of boundary behavior is contained in the Stieltjes inversion formula.

THEOREM 3. Let $\mu(x)$ be a nondecreasing function of real x such that $\int_{-\infty}^{+\infty} (1 + t^2)^{-1} d\mu(t) < \infty$. Then

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{1+t^2}$$

is analytic and has a nonnegative real part in the upper half-plane. If a and b are points of continuity of $\mu(x)$, $a < b$, then

$$\mu(b) - \mu(a) = \lim_{y\searrow 0}\int_a^b \operatorname{Re} f(x+iy)\,dx.$$

Proof of Theorem 3. As in the proof of Theorem 2, $f(z)$ is a well-defined function which is analytic in the upper half-plane and

$$\operatorname{Re} f(z) = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-x)^2+y^2} \geq 0.$$

The proof of the Stieltjes inversion formula requires a change in the order of integration to yield

$$\int_a^b \operatorname{Re} f(x+iy)\,dx = \frac{y}{\pi}\int_{-\infty}^{+\infty}\int_a^b \frac{dx}{(t-x)^2+y^2}\,d\mu(t)$$
$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right]d\mu(t).$$

The interchange is justified by Fubini's theorem since the integrand is nonnegative. To complete the proof we must show that

$$\pi[\mu(b)-\mu(a)] = \lim_{y\searrow 0}\int_{-\infty}^{+\infty}\left[\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right]d\mu(t).$$

If $\epsilon > 0$ is given, choose δ by the continuity of $\mu(x)$ at a and b so that $\mu(a+\delta) - \mu(a-\delta) \leq \epsilon/5$ and $\mu(b+\delta) - \mu(b-\delta) \leq \epsilon/5$. Observe that

$$0 \leq \arctan\frac{b-t}{y} - \arctan\frac{a-t}{y} \leq \pi$$

for all real t and that

$$\arctan\frac{b-t}{y}-\arctan\frac{a-t}{y}=\arctan\frac{y(b-a)}{y^2+(a-t)(b-t)}$$

when $a-t$ and $b-t$ have the same sign. Note also that

$$\arctan x=\int_0^x\frac{dt}{1+t^2}\le\int_0^x dt\le x$$

when $x>0$. It follows that

$$\begin{aligned}
&\left|\mu(b)-\mu(a)-\frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\arctan\frac{b-t}{y}-\arctan\frac{a-t}{y}\right]d\mu(t)\right|\\
&\quad\le\frac{1}{\pi}\int_b^{\infty}\left[\arctan\frac{b-t}{y}-\arctan\frac{a-t}{y}\right]d\mu(t)\\
&\quad\quad+\frac{1}{\pi}\int_a^b\left[\pi-\arctan\frac{b-t}{y}+\arctan\frac{a-t}{y}\right]d\mu(t)\\
&\quad\quad+\frac{1}{\pi}\int_{-\infty}^{a}\left[\arctan\frac{b-t}{y}-\arctan\frac{a-t}{y}\right]d\mu(t)\\
&\quad\le\frac{2\epsilon}{5}+\frac{1}{\pi}\int_{b+\delta}^{\infty}\arctan\frac{y(b-a)}{y^2+(a-t)(b-t)}\,d\mu(t)\\
&\quad\quad+\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\left[\pi-2\arctan\frac{\delta}{y}\right]d\mu(t)\\
&\quad\quad+\frac{1}{\pi}\int_{-\infty}^{a-\delta}\arctan\frac{y(b-a)}{y^2+(a-t)(b-t)}\,d\mu(t)\\
&\quad\le\frac{2\epsilon}{5}+\frac{y}{\pi}\int_{b+\delta}^{\infty}\frac{(b-a)\,d\mu(t)}{y^2+(a-t)(b-t)}+\frac{y}{\pi}\int_{-\infty}^{a-\delta}\frac{(b-a)\,d\mu(t)}{y^2+(a-t)(b-t)}\\
&\quad\quad+\frac{2}{\pi}\left(\frac{\pi}{2}-\arctan\frac{\delta}{y}\right)[\mu(b)-\mu(a)]\le\epsilon
\end{aligned}$$

when y is so small that each of the last three terms is at most $\epsilon/5$. The theorem follows.

4. POISSON REPRESENTATION

The Phragmén-Lindelöf principle and the Stieltjes inversion formula are used to obtain the Poisson representation of functions which are analytic and whose real parts are nonnegative in the upper half-plane.

Th 4

THEOREM 4. If $f(z)$ is analytic and has a nonnegative real part in the upper half-plane, then there exists a nonnegative number p and a nondecreasing function $\mu(x)$ of real x such that

$$\operatorname{Re} f(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}.$$

Proof of Theorem 4. By Problems 2 and 3,

$$\operatorname{Re} f(z+i\epsilon) = p(\epsilon)y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(t+i\epsilon)dt}{(t-x)^2+y^2}$$

for $y > 0$ when $\epsilon > 0$, where $p(\epsilon)$ is a nonnegative constant. Since

$$p(\epsilon) = \lim_{y\to\infty} \operatorname{Re} f(iy+i\epsilon)/y = \lim_{y\to\infty} \operatorname{Re} f(iy)/y$$

by Problem 4, $p(\epsilon) = p$ is independent of ϵ. In terms of

$$\mu_\epsilon(x) = \int_0^x \operatorname{Re} f(t+i\epsilon)dt,$$

the representation becomes

$$\operatorname{Re} f(z+i\epsilon) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu_\epsilon(t)}{(t-x)^2+y^2}.$$

It follows that the numbers $\int_{-\infty}^{+\infty} (1+t^2)^{-1}d\mu_\epsilon(t)$ are bounded independently of ϵ, $0<\epsilon<1$. Since

$$\mu_\epsilon(a) - \mu_\epsilon(-a) \le (1+a^2)\int_{-\infty}^{+\infty} (1+t^2)^{-1}d\mu_\epsilon(t)$$

and since $\mu_\epsilon(-a) \le 0 \le \mu_\epsilon(a)$ for all $a > 0$, the numbers $(\mu_\epsilon(a))$ and $(\mu_\epsilon(-a))$ are bounded independently of ϵ, $0<\epsilon<1$, for each fixed a. By the Helly selection principle, there exists a decreasing sequence (ϵ_n) of positive numbers such that $\mu(x) = \lim \mu_\epsilon(x)$ exists for all real x as $\epsilon \searrow 0$ through the sequence (ϵ_n). The limit $\mu(x)$ is a nondecreasing function of x and

$$\int_a^b \frac{d\mu(t)}{(t-x)^2+y^2} = \lim \int_a^b \frac{d\mu_\epsilon(t)}{(t-x)^2+y^2}$$

for $y > 0$ and $-\infty < a < b < \infty$. Since

$$\operatorname{Re} f(z+i\epsilon) \ge py + \frac{y}{\pi}\int_a^b \frac{d\mu_\epsilon(t)}{(t-x)^2+y^2},$$

it follows that

$$\operatorname{Re} f(z) \ge py + \frac{y}{\pi}\int_a^b \frac{d\mu(t)}{(t-x)^2+y^2}.$$

By the arbitrariness of a and b,

$$\operatorname{Re} f(z) \geq py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}.$$

Consider the function

$$g(z) = -ipz + \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z} + \frac{z}{\pi i}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{1+t^2}$$

which is analytic and has a nonnegative real part for $y > 0$. Since

$$\operatorname{Re} g(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2 + y^2},$$

$\operatorname{Re} f(z) \geq \operatorname{Re} g(z)$ for $y > 0$. The function $h(z) = f(z) - g(z)$ is therefore analytic and has a nonnegative real part in the upper half-plane. If a and b are points of continuity of $\mu(x)$, $a < b$, then

$$\begin{aligned}\lim_{n\to\infty}\int_a^b \operatorname{Re} h(x+i\epsilon_n)\,dx &= \lim_{n\to\infty}\int_a^b \operatorname{Re} f(x+i\epsilon_n)\,dx - \lim_{n\to\infty}\int_a^b \operatorname{Re} g(x+i\epsilon_n)\,dx \\ &= [\mu(b)-\mu(a)] - [\mu(b)-\mu(a)] = 0\end{aligned}$$

by the definition of $\mu(x)$ and the Stieltjes inversion formula. The same conclusion follows for all a and b since any interval (a, b) is contained in an interval (c, d) whose end points are points of continuity of $\mu(x)$. Since $p = \lim \operatorname{Re} g(iy)/y$ as $y \to \infty$ by Problem 4, we obtain $\lim \operatorname{Re} h(iy)/y = 0$ as $y \to \infty$. By Problems 2, 3, and 4,

$$\operatorname{Re} h(z+i\epsilon) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{Re} h(t+i\epsilon)\,dt}{(t-x)^2 + y^2}$$

for $y > 0$ if $\epsilon > 0$. If $-\infty < a < b < \infty$,

$$0 \leq \frac{y}{\pi}\int_a^b \frac{\operatorname{Re} h(t+i\epsilon)\,dt}{(t-x)^2+y^2} \leq \frac{1}{\pi y}\int_a^b \operatorname{Re} h(t+i\epsilon)\,dt$$

where $\lim \int_a^b \operatorname{Re} h(t+i\epsilon_n)\,dt = 0$ as $n \to \infty$. It follows that

$$\operatorname{Re} h(z) = \lim_{n\to\infty}\left[\frac{y}{\pi}\int_{-\infty}^{a} \frac{\operatorname{Re} h(t+i\epsilon_n)\,dt}{(t-x)^2+y^2} + \frac{y}{\pi}\int_b^{\infty} \frac{\operatorname{Re} h(t+i\epsilon_n)\,dt}{(t-x)^2+y^2}\right].$$

If x_1 and x_2 are points in the interval (a, b) and if t lies outside the interval,

$$\begin{aligned}[(t-x_2)^2+y^2]/[(t-x_1)^2+y^2] &= |(t-x_2-iy)/(t-x_1-iy)|^2 \\ &= |1-(x_2-x_1)/(t-x_1-iy)|^2 \\ &\leq [1+|x_2-x_1|/\min(|x_1-a|, |x_1-b|)]^2.\end{aligned}$$

It follows that

$$\operatorname{Re} h(x_1+iy) \leq \operatorname{Re} h(x_2+iy)[1+|x_2-x_1|/\min(|x_1-a|, |x_1-b|)]^2.$$

By the arbitrariness of a and b, $\operatorname{Re} h(x_1 + iy) \le \operatorname{Re} h(x_2 + iy)$. Equality holds since x_1 and x_2 can be interchanged. This implies that the real part of $h'(z)$ is zero and that $h'(z)$ is an imaginary constant for $y > 0$. Since $\lim \operatorname{Re} h(iy)/y = 0$, the constant is zero. So $h(z)$ is a constant. Since $\lim \int_a^b \operatorname{Re} h(t + i\epsilon_n)dt = 0$, the real part of the constant is zero and the theorem follows.

PROBLEM 5. Let $\varphi(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Extend it to the lower half-plane so that $\bar{\varphi}(\bar{z}) = -\varphi(z)$. Let p be the nonnegative number and let $\mu(x)$ be the nondecreasing function of real x such that

$$\operatorname{Re} \varphi(x + iy) = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

when $y > 0$. Show that

$$\frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} = \frac{p}{\pi} + \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})}$$

when z and w are not real.

5. CONSTRUCTION OF THE SPACE $\mathfrak{L}(\varphi)$

A Hilbert space of analytic functions is associated with any function $\varphi(z)$ which is analytic and has a nonnegative real part in the upper half-plane.

THEOREM 5. Let $\varphi(z)$ be a given function which is analytic and has a nonnegative real part in the upper half-plane. Extend $\varphi(z)$ to the lower half-plane so that $\bar{\varphi}(\bar{z}) = -\varphi(z)$. Then there exists a unique Hilbert space $\mathfrak{L}(\varphi)$, whose elements are functions $F(z)$ analytic in the upper and lower half-planes, such that

$$[\varphi(z) + \bar{\varphi}(w)]/[\pi i(\bar{w} - z)]$$

belongs to the space for every nonreal number w, and such that

$$F(w) = \langle F(t), [\varphi(t) + \bar{\varphi}(w)]/[\pi i(\bar{w} - t)]\rangle_{\mathfrak{L}(\varphi)}$$

for every $F(z)$ in $\mathfrak{L}(\varphi)$. The function $[F(z) - F(w)]/(z - w)$ belongs to $\mathfrak{L}(\varphi)$ whenever $F(z)$ belongs to $\mathfrak{L}(\varphi)$, and the identity

$$\begin{aligned} 0 = \langle F(t), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}(\varphi)} - \langle [F(t) - F(\alpha)]/(t - \alpha), G(t)\rangle_{\mathfrak{L}(\varphi)} \\ + (\alpha - \bar{\beta})\langle [F(t) - F(\alpha)]/(t - \alpha), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}(\varphi)} \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathfrak{L}(\varphi)$ and for all nonreal numbers α and β. Let

$$\operatorname{Re} \varphi(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

where p is a nonnegative number and $\mu(x)$ is a nondecreasing function of real x. Then the transformation

$$U\colon f(x) \to \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(t)\,d\mu(t)}{t-z}$$

takes $L^2(\mu)$ isometrically into $\mathfrak{L}(\varphi)$, and the orthogonal complement of the range of the transformation contains only constants. If $U\colon f(x) \to F(z)$ and if w is a nonreal number, then

$$U\colon f(x)/(x-w) \to [F(z)-F(w)]/(z-w).$$

Proof of Theorem 5. The function $\psi(z) = \varphi(z) + ipz$ is analytic in the upper and lower half planes. Since

$$\operatorname{Re} \psi(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

where $\mu(x)$ is a nondecreasing function of real x, $\psi(z)$ has a nonnegative real part in the upper half-plane. We begin by constructing $\mathfrak{L}(\psi)$. If $f(x)$ belongs to $L^2(\mu)$ and has nonnegative values, it determines a nondecreasing function $\nu(x)$ of real x such that $\int_a^b f(t)\,d\mu(t) = \nu(b) - \nu(a)$ whenever $-\infty < a < b < \infty$, and

$$\int_{-\infty}^{+\infty} \frac{f(t)\,d\mu(t)}{t-z} = \int_{-\infty}^{+\infty} \frac{d\nu(t)}{t-z}$$

when z is not real. By the Stieltjes inversion formula,

$$\int_a^b f(t)\,d\mu(t) = \lim_{y \searrow 0} \int_a^b \operatorname{Re} \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{f(t)\,d\mu(t)}{t-z}\,dx$$

whenever a and b are points of continuity of $\mu(x)$, and hence also of $\nu(x)$. Since any element of $L^2(\mu)$ is a linear combination of nonnegative functions, this last identity is valid for any $f(x)$ in $L^2(\mu)$. From this we see that

$$\int_a^b f(t)\,d\mu(t)$$

must vanish for all points of continuity of $\mu(x)$ if $\int_{-\infty}^{+\infty} (t-z)^{-1} f(t)\,d\mu(t)$ vanishes for all nonreal values of z. In this case $f(x)$ vanishes almost everywhere with respect to μ.

Th 5

Let $\mathfrak{L}$ be the set of functions $F(z)$, defined for nonreal values of z, of the form

$$F(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)\,d\mu(t)}{t - z}$$

for some corresponding element $f(x)$ of $L^2(\mu)$. As we have seen, $f(x)$ is essentially uniquely determined by $F(z)$. If we define $\|F\|_{\mathfrak{L}} = \|f\|_{L^2(\mu)}$, then $\mathfrak{L}$ is a Hilbert space. By Problem 5, the identity

$$\frac{\psi(z) + \bar{\psi}(w)}{\pi i(\bar{w} - z)} = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t - z)(-\pi i)(t - \bar{w})}$$

holds whenever z and w are not real. It follows that the expression on the left belongs to $\mathfrak{L}$ as a function of z for every nonreal number w and that the corresponding element of $L^2(\mu)$ is $(-\pi i)^{-1}(x - \bar{w})^{-1}$. If $F(z)$ is in $\mathfrak{L}$ and if $f(x)$ is the corresponding element of $L^2(\mu)$, then

$$\left\langle F(t), \frac{\psi(t) + \bar{\psi}(w)}{\pi i(\bar{w} - t)}\right\rangle_{\mathfrak{L}} = \int_{-\infty}^{+\infty}\frac{f(t)\,d\mu(t)}{\pi i(t - w)} = F(w)$$

since the correspondence between $L^2(\mu)$ and $\mathfrak{L}$ preserves inner products. A space $\mathfrak{L}(\psi)$ therefore exists, and it is equal isometrically to $\mathfrak{L}$. The transformation

$$f(x) \to \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)\,d\mu(t)}{t - z}$$

is an isometry of $L^2(\mu)$ onto $\mathfrak{L}(\psi)$.

If $f(x)$ is in $L^2(\mu)$ and if

$$F(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)\,d\mu(t)}{t - z},$$

then

$$\frac{F(z) - F(w)}{z - w} = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t)\,d\mu(t)}{(t - z)(t - w)}$$

for every nonreal number w. If $g(x)$ is in $L^2(\mu)$ and if

$$G(z) = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{g(t)\,d\mu(t)}{t - z},$$

then we obtain the identity

$$\begin{aligned}
&\langle F(t), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}(\psi)} - \langle [F(t) - F(\alpha)]/(t - \alpha), G(t)\rangle_{\mathfrak{L}(\psi)} \\
&\qquad + (\alpha - \bar{\beta})\langle [F(t) - F(\alpha)]/(t - \alpha), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}(\psi)} \\
&= \int_{-\infty}^{+\infty}\frac{f(t)\bar{g}(t)\,d\mu(t)}{t - \bar{\beta}} - \int_{-\infty}^{+\infty}\frac{f(t)\bar{g}(t)\,d\mu(t)}{t - \alpha} \\
&\qquad\qquad + (\alpha - \bar{\beta})\int_{-\infty}^{+\infty}\frac{f(t)\bar{g}(t)\,d\mu(t)}{(t - \alpha)(t - \bar{\beta})} = 0
\end{aligned}$$

for all nonreal numbers α and β. If $F(z)$ is in $\mathfrak{L}(\psi)$,

$$|F(w)|^2 \leq \|F(t)\|^2 \left\| \frac{\psi(t) + \bar{\psi}(w)}{\pi i(\bar{w} - t)} \right\|^2 = \|F(t)\|^2 \frac{\psi(w) + \bar{\psi}(w)}{\pi i(\bar{w} - w)}$$

by the Schwarz inequality. By Problem 4,

$$\lim_{y \to +\infty} F(iy) = 0.$$

It follows that $\mathfrak{L}(\psi)$ contains no nonzero constant.

If $p = 0$ then $\varphi(z) = \psi(z)$ and the theorem follows. If $p \neq 0$, consider a new Hilbert space whose elements are of the form $\lambda + F(z)$ where $F(z)$ is in $\mathfrak{L}(\psi)$ and λ is a constant. Define an inner product in this space corresponding to the norm

$$\|\lambda + F(t)\|^2 = |\lambda^2| \pi/p + \|F(t)\|^2_{\mathfrak{L}(\psi)}.$$

It is easily verified that this new space is the required space $\mathfrak{L}(\varphi)$. Uniqueness of the space is proved as in SSPS Lemma 11.

6. CHARACTERIZATION OF THE SPACE $\mathfrak{L}(\varphi)$

The stated properties of $\mathfrak{L}(\varphi)$ characterize the space.

THEOREM 6. Let $\mathfrak{L}$ be a Hilbert space whose elements are functions analytic in the upper and lower half-planes. Assume that $[F(z) - F(w)]/(z - w)$ belongs to $\mathfrak{L}$ whenever $F(z)$ belongs to $\mathfrak{L}$, for every nonreal number w, and that the identity

$$0 = \langle F(t), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}} - \langle [F(t) - F(\alpha)]/(t - \alpha), G(t)\rangle_{\mathfrak{L}} + (\alpha - \bar{\beta})\langle [F(t) - F(\alpha)]/(t - \alpha), [G(t) - G(\beta)]/(t - \beta)\rangle_{\mathfrak{L}}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathfrak{L}$ and all nonreal numbers α and β. If the linear functional defined on $\mathfrak{L}$ by $F(z) \to F(w)$ is continuous for some nonreal number w, then $\mathfrak{L}$ is equal isometrically to a space $\mathfrak{L}(\varphi)$.

Proof of Theorem 6. We use the identity first when $F(z) = G(z)$ and when $\alpha = \beta = w$ for some nonreal number w. By the Schwarz inequality,

$$|w - \bar{w}| \; \|[F(t) - F(w)]/(t - w)\|^2_{\mathfrak{L}} \leq 2 \, \|F(t)\|_{\mathfrak{L}} \, \|[F(t) - F(w)]/(t - w)\|_{\mathfrak{L}}.$$

So the transformation $F(z) \to [F(z) - F(w)]/(z - w)$ is continuous in the metric of $\mathfrak{L}$. If $F(z)$ is in $\mathfrak{L}$, the value of $F(z)$ at w is the value of $F(z) - (\alpha - w)[F(z) - F(w)]/(z - w)$ at α. Since we assume that the linear functional $F(z) \to F(\alpha)$ is continuous for some nonreal number α, the linear functional $F(z) \to F(w)$ is continuous for all nonreal numbers w.

Since $\mathfrak{L}$ is a Hilbert space, there exists a unique element $L(w, z)$ of $\mathfrak{L}$ such that $F(w) = \langle F(t), L(w, t)\rangle_{\mathfrak{L}}$ for every $F(z)$ in $\mathfrak{L}$. If α and β are nonreal numbers, $[L(\alpha, z) - L(\alpha, \beta)]/(z - \beta)$ belongs to $\mathfrak{L}$. If $F(z)$ is in $\mathfrak{L}$,

$$\begin{aligned}\langle F(t), [L(\alpha, t) - L(\alpha, \beta)]/(t - \beta)\rangle_{\mathfrak{L}} &= \langle [F(t) - F(\bar\beta)]/(t - \bar\beta), L(\alpha, t)\rangle_{\mathfrak{L}} \\ &= [F(\alpha) - F(\bar\beta)]/(\alpha - \bar\beta) = \langle F(t), [L(\alpha, t) - L(\bar\beta, t)]/(\bar\alpha - \beta)\rangle_{\mathfrak{L}}.\end{aligned}$$

By the arbitrariness of $F(z)$,

$$[L(\alpha, z) - L(\alpha, \beta)]/(z - \beta) = [L(\alpha, z) - L(\bar\beta, z)]/(\bar\alpha - \beta).$$

On the other hand,

$$L(\alpha, \beta) = \langle L(\alpha, t), L(\beta, t)\rangle_{\mathfrak{L}} = \langle L(\beta, t), L(\alpha, t)\rangle_{\mathfrak{L}}^{-} = L(\beta, \alpha)^{-}.$$

It follows that

$$\pi i(\bar\alpha - z)L(\alpha, z) = \pi i(\bar\alpha - \beta)L(\alpha, \beta) + \pi i(\beta - z)L(\bar\beta, z).$$

If

$$\varphi(z) = \pi i(\alpha - z)L(\bar\alpha, z) + \tfrac{1}{2}\pi i(\bar\alpha - \alpha)L(\alpha, \alpha)$$

for some nonreal number α, then $\varphi(z)$ is defined for nonreal z, $\bar\varphi(\bar z) = -\varphi(z)$, and

$$L(w, z) = [\varphi(z) + \bar\varphi(w)]/[\pi i(\bar w - z)].$$

Since the elements of $\mathfrak{L}$ are analytic in the upper and lower half-planes, $\varphi(z)$ is analytic in each half-plane. Since

$$L(w, w) = \|L(w, t)\|_{\mathfrak{L}}^2 \geq 0$$

for nonreal w, the real part of $\varphi(z)$ is nonnegative in the upper half-plane. The theorem follows from the uniqueness part of Theorem 5.

7. FACTORIZATION OF FUNCTIONS OF PÓLYA CLASS

An entire function $E(z)$ is said to be of Pólya class if it has no zeros in the upper half-plane, if $|E(x - iy)| \leq |E(x + iy)|$ for $y > 0$, and if $|E(x + iy)|$ is a nondecreasing function of $y > 0$ for each fixed x.

PROBLEM 6. Show that a polynomial is of Pólya class if it has no zeros in the upper half-plane.

PROBLEM 7. Show that

$$\left| (1 - z) \exp\left(z + \tfrac{1}{2}z^2 + \cdots + \frac{1}{r}z^r\right) - 1 \right| \leq \exp(|z|^{r+1}) - 1$$

for all complex z, $r = 1, 2, 3, \cdots$. *Hint:* Show that the derivative of the function $(1 - z) \exp\left(z + \frac{1}{2}z^2 + \cdots + \frac{1}{r} z^r\right)$ is the function

$$-z^r \exp\left(z + \tfrac{1}{2}z^2 + \cdots + \frac{1}{r} z^r\right)$$

and that the inequality

$$x + \tfrac{1}{2}x^2 + \cdots + \frac{1}{r} x^r \leq x^{r+1} + \log(1 + r)$$

holds for $x > 0$.

PROBLEM 8. Show that

$$1 + |ab - 1| \leq (1 + |a - 1|)(1 + |b - 1|)$$

for all complex numbers a and b.

PROBLEM 9. Let (z_n) be a sequence of numbers such that $y_n \geq 0$ for every n and

$$\sum_1^\infty \frac{1 + y_n}{x_n^2 + y_n^2} < \infty.$$

Show that the product

$$E(z) = \prod_1^\infty (1 - z/\bar{z}_n)e^{h_n z}$$

converges uniformly on bounded sets if

$$h_n = \frac{x_n}{x_n^2 + y_n^2}.$$

Show that the limit is an entire function of Pólya class. *Hint:* If

$$P_r(z) = \prod_{n=1}^r (1 - z/\bar{z}_n)e^{z/\bar{z}_n},$$

then

$$|P_s(z) - P_r(z)| \leq \exp\left\{\sum_{n=1}^s |z/\bar{z}_n|^2\right\} - \exp\left\{\sum_{n=1}^r |z/\bar{z}_n|^2\right\}$$

when $r < s$.

The Phragmén-Lindelöf principle is used to obtain a factorization theorem for functions of Pólya class.

THEOREM 7. If $E(z)$ is an entire function of Pólya class which has a zero of order r at the origin, then

$$E(z) = E^{(r)}(0)(z^r/r!)e^{-az^2}e^{-ibz} \prod (1 - z/\bar{z}_n)e^{h_n z}$$

where $a \geq 0$, $\operatorname{Re} b \geq 0$, $(\bar{z}_n)$ is the sequence of nonzero zeros of $E(z)$, and
$h_n = \dfrac{x_n}{x_n^2 + y_n^2}$ for every n.

We show in the proof that the convergence condition

$$\sum_1^\infty \frac{1 + y_n}{x_n^2 + y_n^2} < \infty$$

is satisfied if there are an infinite number of zeros.

LEMMA 1. If an entire function $E(z)$ is of Pólya class and has a zero $\bar{w}$, then $E(z)/(z - \bar{w})$ is of Pólya class.

LEMMA 2. If an entire function $E(z)$ is of Pólya class and has no zeros, then $E(z) = E(0)e^{-az^2}e^{-ibz}$ where $a \geq 0$ and $\operatorname{Re} b \geq 0$.

Proof of Lemma 1. Since $E(z)$ is of Pólya class,

$$|E(x + ih - iy)| \leq |E(x + ih + iy)|$$

when $y \geq 0$ and $h \geq 0$. For each fixed h,

$$\bar{E}(x + ih - iy)/E(x + ih + iy) = \bar{E}(\bar{z} + ih)/E(z + ih)$$

is analytic and bounded for $y > 0$. Consider $F(z) = E(z)/(z - \bar{w})$. Since $\bar{w}$ is a zero of $E(z)$, $F(z)$ is an entire function. Since $E(z)$ has no zeros in the upper half-plane, neither does $F(z)$. The function

$$\bar{F}(\bar{z} + ih)/F(z + ih) = [\bar{E}(\bar{z} + ih)/E(z + ih)][(z + ih - \bar{w})/(z - ih - w)]$$

is analytic and bounded in the upper half-plane, and it is bounded by 1 on the real axis. By the Phragmén-Lindelöf principle, it is bounded by 1 in the upper half-plane. It follows that

$$|F(x + ih - iy)| \leq |F(x + ih + iy)|$$

for $y \geq 0$ and $h \geq 0$, and this implies that $F(z)$ is of Pólya class.

Proof of Lemma 2. Since $E(z)$ has no zeros, we can write $E(z) = E(0) \exp F(z)$ for an entire function $F(z)$ which has a zero at the origin. Since $E(z)$ is of Pólya class,

$$\operatorname{Re} F(x + ih - iy) \leq \operatorname{Re} F(x + ih + iy)$$

for $y \geq 0$ and $h \geq 0$. So if $h \geq 0$, $F(z + ih) - \bar{F}(\bar{z} + ih)$ is an entire function whose real part is nonnegative in the upper half-plane and zero on the real axis. By Problem 3,

$$\operatorname{Re} [F(z + ih) - \bar{F}(\bar{z} + ih)] = p(h)y$$

for $y \geq 0$, where $p(h)$ is a nonnegative constant. Since the real part of the entire function $F(z + ih) - \bar{F}(\bar{z} + ih) + ip(h)z$ vanishes in the upper half-plane, the function is a constant. Since the second derivative of the function must vanish identically, $F''(z + ih) = \bar{F}''(\bar{z} + ih)$. When $y = 0$, we have $F''(x + ih) = \bar{F}''(x + ih)$ for all $h \geq 0$. Since $F''(z)$ is then a real valued entire function, it is a constant. Since $F(z)$ vanishes at the origin, $F(z) = -az^2 - ibz$ for some numbers a and b, a real. Since

$$\operatorname{Re}[F(z + ih) - \bar{F}(\bar{z} + ih)] = 4ahy + 2y \operatorname{Re} b \geq 0$$

for $h \geq 0$ and $y \geq 0$, we must have $a \geq 0$ and $\operatorname{Re} b \geq 0$.

Proof of Theorem 7. By repeated application of Lemma 1, we can write $E(z) = E^{(r)}(0)(z^r/r!)F(z)$ where $F(z)$ is an entire function of Pólya class and $F(0) = 1$. If $F(z)$ has no zeros, the theorem follows from Lemma 2. Otherwise let $\bar{z}_0$ be the choice of a zero of $F(z)$ nearest the origin. By Lemma 1, $F(z)/(1 - z/\bar{z}_0)$ is of Pólya class. Let $h_0 = x_0/(x_0^2 + y_0^2)$. Since the modulus of $e^{h_0 z}$ is constant on every vertical line, $F_1(z) = F(z)e^{-h_0 z}/(1 - z/\bar{z}_0)$ is of Pólya class. If $F_1(z)$ has no zeros, the theorem follows from Lemma 2. Otherwise continue inductively in the obvious way. At the nth stage, $F_n(z)$ will be an entire function of Pólya class which has value 1 at the origin. The theorem follows immediately if this function has no zeros. Otherwise let $\bar{z}_n$ be the choice of a zero of $F_n(z)$ nearest the origin and let

$$F_{n+1}(z) = F_n(z)e^{-h_n z}/(1 - z/\bar{z}_n)$$

where $h_n = x_n/(x_n^2 + y_n^2)$. In the worst case, $F_n(z)$ is defined for every $n = 1, 2, 3, \cdots$. Let

$$P_n(z) = \prod_{k=0}^{n} (1 - z/\bar{z}_k)e^{h_k z}.$$

Then $P_n(z)$ is of Pólya class and $F(z) = P_n(z)F_{n+1}(z)$. It follows that

$$iF'(z)/F(z) = iP_n'(z)/P_n(z) + iF_{n+1}'(z)/F_{n+1}(z)$$

where each term is analytic in the upper half-plane. Since $F_{n+1}(z)$ is of Pólya class,

$$\operatorname{Re} iF_{n+1}'(z)/F_{n+1}(z) = \partial/\partial y \log |F_{n+1}(x + iy)| \geq 0$$

for $y > 0$. It follows that

$$\operatorname{Re} iP_n'(z)/P_n(z) \leq \operatorname{Re} iF'(z)/F(z)$$

for $y > 0$. Since

$$\operatorname{Re} \frac{iP_n'(z)}{P_n(z)} = \sum_{k=0}^{n} \frac{y + y_k}{(x - x_k)^2 + (y + y_k)^2}$$

and since n is arbitrary,

$$\sum_{0}^{\infty} \frac{y + y_k}{(x - x_k)^2 + (y + y_k)^2} \leq \text{Re} \frac{iF'(z)}{F(z)}$$

for $y > 0$. It follows that

$$\sum \frac{1 + y_n}{x_n^2 + y_n^2} < \infty.$$

By Problem 9, $P_\infty(z) = \lim P_n(z)$ converges uniformly on bounded sets and the limit is an entire function of Pólya class. It follows that $\lim F_n(z) = F_\infty(z)$ exists uniformly on bounded sets. Since $F_n(z)$ is of Pólya class for every n, $F_\infty(z)$ is an entire function of Pólya class. Since we always chose $\bar{z}_n$ to be a zero of $F_n(z)$ nearest the origin, $F_\infty(z)$ has no zeros. The theorem now follows from Lemma 2 since $F(z) = F_\infty(z)P_\infty(z)$.

PROBLEM 10. Let $E(z)$ be a polynomial of Pólya class such that $E(0) = 1$, and let $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are polynomials which are real for real z. Show that

$$\log |E(z)| \leq xA'(0) + yB'(0) + \tfrac{1}{2}[A'(0)^2 - A''(0) + B'(0)^2]\,|z|^2$$

for all complex z.

PROBLEM 11. If $a > 0$ is given, find a sequence $(P_n(z))$ of polynomials, which have only real zeros, such that $e^{-az^2} = \lim P_n(z)$ uniformly on bounded sets.

PROBLEM 12. If b is a given number, $\text{Re}\, b > 0$, find a sequence $(P_n(z))$ of polynomials of Pólya class such that $e^{-ibz} = \lim P_n(z)$ uniformly on bounded sets.

PROBLEM 13. If $E(z)$ is a given entire function of Pólya class, show that there exists a sequence $(P_n(z))$ of polynomials of Pólya class such that $E(z) = \lim P_n(z)$ uniformly on bounded sets.

PROBLEM 14. Let $E(z)$ be an entire function which has no zeros for $y > 0$, such that $|E(x - iy)| \leq |E(x + iy)|$ for $y > 0$. Show that

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$ unless $E(z)$ and $\bar{E}(\bar{z})$ are linearly dependent.

PROBLEM 15. If $E(z)$ is an entire function of Pólya class, show that $|E(x + iy)|$ is an increasing function of $y > 0$ for each fixed x unless $E(z) = E(0)e^{hz}$ for some real number h.

PROBLEM 16. Let $E(z)$ be an entire function of Pólya class such that $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. Show that $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions of Pólya class which are real for real z.

PROBLEM 17. Let $E(z)$ be an entire function of Pólya class which is not a constant. Show that $E'(z)$ is of Pólya class.

PROBLEM 18. Show that $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ is of Pólya class. Determine the factorization given by Theorem 7. By computing the second derivative of $\cos z$ at the origin, show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

PROBLEM 19. The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

for $x > 0$. Show that $z\Gamma(z) = \Gamma(z + 1)$. Show that $\Gamma(z)$ has an analytic continuation in the complex plane except for simple poles at zero and the negative integers. Show that

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n (1 - t/n)^n t^{z-1} dt$$

where

$$z(z+1)\cdots(z+n)\int_0^n (1 - t/n)^n t^{z-1} dt = n^z n!.$$

Show that the reciprocal of $\Gamma(z)$ is an entire function of Pólya class. Show that there exists a number $a \geq 0$ and a real number γ such that

$$1/\Gamma(z) = z e^{\gamma z} e^{-az^2} \prod_1^\infty (1 + z/n) e^{-z/n}.$$

Show that $a = 0$ and that Euler's constant γ is given by

$$\gamma = \lim_{n\to\infty} \left(1 + \tfrac{1}{2} + \cdots + \frac{1}{n} - \log n\right).$$

Hint: Use the identities

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma - 2az + \sum_1^\infty \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{z} + \frac{\Gamma'(z)}{\Gamma(z)}$$

and the fact that $\Gamma(1) = 1$.

8. FACTORIZATION OF FUNCTIONS OF BOUNDED TYPE

A function $F(z)$, which is analytic in a region, is said to be of bounded type in the region if $F(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are analytic and bounded in the region and $Q(z)$ is not identically zero.

PROBLEM 20. Show that a function $F(z)$, which is analytic in the upper half-plane, is of bounded type in the upper half-plane if its real part is nonnegative in the half-plane.

PROBLEM 21. Show that the sum and product of two functions which are of bounded type in the upper half-plane are functions of bounded type in the half-plane.

PROBLEM 22. Show that a polynomial is a function of bounded type in the upper half-plane.

PROBLEM 23. Let (z_n) be a sequence of numbers such that $y_n > 0$ for every n and

$$\sum \frac{y_n}{x_n^2 + y_n^2} < \infty.$$

Show that the Blaschke product

$$B(z) = \prod_1^\infty (1 - z/z_n)/(1 - z/\bar{z}_n)$$

converges uniformly on every bounded set which lies at a positive distance from the numbers $(\bar{z}_n)$. Show that $B(z)$ is analytic and bounded by 1 in the upper half-plane and that $B(z)\bar{B}(\bar{z}) = 1$. *Hint:* If

$$\rho(z) = \inf_n |1/z - 1/\bar{z}_n|,$$

show that

$$1 + |(1 - z/z_n)/(1 - z/\bar{z}_n) - 1| \le \exp\left(\frac{2}{\rho(z)} \frac{y_n}{x_n^2 + y_n^2}\right).$$

If

$$B_n(z) = \prod_{k=1}^n (1 - z/z_k)/(1 - z/\bar{z}_k),$$

show that

$$|B_r(z) - B_n(z)| \le \exp\left(\frac{2}{\rho(z)} \sum_{k=1}^r \frac{y_k}{x_k^2 + y_k^2}\right) - \exp\left(\frac{2}{\rho(z)} \sum_{k=1}^n \frac{y_k}{x_k^2 + y_k^2}\right)$$

when $n < r$.

If a function is analytic and of bounded type in the upper half-plane and if the function has no zeros in a neighborhood of the origin, then its zeros coincide with those of a Blaschke product.

THEOREM 8. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane, such that the origin is not a limit point of zeros of $F(z)$. Then

$$F(z) = G(z) \prod (1 - z/z_n)/(1 - z/\bar{z}_n)$$

where $z_1, z_2, z_3, \cdots$, are the zeros of $F(z)$ in the upper half-plane, repeated according to multiplicity, and $G(z)$ is a function which is analytic and of bounded type in the upper half-plane and which has no zeros in the half-plane.

If $F(z)$ has no zeros in the upper half-plane, the product is taken equal to 1. If there are an infinite number of zeros, then we show in the proof that the convergence condition

$$\sum \frac{y_n}{x_n^2 + y_n^2} < \infty$$

is satisfied.

Proof of Theorem 8. Since $F(z)$ is assumed to be of bounded type in the upper half-plane, there exists a nonzero function $Q(z)$, which is analytic and bounded by 1 in the upper half-plane, such that $Q(z)F(z)$ is analytic and bounded by 1 in the upper half-plane. Since the zeros of a nonzero analytic function are isolated and have finite multiplicities, they are countable. Let (z_n) be an enumeration of the zeros of $F(z)$ in the upper half-plane, repeated according to multiplicity. The theorem is immediate when $F(z)$ has no zeros and is easily obtained when $F(z)$ has only a finite number of zeros. Define a sequence $(F_n(z))$ of analytic functions inductively by $F_1(z) = F(z)$ and $F_{n+1}(z) = F_n(z)(1 - z/\bar{z}_n)/(1 - z/z_n)$. We show by induction that $Q(z)F_n(z)$ is bounded by 1 in the upper half-plane. We know that $Q(z)F(z)$ is bounded by 1. Assume it known that $Q(z)F_n(z)$ is bounded by 1. We show that $Q(z)F_{n+1}(z)$ is bounded by 1. By construction,

$$Q(z)F_{n+1}(z) = Q(z)F_n(z)(1 - z/\bar{z}_n)/(1 - z/z_n)$$

where the last factor is bounded on any set which lies at a positive distance from the point z_n. Since $Q(z)F_{n+1}(z)$ is bounded in a neighborhood of z_n by continuity, it is bounded in the upper half-plane. On the line $y = h$, $h > 0$, it is bounded by

$$\max_x |[1 - (x + ih)/\bar{z}_n]/[1 - (x + ih)/z_n]|$$
$$= \max \{1 + 4hy_n/[(x - x_n)^2 + (h - y_n)^2]\}^{\frac{1}{2}}$$
$$= [1 + 4hy_n/(h - y_n)^2]^{\frac{1}{2}} = (h + y_n)/|h - y_n|.$$

By the Phragmén-Lindelöf principle, $Q(z)F_{n+1}(z)$ has the same bound in the half-plane $y \geq h$. By the arbitrariness of h, $Q(z)F_{n+1}(z)$ is bounded by 1 in the upper half-plane.

The theorem follows immediately if there are only a finite number of zeros. In the infinite case, we show that the convergence condition of Problem 23 is satisfied. Write $F(z) = B_n(z)F_{n+1}(z)$ where

$$B_n(z) = \prod_{k=1}^{n} (1 - z/z_k)/(1 - z/\bar{z}_k).$$

Since $Q(z)F_{n+1}(z)$ is bounded by 1 in the upper half-plane, $|Q(z)F(z)| \leq |B_n(z)|$, or equivalently,

$$-\log |Q(z)F(z)| \geq \sum_{k=1}^{n} \log |(1 - z/\bar{z}_k)/(1 - z/z_k)|$$

$$\geq \tfrac{1}{2}\sum_{k=1}^{n} \log \{1 + 4yy_k/[(x - x_k)^2 + (y - y_k)^2]\}.$$

By the arbitrariness of n,

$$\sum_{1}^{\infty} \log \{1 + 4yy_k/[(x - x_k)^2 + (y - y_k)^2]\} \leq -2 \log |Q(z)F(z)|.$$

Since $Q(z)F(z)$ does not vanish identically and since $\log (1 + x) \sim x$ for small positive x, it follows that

$$\sum_{1}^{\infty} \frac{yy_k}{x_k^2 + (y - y_k)^2} < \infty$$

for some $y > 0$. Since the origin is not a limit point of zeros of $F(z)$,

$$\sum_{1}^{\infty} \frac{y_k}{x_k^2 + y_k^2} < \infty.$$

By Problem 23, $\lim B_n(z) = B(z)$ exists uniformly on any bounded set at a positive distance from the real axis. The limit function is analytic and bounded by 1 in the upper half-plane, and $F(z) = B(z)G(z)$ where $G(z) = \lim F_n(z)$ uniformly on any bounded set at a positive distance from the real axis. It follows that $G(z)$ is analytic in the upper half-plane and that $Q(z)G(z)$ is bounded by 1. Since $Q(z)$ is bounded by 1, $G(z)$ is of bounded type in the upper half-plane. Since the sequence (z_n) is chosen so as to exhaust the zeros of $F(z)$, $G(z)$ has no zeros in the half-plane.

PROBLEM 24. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. Show that there exists a function $Q(z)$, which is analytic and bounded by 1 and which has no zeros in the upper half-plane, such that $P(z) = Q(z)F(z)$ is bounded by 1 in the half-plane.

9. NEVANLINNA'S FACTORIZATION

These results are used to obtain Nevanlinna's factorization of functions of bounded type in a half-plane.

THEOREM 9. Let $F(z)$ be a function which is analytic in the upper half-plane and which does not have the origin as a limit point of zeros. A necessary and sufficient condition that $F(z)$ be of bounded type in the half-plane is that

$$F(z) = B(z) \exp(-ihz) \exp G(z) \,.$$

where $B(z)$ is a Blaschke product, h is a real number, and $G(z)$ is a function analytic in the upper half-plane such that

$$\operatorname{Re} G(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

for some real valued function $\mu(x)$ such that

$$\int_{-\infty}^{+\infty} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

Blaschke products are defined as in Problem 23. The last integral is defined as a supremum of sums

$$\sum_{1}^{r} \frac{|\mu(t_k) - \mu(t_{k-1})|}{1+\max(t_{k-1}^2, t_k^2)}$$

taken over all finite subsets $t_0 < t_1 < \cdots < t_r$ of the real line.

LEMMA 3. Let $\mu(x)$ be a real valued function of real x such that

$$\int_{-\infty}^{+\infty} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

Then $\mu(x) = \sigma(x) - \nu(x)$ for some nondecreasing functions $\sigma(x)$ and $\nu(x)$ such that

$$\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{1+t^2} < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{d\nu(t)}{1+t^2} < \infty.$$

Proof of Lemma 3. We construct a nondecreasing function $\tau(x)$ of real x such that $|\mu(b) - \mu(a)| \le \tau(b) - \tau(a)$ whenever $a < b$ and such that $\int_{-\infty}^{+\infty} (1+t^2)^{-1} d\tau(t) < \infty$. The lemma then follows with

$$\sigma(x) = \tfrac{1}{2}[\tau(x) + \mu(x)] \quad \text{and} \quad \nu(x) = \tfrac{1}{2}[\tau(x) - \mu(x)].$$

If $a > 0$, define

$$\tau(a) = \sup \sum |\mu(t_k) - \mu(t_{k-1})|$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \cdots < t_r = a$ of the interval $(0, a)$. For any such partition,

$$\sum |\mu(t_k) - \mu(t_{k-1})| \leq (1 + a^2) \sum \frac{|\mu(t_k) - \mu(t_{k-1})|}{1 + \max(t_{k-1}^2, t_k^2)}$$
$$\leq (1 + a^2) \int_{-\infty}^{+\infty} (1 + t^2)^{-1} |d\mu(t)|.$$

Therefore $\tau(a) < \infty$. Obviously $\tau(a)$ is a nondecreasing function of $a > 0$. Let $\tau(0) = 0$. Define

$$-\tau(-a) = \sup \sum |\mu(t_k) - \mu(t_{k-1})|$$

where the supremum is taken over all partitions $-a = t_0 < t_1 < \cdots < t_r = 0$ of the interval $(-a, 0)$. Then $\tau(x)$ is a nondecreasing function of real x. If $0 < a < b$ and if $0 = u_0 < u_1 < \cdots < u_r = a$, $a = v_0 < v_1 < \cdots < v_s = b$, then

$$\sum_1^r |\mu(u_i) - \mu(u_{i-1})| + \sum_1^s |\mu(v_j) - \mu(v_{j-1})| \leq \tau(b)$$

by the definition of $\tau(b)$. Since the partition of $(0, a)$ is arbitrary,

$$\tau(a) + \sum_1^s |\mu(v_j) - \mu(v_{j-1})| \leq \tau(b).$$

Therefore $\sup \Sigma |\mu(v_j) - \mu(v_{j-1})| \leq \tau(b) - \tau(a)$ where the supremum is taken over all partitions of (a, b). We show that equality holds.

If $\epsilon > 0$ is given, choose a partition of $(0, b)$ of the form $0 = u_0 < u_1 < \cdots < u_r = a$, $a = v_0 < v_1 < \cdots < v_s = b$, so that

$$\sum_1^r |\mu(u_i) - \mu(u_{i-1})| + \sum_1^s |\mu(v_j) - \mu(v_{j-1})| \geq \tau(b) - \epsilon.$$

This is possible by the definition of $\tau(b)$ since the insertion of the point a in the partition does not decrease the corresponding sum. Since

$$\sum_1^r |\mu(u_i) - \mu(u_{i-1})| \leq \tau(a),$$

we have $\sum_1^s |\mu(v_j) - \mu(v_{j-1})| \geq \tau(b) - \tau(a) - \epsilon$. Since ϵ is arbitrary, $\tau(b) - \tau(a) = \sup \Sigma |\mu(v_j) - \mu(v_{j-1})|$ where the supremum is taken over all partitions of (a, b). A similar argument will show that the same formula holds when $a < b < 0$. It then follows in an obvious way for unrestricted numbers $a < b$. By choosing the trivial partition of (a, b), we find in particular that $|\mu(b) - \mu(a)| \leq \tau(b) - \tau(a)$. We now show that

$$\int_a^b \frac{d\tau(t)}{1 + t^2} = \int_a^b \frac{|d\mu(t)|}{1 + t^2}$$

for $-\infty < a < b < \infty$.

If $a = t_0 < t_1 < \cdots < t_r = b$, then

$$\sum_1^r \frac{|\mu(t_k) - \mu(t_{k-1})|}{1 + \max(t_{k-1}^2, t_k^2)} \leq \sum_1^r \frac{\tau(t_k) - \tau(t_{k-1})}{1 + \max(t_{k-1}^2, t_k^2)}$$

$$\leq \sum_1^r \int_{t_{k-1}}^{t_k} \frac{d\tau(t)}{1 + t^2} \leq \int_a^b \frac{d\tau(t)}{1 + t^2}.$$

Since the partition of (a, b) is arbitrary,

$$\int_a^b \frac{|d\mu(t)|}{1 + t^2} \leq \int_a^b \frac{d\tau(t)}{1 + t^2}.$$

To obtain the reverse inequality, let $\epsilon > 0$ be given. Choose $\delta > 0$ by the uniform continuity of $(1 + x^2)^{-1}$ on (a, b) so that $|(1 + x^2)^{-1} - (1 + t^2)^{-1}| \leq \epsilon$ whenever $|x - t| \leq \delta$ and x and t belong to (a, b). If $a = t_0 < t_1 < \cdots < t_r = b$ is a partition of (a, b) of mesh at most δ, then

$$\int_a^b \frac{|d\mu(t)|}{1 + t^2} = \sum \int_{t_{k-1}}^{t_k} \frac{|d\mu(t)|}{1 + t^2}$$

$$\geq \sum \int_{t_{k-1}}^{t_k} \frac{|d\mu(t)|}{1 + \max(t_{k-1}^2, t_k^2)}$$

$$\geq \sum \frac{\tau(t_k) - \tau(t_{k-1})}{1 + \max(t_{k-1}^2, t_k^2)}$$

$$\geq \sum \left\{ \int_{t_{k-1}}^{t_k} \frac{d\tau(t)}{1 + t^2} - \epsilon[\tau(t_k) - \tau(t_{k-1})] \right\}$$

$$\geq \int_a^b \frac{d\tau(t)}{1 + t^2} - \epsilon[\tau(b) - \tau(a)].$$

The desired identity now follows from the arbitrariness of ϵ. Since a and b are arbitrary,

$$\int_{-\infty}^{+\infty} \frac{d\tau(t)}{1 + t^2} = \int_{-\infty}^{+\infty} \frac{|d\mu(t)|}{1 + t^2} < \infty.$$

Proof of Theorem 9, *the necessity*. By Theorem 8, $F(z) = B(z)L(z)$ where $B(z)$ is a Blaschke product and $L(z)$ is an analytic function which is of bounded type and has no zeros in the upper half-plane. By Problem 24, $L(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are analytic functions which are bounded by 1 and have no zeros in the upper half-plane. We can therefore write $P(z) = 1/\exp V(z)$ and $Q(z) = 1/\exp U(z)$ where $U(z)$ and $V(z)$ are analytic functions whose real parts are nonnegative in the upper half-plane. By Theorem 4 there exist nonnegative numbers p and q and nondecreasing functions $\sigma(x)$ and $\nu(x)$ such that

$$\operatorname{Re} U(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{(t-x)^2+y^2},$$

$$\operatorname{Re} V(x+iy) = qy + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\nu(t)}{(t-x)^2+y^2}$$

for $y > 0$. Let $h = p - q$ and let $G(z) = ihz + U(z) - V(z)$. Then $F(z) = B(z)\exp(-ihz)\exp G(z)$ and

$$\begin{aligned}\operatorname{Re} G(x+iy) &= -hy + \operatorname{Re} U(x+iy) - \operatorname{Re} V(x+iy)\\ &= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{(t-x)^2+y^2} - \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\nu(t)}{(t-x)^2+y^2}\\ &= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}\end{aligned}$$

where $\mu(x) = \sigma(x) - \nu(x)$. If $t_0 < t_1 < \cdots < t_r$ is a finite subset of the real line,

$$\begin{aligned}\sum_1^r \frac{|\mu(t_k)-\mu(t_{k-1})|}{1+\max(t_{k-1}^2, t_k^2)} &\le \sum_1^r \frac{\sigma(t_k)-\sigma(t_{k-1})}{1+\max(t_{k-1}^2, t_k^2)} + \sum_1^r \frac{\nu(t_k)-\nu(t_{k-1})}{1+\max(t_{k-1}^2, t_k^2)}\\ &\le \int_{-\infty}^{+\infty} \frac{d\sigma(t)}{1+t^2} + \int_{-\infty}^{+\infty} \frac{d\nu(t)}{1+t^2} < \infty.\end{aligned}$$

Proof of Theorem 9, *the sufficiency*. Since h is real, we can write $h = p - q$ where p and q are nonnegative. By Lemma 3, $\mu(x) = \sigma(x) - \nu(x)$ where $\sigma(x)$ and $\nu(x)$ are nondecreasing functions of real x such that

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} d\sigma(t) < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} (1+t^2)^{-1} d\nu(t) < \infty.$$

Let $U(z)$ and $V(z)$ be a choice of functions which are analytic in the upper half-plane such that

$$\operatorname{Re} U(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{(t-x)^2+y^2}$$

$$\operatorname{Re} V(x+iy) = qy + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\nu(t)}{(t-x)^2+y^2}$$

for $y > 0$. (See the proof of Theorem 2.) Choose the arbitrary constants in these functions so that $G(z) = ihz + U(z) - V(z)$. Let $P(z) = 1/\exp V(z)$ and $Q(z) = 1/\exp U(z)$. Since the real parts of $U(z)$ and $V(z)$ are nonnegative in the upper half-plane, $P(z)$ and $Q(z)$ are bounded by 1 in the half-plane. Since $F(z) = B(z)P(z)/Q(z)$ where $B(z)P(z)$ and $Q(z)$ are analytic and bounded by 1 in the upper half-plane, $F(z)$ is of bounded type in the half-plane.

PROBLEM 25. Show that

$$y = \frac{1 - |z|^2}{2\pi} \int_0^\pi \frac{4y \sin^2 \theta \, d\theta}{|e^{i\theta} - z|^2 \, |e^{-i\theta} - z|^2}$$

for $|z| < 1$ and $y > 0$. *Hint:* See the proof of Theorem 1.

10. FORMULAS FOR MEAN TYPE

In work with functions of bounded type, it is frequently necessary to refer to the number h which is associated with $F(z)$ by Theorem 9. There is no accepted name for this number in the literature. It is however closely related to the concept of exponential type, and we will call it the mean type of $F(z)$ in the upper half-plane. An analogous concept of mean type can be made for functions which are of bounded type in the lower half-plane $y < 0$. An entire function is said to be of exponential type if

$$\limsup_{|z|\to\infty} \frac{\log |F(z)|}{|z|} < \infty.$$

This limit is then taken as the definition of the exponential type of $F(z)$. By a theorem of M. G. Kreĭn, an entire function is of exponential type if it is of bounded type in the upper and lower half-planes. In this case the exponential type of the function is equal to the maximum of its mean types in the upper and lower half-planes. Thus mean type is a generalization of exponential type to functions which are not necessarily entire. Two useful formulas for mean type are known. One of these gives mean type as an average radial limit in the upper half-plane. The second formula shows that mean type is determined purely by what happens on the imaginary axis.

THEOREM 10. In Theorem 9,

$$h = \lim_{r\to\infty} (2/\pi) r^{-1} \int_0^\pi \log |F(re^{i\theta})| \sin \theta \, d\theta$$

and

$$h = \limsup_{y\to\infty} y^{-1} \log |F(iy)|.$$

These formulas have obvious proofs when $F(z)$ has no zeros. The following lemmas are used to show that the presence of the Blaschke product has a negligible effect on the limits.

LEMMA 4. Let $F(z)$ be a function which is analytic and bounded by 1 in the upper half-plane and which does not vanish identically. If

$$h = \liminf_{r\to\infty} (2/\pi) r^{-1} \int_0^\pi \log |F(re^{i\theta})| \sin \theta \, d\theta,$$

then $h > -\infty$ and $F(z) \exp(ihz)$ is bounded by 1 in the upper half-plane.

LEMMA 5. Let $F(z)$ be a function which is analytic and bounded by 1 in the upper half-plane and which does not vanish identically. If

$$h = \limsup_{y\to\infty} y^{-1} \log |F(iy)|,$$

then $h > -\infty$ and $F(z) \exp(ihz)$ is bounded by 1 in the upper half-plane.

LEMMA 6. If $B(z)$ is a Blaschke product and if h is a real number such that $B(z) \exp(ihz)$ is bounded by 1 in the upper half-plane, then $h \geq 0$.

Proof of Lemma 4. If $\epsilon > 0$ and $p > 0$, the function $F(z + i\epsilon) \exp(-ipz)$ is analytic in the upper half-plane, it has a continuous extension to the closed half-plane, and it is bounded by 1 on the real axis. By the proof of the Phragmén-Lindelöf principle, the inequality

$$\log |F(z + i\epsilon) \exp(-ipz)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log^+ |F(ae^{i\theta} + i\epsilon) \exp(-ipae^{i\theta})| \sin\theta\, d\theta}{|ae^{i\theta} - z|^2\, |ae^{-i\theta} - z|^2}$$

holds for $|z| < a$ and $y > 0$. By Problem 25, the inequality can be written

$$\log |F(z + i\epsilon)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \max\{-pa\sin\theta, \log |F(ae^{i\theta} + i\epsilon)|\} \sin\theta\, d\theta}{|ae^{i\theta} - z|^2\, |ae^{-i\theta} - z|^2}.$$

By the arbitrariness of p and Fatou's theorem,

$$\log |F(z + i\epsilon)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log |F(ae^{i\theta} + i\epsilon)| \sin\theta\, d\theta}{|ae^{i\theta} - z|^2\, |ae^{-i\theta} - z|^2}.$$

Let $\epsilon \searrow 0$. Again by Fatou's theorem,

$$\log |F(z)| \leq \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay \log |F(ae^{i\theta})| \sin\theta\, d\theta}{|ae^{i\theta} - z|^2\, |ae^{-i\theta} - z|^2}$$

for $|z| < a$ and $y > 0$. Let $a \to \infty$ with z fixed. If $|z/a| < \epsilon$ where $\epsilon < 1$, then

$$(1 - \epsilon^2)^{-1}(1 + \epsilon)^4 \log |F(z)| \leq (2y/\pi)a^{-1} \int_0^\pi \log |F(ae^{i\theta})| \sin\theta\, d\theta.$$

By the arbitrariness of a and ϵ, $\log |F(z)| \leq hy$.

Proof of Lemma 5. Explicit proof is restricted to the special case in which $F(z)$ is continuous in the closed half-plane. The general case follows on considering $F(z + i\epsilon)$ where $\epsilon > 0$. The lemma is immediate if $h = 0$. If $h < 0$, consider any number $a > 0$ such that $h < -a$. By the definition of h, $y^{-1} \log |F(iy)| < -a$ for large values of y. For these values of y, $e^{ay} |F(iy)| < 1$. Since $F(iy)$ is bounded by 1, there exists a number $M \geq 1$ such that $|F(iy)| \leq Me^{-ay}$ for $y > 0$. We use this inequality to obtain a rough estimate of $F(z)$ in the upper half-plane.

When $y > 0$, let $\sqrt{z}$ be the choice of square root which lies in the first quadrant. Then

$$G(z) = M^{-1} F(\sqrt{z}) \exp(-ia\sqrt{z})$$

is analytic in the upper half-plane, it has a continuous extension to the closed half-plane, and it is bounded by 1 on the real axis. Since $F(\sqrt{z})$ is bounded by 1 and since

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \sqrt{r} \sin(\tfrac{1}{2}\theta) \sin\theta \, d\theta = 0,$$

we obtain

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \log^+ |G(re^{i\theta})| \sin\theta \, d\theta = 0.$$

By the Phragmén-Lindelöf principle, $G(z)$ is bounded by 1 in the upper half-plane. It follows that $F(z) \exp(-iaz)$ is bounded by M in the first quadrant. The same argument applied to $\bar{F}(-\bar{z})$ will show that $F(z) \exp(-iaz)$ is bounded by M in the second quadrant. So $F(z) \exp(-iaz)$ is bounded by M in the upper half-plane. But the function is continuous in the closed half-plane, and it is bounded by 1 on the real axis. By the Phragmén-Lindelöf principle, it is bounded by 1 in the upper half-plane. The lemma follows by the arbitrariness of a.

Proof of Lemma 6. If $B(z) = \Pi\,(1 - z/z_k)/(1 - z/\bar{z}_k)$, let

$$B_n(z) = \prod_{k>n} (1 - z/z_k)/(1 - z/\bar{z}_k).$$

By the convergence of the product for $B(z)$, $\lim B_n(z) = 1$ as $n \to \infty$. Since

$$B(z) = B_1(z)(1 - z/z_1)/(1 - z/\bar{z}_1)$$

and since $B(z) \exp(ihz)$ is bounded by 1 in the upper half-plane, $B_1(z) \exp(ihz)$ is bounded in the upper half-plane. If $\epsilon > 0$, then $B_1(z + i\epsilon) \exp(ihz)$ is bounded in the upper half-plane and continuous in the closed half-plane. By the proof of Theorem 8, the function is bounded by $(\epsilon + y_1)/|\epsilon - y_1|$ on the real axis. By the Phragmén-Lindelöf principle, the function has the same bound in the upper half-plane. By the arbitrariness of ϵ, $B_1(z) \exp(ihz)$ is bounded by 1 in the upper half-plane. Continue inductively in the obvious way. Since $B_n(z) \exp(ihz)$ is bounded by 1 in the upper half-plane for every n, $\exp(ihz)$ is bounded by 1 in the upper half-plane, and $h \geq 0$.

Proof of Theorem 10. By the proof of Theorem 9, we can write

$$F(z) = B(z) \exp(-ihz)\, P(z)/Q(z)$$

where $P(z)$ and $Q(z)$ are functions which are analytic and bounded by 1 in the upper half-plane, which have no zeros in the half-plane, and which satisfy

$$-\log|Q(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\sigma(t)}{(t-x)^2+y^2},$$

$$-\log|P(x+iy)| = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\nu(t)}{(t-x)^2+y^2}$$

for some nondecreasing functions $\sigma(x)$ and $\nu(x)$. By Lemmas 4, 5, and 6,

$$0 = \lim_{r\to\infty} (2/\pi)r^{-1}\int_0^\pi \log|B(re^{i\theta})|\sin\theta\, d\theta$$

and

$$0 = \limsup_{y\to\infty} y^{-1}\log|B(iy)|.$$

The theorem follows once we show that

$$0 = \lim_{r\to\infty} (2/\pi)r^{-1}\int_0^\pi \log|P(re^{i\theta})|\sin\theta\, d\theta$$

and

$$0 = \lim_{y\to\infty} y^{-1}\log|P(iy)|$$

and that the same formulas hold with $P(z)$ replaced by $Q(z)$. The second of these formulas is true by Problem 4. The first formula now follows by Lemma 4.

PROBLEM 26. If $B(z)$ is a Blaschke product, show that

$$\lim_{\epsilon\searrow 0}\int_{-\infty}^{+\infty}(1+t^2)^{-1}\log|B(t+i\epsilon)|dt = 0.$$

Hint: If $B_n(z)$ is defined as in the proof of Lemma 6, show that

$$\pi\log|B_n(i)| \le \liminf_{\epsilon\searrow 0}\int_{-\infty}^{+\infty}(1+t^2)^{-1}\log|B(t+i\epsilon)|dt$$

for every n.

PROBLEM 27. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. Assume that $|F(z)|$ has a continuous extension to the closed half-plane. If $\mu(x)$ and $G(z)$ are defined as in Theorem 9, show that

$$\mu(b) - \mu(a) = \lim_{y\searrow 0}\int_a^b \log|F(x+iy)|dx$$

whenever a and b are points of continuity of $\mu(x)$, $a < b$. Show that

$$\mu(b) - \mu(a) \le \int_a^b \log|F(t)|dt$$

whenever $a < b$ and that equality holds if $|F(x)| \neq 0$ for $a \leq x \leq b$. Show that

$$\int_a^b |\log|F(t)||dt \leq \int_a^b |d\mu(t)|.$$

Show that

$$\operatorname{Re} G(x+iy) \leq \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\log|F(t)|dt}{(t-x)^2+y^2}$$

for $y > 0$ and that equality holds if $|F(x)| \neq 0$ for all real x.

The mean type has been defined only for functions which are nonzero in a neighborhood of the origin. If a function $F(z)$ is analytic and of bounded type in the upper half-plane, then the mean type of $F(z + i\epsilon)$ is defined for every positive ϵ, and it does not depend on ϵ. This number is equal to the mean type of $F(z)$ if the origin is not a limit point of zeros of $F(z)$. Otherwise we take it as the definition of the mean type of $F(z)$. The mean type of the function which is identically zero is taken to be $-\infty$.

PROBLEM 28. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. Show that the mean type of $F(z - a)$ is equal to the mean type of $F(z)$ for every real number a.

PROBLEM 29. Let $F(z)$ and $G(z)$ be functions which are analytic and of bounded type in the upper half-plane. Show that the mean type of $F(z) + G(z)$ does not exceed the maximum of the mean types of $F(z)$ and $G(z)$. Show that the mean type of $F(z)G(z)$ is the sum of the mean types of $F(z)$ and $G(z)$.

PROBLEM 30. Show that a function which is analytic and has a nonnegative real part in the upper half-plane has zero mean type in the half-plane if it does not vanish identically.

PROBLEM 31. Show that a nonzero polynomial has zero mean type in the upper half-plane.

11. CONDITIONS FOR BOUNDED TYPE

The following condition for bounded type is often used.

THEOREM 11. Let $F(z)$ be a function which is analytic in the upper half-plane, such that $|F(z)|$ has a continuous extension to the closed half-plane. Then $F(z)$ is of bounded type in the half-plane if

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} \log^+ |F(t)|dt < \infty,$$

if

$$\liminf_{r\to\infty} r^{-2} \int_0^\pi \log^+ |F(re^{i\theta})| \sin\theta \, d\theta = 0,$$

and if

$$\limsup_{y\to\infty} y^{-1} \log |F(iy)| < \infty.$$

Proof of Theorem 11. By Theorem 2, there exists a function $Q(z)$, which is analytic and has no zeros for $y > 0$, such that

$$-\log |Q(x+iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\log^+ |F(t)| dt}{(t-x)^2 + y^2}$$

for $y > 0$. The function $Q(z)$ is bounded by 1 in the upper half-plane, $|Q(z)|$ has a continuous extension to the closed half-plane, and $|1/Q(x)| = \max(1, |F(x)|)$ for all real x. Let h be the choice of a real number such that

$$h > \limsup_{y\to\infty} y^{-1} \log |F(iy)|.$$

Then the function $P(z) = Q(z)e^{ihz}F(z)$ is analytic in the upper half-plane, $|P(z)|$ is continuous in the closed half-plane, and $|P(x)| \le 1$ for all real x. Since

$$0 = \lim_{y\to\infty} y^{-1} \log |Q(iy)|$$

by Problem 4, we have

$$\limsup_{y\to\infty} y^{-1} \log |P(iy)| < 0.$$

It follows that $P(z)$ is bounded on the imaginary axis. Since $Q(z)$ is bounded by 1, the hypotheses imply that

$$\liminf_{r\to\infty} r^{-2} \int_0^\pi \log^+ |P(re^{i\theta})| \sin\theta \, d\theta = 0.$$

When $y > 0$, let $\sqrt{z}$ be the choice of square root which lies in the first quadrant. Then $P(\sqrt{z})$ is analytic in the upper half-plane. Its modulus is continuous in the closed half-plane and is bounded on the real axis. It follows from the last written limit that

$$\liminf_{r\to\infty} r^{-1} \int_0^\pi \log^+ |P(\sqrt{r}\, e^{\frac{1}{2}i\theta})| \sin\theta \, d\theta = 0.$$

By the Phragmén-Lindelöf principle, $P(\sqrt{z})$ is bounded in the upper half-plane. In other words, $P(z)$ is bounded in the first quadrant. The same argument with $P(z)$ replaced by $P^*(-z)$ will show that $P(z)$ is bounded in the upper half-plane. But $|P(z)|$ is continuous in the closed half-plane and is bounded by 1 on the real axis. By the Phragmén-Lindelöf principle, $P(z)$ is bounded by 1 in the upper half-plane. The theorem follows.

PROBLEM 32. Show that, for $y > 0$,

$$\left(\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|dt}{(t-x)^2+y^2}\right)^2 \leq \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|^2dt}{(t-x)^2+y^2}$$

if $f(x)$ is a Borel measurable function of real x. Show also that Jensen's inequality

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f(t)|dt}{(t-x)^2+y^2} \leq \log\left(\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|dt}{(t-x)^2+y^2}\right)$$

holds for $y > 0$. *Hint:*

$$\log x = \int_1^x t^{-1}dt = \lim_{h\searrow 0}(x^h - 1)/h.$$

12. CAUCHY'S FORMULA IN A HALF-PLANE

The bounded type theory is used to establish Cauchy's formula in the upper half-plane.

THEOREM 12. Let $f(z)$ be a function which is analytic and of bounded type in the upper half-plane, and which has a continuous extension to the closed half-plane. Assume that the mean type of $f(z)$ is not positive and that $\int_{-\infty}^{+\infty}|f(t)|^2dt < \infty$. Then

$$2\pi i f(z) = \int_{-\infty}^{+\infty}(t-z)^{-1}f(t)dt$$

for $y > 0$ and

$$0 = \int_{-\infty}^{+\infty}(t-z)^{-1}f(t)dt$$

for $y < 0$.

Proof of Theorem 12. By Problem 27 the hypotheses imply that

$$\log|f(x+iy)| \leq \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f(t)|dt}{(t-x)^2+y^2}$$

for $y > 0$. Since the inequality

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log|f(t)|dt}{(t-x)^2+y^2} \leq \log\left(\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|dt}{(t-x)^2+y^2}\right)$$

holds by Problem 32, we obtain

$$|f(x+iy)| \leq \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|f(t)|dt}{(t-x)^2+y^2}$$

for $y > 0$. By the proof of Theorem 2, there exists a function $g(z)$, which is analytic in the upper half-plane, such that

$$\operatorname{Re} g(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|f(t)|\,dt}{(t-x)^2+y^2}$$

for $y > 0$. Then $|f(x+iy)| \leq \operatorname{Re} g(x+iy)$ for $y > 0$. Therefore the function $g(z) - f(z)$ is analytic and has a nonnegative real part in the upper half-plane. By the proof of Theorem 2, $\operatorname{Re} g(z)$ has a continuous extension to the closed half-plane and $|f(x)| = \lim \operatorname{Re} g(x+iy)$ as $y \searrow 0$ for all real x. Since $f(z)$ has a continuous extension to the closed half-plane, $\operatorname{Re}[g(z) - f(z)]$ has a continuous extension to the closed half-plane and is equal to $|f(x)| - \operatorname{Re} f(x)$ on the boundary. By Problems 2 and 3, there is a number $p \geq 0$ such that

$$\operatorname{Re}[g(z) - f(z)] = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|f(t)| - \operatorname{Re} f(t)}{(t-x)^2+y^2}\,dt$$

for $y > 0$. For the same reasons there is a number $q \geq 0$ such that

$$\operatorname{Re}[g(z) + f(z)] = qy + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|f(t)| + \operatorname{Re} f(t)}{(t-x)^2+y^2}\,dt$$

for $y > 0$. By the definition of $g(z)$, we can conclude that $p = q = 0$ and that

$$\operatorname{Re} f(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{Re} f(t)\,dt}{(t-x)^2+y^2}$$

for $y > 0$. Since the functions $g(z) + if(z)$ and $g(z) - if(z)$ are analytic in the upper half-plane and since the real parts of the functions are nonnegative, the same argument will show that

$$\operatorname{Re} if(x+iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{Re} if(t)\,dt}{(t-x)^2+y^2}$$

for $y > 0$. It follows that

$$\begin{aligned} f(z) &= \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(t)\,dt}{(t-x)^2+y^2} \\ &= (2\pi i)^{-1}\int_{-\infty}^{+\infty} (t-z)^{-1} f(t)\,dt - (2\pi i)^{-1}\int_{-\infty}^{+\infty} (t-\bar{z})^{-1} f(t)\,dt \end{aligned}$$

for $y > 0$. But the first integral represents a function analytic in the upper half-plane and the second integral represents a function whose complex conjugate is analytic. But if a function and its conjugate are both analytic, then the derivative of the function vanishes identically and the function is a constant. Since $\int_{-\infty}^{+\infty} (t - \bar{z})^{-1} f(t)\,dt$ is a constant for $y > 0$ and since

$$\left|\int_{-\infty}^{+\infty} (t - \bar{z})^{-1} f(t)\,dt\right|^2 \le \int_{-\infty}^{+\infty} |f(t)|^2 dt \int_{-\infty}^{+\infty} |t - z|^{-2} dt \le (\pi/y) \int_{-\infty}^{+\infty} |f(t)|^2 dt,$$

the integral vanishes identically.

13. FACTORIZATION OF POSITIVE FUNCTIONS

If $F(z)$ is any entire function, define $F^*(z) = \bar{F}(\bar{z})$. Note that $F^*(z)$ is also an entire function and that it coincides with the conjugate of $F(z)$ on the real axis. If an entire function $F(z)$ is real for real z, then $F(z)$ coincides with $F^*(z)$ on the real axis, and hence, by analytic continuation, in the complex plane. Thus an entire function $F(z)$ is real for real z if, and only if, $F^*(z) = F(z)$. If an entire function $P(z)$ is of the form $P(z) = Q(z)Q^*(z)$ for some entire function $Q(z)$, then the values of $P(z)$ are nonnegative on the real axis. The converse is true if the zeros of $P(z)$ are sufficiently near the real axis.

THEOREM 13. Let $P(z)$ be an entire function which has nonnegative values on the real axis and which does not vanish identically. Let (z_n) be the zeros of $P(z)$ in the upper half-plane, zeros repeated according to multiplicity. If

$$\sum \frac{y_n}{x_n^2 + y_n^2} < \infty,$$

then $P(z) = Q(z)Q^*(z)$ for some entire function $Q(z)$, which has no zeros in the upper half-plane, such that $|Q(x - iy)| \le |Q(x + iy)|$ for $y > 0$.

Proof of Theorem 13. By Problem 23 the Blaschke product

$$B(z) = \prod (1 - z/z_n)/(1 - z/\bar{z}_n)$$

converges. Since the zeros of $B(z)$ are zeros of $P(z)$, the function $F(z) = P(z)/B(z)$ is entire. Since $P^*(z) = P(z)$, the nonreal zeros of $P(z)$ occur in conjugate pairs. It follows that the nonreal zeros of $F(z)$ have even multiplicities. Since $P(z)$ is nonnegative on the real axis, its real zeros have even multiplicities. Therefore all zeros of $F(z)$ have even multiplicities, and we can write $F(z) = Q(z)^2$ for some entire function $Q(z)$. Since the zeros of $B(z)$ exhaust the zeros of $P(z)$ in the upper half-plane, $Q(z)$ has no zeros in the half-plane. Since $P^*(z) = P(z)$, since $B^*(z)B(z) = 1$, and since $B(z)$ is bounded by 1 in the upper half-plane, $|Q(x - iy)| \le |Q(x + iy)|$ for $y > 0$. By construction, $P(z)^2 = [Q(z)Q^*(z)]^2$. Since $P(z)$ and $Q(z)Q^*(z)$ are nonnegative on the real axis, they are identical in the complex plane.

14. CONDITIONS FOR PÓLYA CLASS

The next result is a method for proving that a given entire function is of Pólya class.

THEOREM 14. Let $E(z)$ be an entire function which has no zeros in the upper half-plane, which satisfies the inequality

$$|E(x - iy)| \leq |E(x + iy)|$$

for $y > 0$, and which has value one at the origin. Let $\log E(z)$ be defined continuously in the upper half-plane so as to have limit zero at the origin. Then a necessary and sufficient condition that $E(z)$ be of Pólya class is that

$$\operatorname{Re} i[\log E(z)]/z \geq 0$$

for $y > 0$.

Proof of Theorem 14, *the necessity.* By Problem 13 there exists a sequence $(E_n(z))$ of polynomials of Pólya class such that $E(z) = \lim E_n(z)$ uniformly on bounded sets. Since $E(z)$ has value one at the origin, we can choose the approximating polynomials to have value one at the origin. Since

$$\log E(z) = \lim_{n\to\infty} \log E_n(z)$$

for $y > 0$, it is sufficient to prove necessity in the case that $E(z)$ is a polynomial. In this case

$$E(z) = (1 - z/\bar{w}_1) \cdots (1 - z/\bar{w}_r)$$

where $w_1, \cdots, w_r$ are nonzero numbers which lie on or above the real axis. Since

$$\log E(z) = \log (1 - z/\bar{w}_1) + \cdots + \log (1 - z/\bar{w}_r),$$

it is sufficient to prove the theorem in the case that $E(z) = 1 - z/\bar{w}$ is a linear function. It remains to show that the expression

$$\operatorname{Re} i[\log (1 - z/\bar{w})]/z$$

is nonnegative when z is in the upper half-plane and w lies on or above the real axis. To see this, write $w = \rho e^{i\varphi}$ where $0 \leq \varphi \leq \pi$. Since

$$\partial/\partial\rho \operatorname{Re} i[\log (1 - ze^{i\varphi}/\rho)]/z = -(\sin \varphi + y/\rho)/|z - \bar{w}|^2 < 0,$$

the expression decreases as ρ increases for any fixed z and φ. Since the expression has limit zero as $\rho \to \infty$, it is positive for all $\rho > 0$.

Proof of Theorem 14, *the sufficiency.* Define

$$\varphi(x) = \lim_{y \searrow 0} \operatorname{Re} i \log E(x + iy)$$

when $E(x) \neq 0$. Since the function is differentiable at all such points, since

$$\varphi'(x) = \lim_{y \searrow 0} \partial/\partial y \log |E(x + iy)|,$$

and since $|E(x - iy)| \leq |E(x + iy)|$ for $y > 0$, $\varphi'(x) \geq 0$ whenever $\varphi(x)$ is defined. If h is a real zero of $E(z)$ of order r, then $\varphi(h+)$ and $\varphi(h-)$ exist and $\varphi(h+) - \varphi(h-) = \pi r$. It follows that $\varphi(x)$ is a nondecreasing function of real x. Since $\varphi(x)$ vanishes at the origin, $\varphi(x)/x \geq 0$ at all points of continuity. By the Poisson representation, Theorem 4, there exists a number $a \geq 0$ such that

$$\operatorname{Re} i \frac{\log E(z)}{z} = ay + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(t)/t \, dt}{(t - x)^2 + y^2}$$

for $y > 0$.

If $E(z)$ has no zeros, then $\log E(z)$ is an entire function, and $E^*(z)/E(z)$ is an entire function which has no zeros, which is bounded by one in the upper half-plane, which has absolute value one on the real axis, and which has value one at the origin. By Nevanlinna's factorization, Theorem 9, $E^*(z)/E(z) = \exp(2ihz)$ for some number $h \geq 0$. The function $E(z) \exp(ihz)$ is real for real z, $\varphi(x) = hx$ for all real x, and

$$\operatorname{Re} i[\log E(z)]/z = h + ay$$

for $y > 0$. It follows that there exists a real number k such that

$$i[\log E(z)]/z = h + ik - iaz.$$

The function

$$E(z) = \exp(kz - ihz - az^2)$$

is then of Pólya class.

If $E^*(z)$ has a zero w_1, then $w_1 \neq 0$ and $E(z) = E_1(z)(1 - z/\bar{w}_1)$ where $E_1(z)$ is an entire function which has no zeros in the upper half-plane, which satisfies the inequality $|E_1(x - iy)| \leq |E_1(x + iy)|$ for $y > 0$, and which has value one at the origin. As we have seen,

$$\varphi_1(x) = \lim_{y \searrow 0} \operatorname{Re} i \log E_1(x + iy)$$

is a nondecreasing function of x. Since

$$\operatorname{Re} i \frac{\log(1 - z/\bar{w}_1)}{z} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{[\varphi(t) - \varphi_1(t)]/t}{(t - x)^2 + y^2} \, dt$$

for $y > 0$, we can conclude that

$$\operatorname{Re} i \frac{\log E_1(z)}{z} = ay + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi_1(t)/t}{(t - x)^2 + y^2} \, dt$$

is nonnegative for $y > 0$. We have seen that such a function $E_1(z)$ is of Pólya class if it has no zeros. In this case it follows immediately that $E(z)$ is of Pólya class. An inductive argument will show that $E(z)$ is of Pólya class if it has only a finite number of zeros.

If $E(z)$ has an infinite number of zeros, let (w_n) be an enumeration of the zeros of $E^*(z)$. For every n,

$$E(z) = E_n(z)(1 - z/\bar{w}_1) \cdots (1 - z/\bar{w}_n)$$

where $E_n(z)$ is an entire function which has no zeros in the upper half-plane, which satisfies the inequality $|E_n(x - iy)| \leq |E_n(x + iy)|$ for $y > 0$, which has value one at the origin, and which satisfies the inequality

$$\operatorname{Re} i[\log E_n(z)]/z \geq 0$$

for $y > 0$. It follows that

$$\operatorname{Re} i[\log (1 - z/\bar{w}_1) + \cdots + \log (1 - z/\bar{w}_n)]/z \leq \operatorname{Re} i[\log E(z)]/z$$

for $y > 0$. When $z = i$ the inequality implies that

$$\sum_{n=1}^{\infty} (1 + i\bar{w}_n - iw_n)/|w_n|^2 < \infty.$$

By Problem 9, the product

$$F(z) = \prod_{n=1}^{\infty} (1 - z/\bar{w}_n)e^{h_n z}$$

converges uniformly on bounded sets and represents an entire function of Pólya class if we choose $h_n = \operatorname{Re} 1/w_n$ for every n. It follows that $E(z) = F(z)G(z)$ where

$$G(z) = \lim_{n \to \infty} E_n(z) \exp (-h_1 z - \cdots - h_n z)$$

is an entire function which has no zeros, which satisfies the inequality $|G(x - iy)| \leq |G(x + iy)|$ for $y > 0$, which has value one at the origin, and which satisfies the inequality

$$\operatorname{Re} i[\log G(z)]/z \geq 0$$

for $y > 0$. Since we have shown that such a function $G(z)$ is of Pólya class, we can conclude that $E(z)$ is of Pólya class.

PROBLEM 33. If a function $F(z)$ is analytic and of bounded type in the upper half-plane, if it has no zeros in the half-plane, and if $\log F(z)$ is defined continuously in the half-plane, show that

$$|z - \bar{z}|\,|\log F(z)|/|z + i|^2$$

is bounded in the half-plane. *Hint:* Reduce the problem to the case in which $\operatorname{Re} \log F(z) \geq 0$ for $y > 0$, apply the Poisson representation, and use the inequality

$$\left|\frac{t - i}{t - z}\right| \leq \frac{|z - i| + |z + i|}{|z - \bar{z}|}.$$

PROBLEM 34. Let $E(z)$ be an entire function which has no zeros in the upper half-plane and which satisfies the inequality $|E(x - iy)| \leq |E(x + iy)|$ for $y > 0$. Show that $E(z)$ is of Pólya class if there exists an entire function $F(z)$ of Pólya class such that $E(z)/F(z)$ is of bounded type in the upper half-plane.

PROBLEM 35. If $E(z)$ is an entire function of Pólya class such that $E^*(z) = E(-z)$, show that

$$|E(z)| \leq \big|E(\,i\,|z|)\big|$$

for all complex z.

PROBLEM 36. Let $F(z)$ be an entire function such that $F(z)$ and $F^*(z)$ are of bounded type in the upper half-plane. Show that there exists an entire function $E(z)$ of Pólya class such that

$$F(z)F^*(z) + F(-z)F^*(-z) = E(z)E^*(z),$$

such that $E^*(z) = E(-z)$, and such that

$$|F(x + iy)| \leq \big|E(x + i\,|y|)\big|$$

for all complex z. Show that $E(z)$ can be chosen of bounded type in the upper half-plane and of mean type equal to the maximum of the mean types of $F(z)$ and $F^*(z)$ in the upper half-plane.

PROBLEM 37. Prove Kreĭn's theorem that an entire function $F(z)$ is of exponential type if it is of bounded type in the upper half-plane and if $F^*(z)$ is of bounded type in the upper half-plane. Show that the exponential type of $F(z)$ is the maximum of the mean types of $F(z)$ and $F^*(z)$ in the upper half-plane.

PROBLEM 38. Show that an entire function of zero exponential type is bounded in the complex plane, and hence is a constant, if it is bounded on the real axis.

PROBLEM 39. Show that an entire function $F(z)$ is a constant if $F(z)$ and $F^*(z)$ are of bounded type in the upper half-plane and if $F(z)$ is bounded on the imaginary axis.

15. ANOTHER FORMULA FOR MEAN TYPE

A theorem of Titchmarsh and Valiron evaluates mean type for functions of Pólya class. The method applies also to certain functions which are not entire.

THEOREM 15. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane and which has no zeros in the half-plane. Assume that $|F(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x. Then there exists a nondecreasing function $\psi(x)$ of real x such that

$$\operatorname{Re} i\,\frac{F'(z)}{F(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\psi(t)}{(t-x)^2+y^2}$$

for $y > 0$. The mean type τ of $F(z)$ in the upper half-plane is given by

$$\tau = \lim_{|x|\to\infty} \psi(x)/x.$$

Proof of Theorem 15. Since we assume that $|F(x+iy)|$ is a nondecreasing function of $y > 0$ for each fixed x, the real part of $iF'(z)/F(z)$ is nonnegative in the upper half-plane. By Theorem 4 there exists a nonnegative number p and a nondecreasing function $\psi(x)$ of real x such that $\psi(0) = 0$ and such that

$$\operatorname{Re} i\,\frac{F'(z)}{F(z)} = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\psi(t)}{(t-x)^2+y^2}$$

for $y > 0$. Since $\operatorname{Re} iF'(z)/F(z) \geq py$, we obtain

$$\log|F(iy)/F(i)| \geq \tfrac{1}{2}py^2 - \tfrac{1}{2}p$$

for $y > 1$. Since

$$\lim_{y\to+\infty} y^{-1}\log|F(iy)| = \tau < \infty$$

by Theorem 10, we must have $p = 0$. Let $\varphi(x)$ be a nondecreasing function of real x, whose values are integer multiples of π, such that $|\psi(x) - \varphi(x)| \leq \frac{1}{2}\pi$ for all real x. Let $G(z)$ be an entire function of Pólya class, which is real for real z, such that $G(0) = 1$ and such that

$$\operatorname{Re} i\,\frac{G'(z)}{G(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\varphi(t)}{(t-x)^2+y^2}$$

for $y > 0$. Then

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\psi(t)-\varphi(t)}{(t-x)^2+y^2}\,dt$$

is an absolutely convergent integral which represents the real part of a function analytic in the upper half-plane, and

$$\begin{aligned}\frac{\partial}{\partial x}\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\psi(t)-\varphi(t)}{(t-x)^2+y^2}\,dt &= \frac{y}{\pi}\int_{-\infty}^{+\infty}[\psi(t)-\varphi(t)]\frac{\partial}{\partial x}\frac{1}{(t-x)^2+y^2}\,dt\\ &= -\frac{y}{\pi}\int_{-\infty}^{+\infty}[\psi(t)-\varphi(t)]\frac{\partial}{\partial t}\frac{1}{(t-x)^2+y^2}\,dt\\ &= \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d[\psi(t)-\varphi(t)]}{(t-x)^2+y^2}\\ &= \operatorname{Re}\,[iF'(z)/F(z)-iG'(z)/G(z)]\end{aligned}$$

for $y > 0$. Differentiation under the integral sign is justified by absolute convergence of the equivalent integral. Integration by parts is permissible because $\psi(t)-\varphi(t)$ is bounded. If $\log F(z)$ and $\log G(z)$ are defined continuously in the upper half-plane, then

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\psi(t)-\varphi(t)}{(t-x)^2+y^2}\,dt \quad\text{and}\quad \operatorname{Re}\,[i\log F(z)-i\log G(z)]$$

differ by a real constant. Since $F(z)$ can be multiplied by a constant of absolute value 1 without changing the hypotheses or conclusion of the theorem, we can restrict explicit proof to the case in which

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\psi(t)-\varphi(t)}{(t-x)^2+y^2}\,dt = \operatorname{Re}\,[i\log F(z)-i\log G(z)]$$

for $y > 0$. Since $|\psi(x)-\varphi(x)| \le \frac{1}{2}\pi$ for all real x,

$$|\operatorname{Re}\,[i\log F(z)-i\log G(z)]| \le \tfrac{1}{2}\pi$$

for $y > 0$. It follows that

$$\operatorname{Re}\,[G(z)/F(z)] \ge 0$$

for $y > 0$. By Problems 20 and 30, $G(z)/F(z)$ is of bounded type and of zero mean type in the upper half-plane. Since we assume that $F(z)$ is of bounded type and of mean type τ in the half-plane, $G(z)$ is of bounded type and of mean type τ in the half-plane. By Problem 27,

$$\log|G(x+iy)| = \tau y + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{\log|G(t)|\,dt}{(t-x)^2+y^2}$$

for $y > 0$, the integral being absolutely convergent. It follows that

$$\frac{\log G(z) + \log \bar{G}(w)}{i(\bar{w} - z)} = \tau + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\log |G(t)|}{(t - z)(t - \bar{w})}\, dt$$

when z and w are in the upper half-plane. By the Lebesgue dominated convergence theorem,

$$\lim_{r\to\infty} \frac{\log G(re^{i\theta})}{r} = -i\tau e^{i\theta}$$

for $0 < \theta < \pi$. Convergence is uniform in any sector $\delta < \theta < \pi - \delta$ where $0 < \delta < \frac{1}{2}\pi$. The function $i[\log G(z)]/z$ is analytic in the upper half-plane and continuous in the closed half-plane except for isolated singularities at the discontinuities of $\varphi(x)$. If $a > 0$ and if $0 < \delta < \frac{1}{2}\pi$, then

$$i\int_0^a \frac{\log G(re^{i\delta})}{r}\, dr - \int_\delta^{\frac{1}{2}\pi} \log G(ae^{i\theta})\, d\theta - i\int_0^a \frac{\log G(ir)}{r}\, dr = 0$$

by Cauchy's formula. In the limit as $\delta \searrow 0$ we obtain

$$\int_0^a \frac{\varphi(x)}{x}\, dx = \int_0^a \operatorname{Re} i\,\frac{\log G(iy)}{y}\, dy + \int_0^{\frac{1}{2}\pi} \log |G(ae^{i\theta})|\, d\theta.$$

It follows that

$$\limsup_{a\to\infty} \frac{1}{a}\int_0^a \frac{\varphi(t)}{t}\, dt = \limsup_{a\to\infty} \int_0^{\frac{1}{2}\pi} \frac{\log |G(ae^{i\theta})|}{a}\, d\theta,$$

and similarly for lower limits. Since $G(z)$ is of exponential type by Kreĭn's theorem, Problem 37, the integrand on the right has an upper bound. By Fatou's theorem,

$$\limsup_{a\to\infty} \frac{1}{a}\int_0^a \frac{\varphi(t)}{t}\, dt \le \int_0^{\frac{1}{2}\pi} \tau \sin\theta\, d\theta = \tau.$$

Since $\varphi(x)/x \ge 0$,

$$\limsup_{a\to\infty} \frac{1}{a}\int_{\frac{1}{2}a}^a \frac{\varphi(t)}{t}\, dt \le \tau.$$

Since $\varphi(x)$ is a nondecreasing function of x,

$$\limsup_{a\to\infty} \frac{\varphi(\frac{1}{2}a)}{a} \log 2 \le \tau.$$

From this we see that $\varphi(x)/x \le M$ is bounded on the half-line $(0, \infty)$. The set of points $x > 0$ such that $\log |G(x)| < -x$ is a union of disjoint intervals (u_n, v_n) such that

$$\tfrac{1}{2}\sum \log \frac{1 + v_n^2}{1 + u_n^2} = \sum \int_{u_n}^{v_n} \frac{t\, dt}{1 + t^2} \le \int_0^\infty \frac{\log^+ |1/G(t)|}{1 + t^2}\, dt < \infty.$$

From this we see that $\lim v_n/u_n = 1$ as $n \to \infty$. If $0 < \delta < \frac{1}{2}\pi$, then

$$\lim_{a\to\infty} \int_\delta^{\frac{1}{2}\pi} \frac{\log |G(ae^{i\theta})|}{a}\, d\theta = \int_\delta^{\frac{1}{2}\pi} \tau \sin \theta\, d\theta = \tau \cos \delta$$

by uniform convergence of the integrand. On the other hand,

$$\lim_{a\to\infty} \int_0^\delta \frac{\log |G(ae^{i\theta})|}{a}\, d\theta = \int_0^\delta \tau \sin \theta\, d\theta = \tau(1 - \cos \delta)$$

by the Lebesgue dominated convergence theorem if the limit is taken with a outside the union of the intervals $(u_n \cos \delta, v_n/\cos \delta)$. For

$$\frac{\log |G(ae^{i\theta})|}{a} \geq \frac{\log |G(a \cos \theta)|}{a} \geq -1$$

and the integrand is bounded on this set. It follows that

$$\lim_{a\to\infty} \frac{1}{a} \int_0^a \frac{\varphi(t)}{t}\, dt = \lim_{a\to\infty} \int_0^{\frac{1}{2}\pi} \frac{\log |G(ae^{i\theta})|}{a}\, d\theta = \tau$$

if the limit is taken with a outside the union of the intervals $(u_n \cos \delta, v_n/\cos \delta)$. But if $a < b$,

$$\left|b^{-1} \int_0^b \varphi(t)/t\, dt - a^{-1} \int_0^a \varphi(t)/t\, dt\right|$$

$$\leq b^{-1} \int_a^b \varphi(t)/t\, dt + (a^{-1} - b^{-1}) \int_0^a \varphi(t)/t\, dt$$

$$\leq 2M(1 - a/b).$$

It follows that

$$\tau = \lim_{a\to\infty} a^{-1} \int_0^a \varphi(t)/t\, dt.$$

If $s > 1$, we have

$$(s - 1)\tau = \lim_{a\to\infty} a^{-1} \int_a^{sa} \varphi(t)/t\, dt.$$

Since $\varphi(x)$ is nondecreasing,

$$\log s \limsup_{a\to\infty} \varphi(a)/a \leq (s - 1)\tau \leq s \log s \liminf_{a\to\infty} \varphi(a)/a.$$

By the arbitrariness of s, $\tau = \lim \varphi(x)/x$ as $x \to +\infty$. Since the same conclusion holds with $G(z)$ replaced by $G(-z)$, we have $\tau = \lim \varphi(x)/x$ as $x \to -\infty$. The theorem follows because the difference $\psi(x) - \varphi(x)$ is bounded.

CHAPTER 2

Eigenfunction Expansions

16. CONSTRUCTION OF PALEY-WIENER SPACES

The theory of Hilbert spaces of entire functions is a generalization of Fourier analysis. If $f(x)$ belongs to $L^2(-\infty, +\infty)$, its Fourier transform $F(x)$ is the function defined formally by

$$2\pi F(x) = \int_{-\infty}^{+\infty} e^{-ixt} f(t)\, dt.$$

In general the integral does not converge. But if $f(x)$ belongs to $L^1(-\infty, +\infty)$, the Fourier transform $F(x)$ is a well-defined, continuous function of real x. The definition of $F(x)$ in the general case depends on Plancherel's formula, which states that

$$2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt.$$

Since a dense set of elements of $L^2(-\infty, +\infty)$ belong to $L^1(-\infty, +\infty)$, there exists a unique way to define the Fourier transformation in $L^2(-\infty, +\infty)$ which preserves the Plancherel formula and is consistent with the Fourier transformation for elements of $L^1(-\infty, +\infty)$.

An example of a function $f(x)$ in $L^2(-\infty, +\infty)$ which also belongs to $L^1(-\infty, +\infty)$ is one which vanishes outside of a finite interval $(-a, a)$. In this case its Fourier transform $F(x)$ can be extended to a function of a complex variable z by

$$2\pi F(z) = \int_{-\infty}^{+\infty} e^{-izt} f(t)\, dt.$$

The integral is absolutely convergent for all complex z and represents an entire function. An arbitrary element $f(x)$ of $L^2(-\infty, +\infty)$ can be approximated by functions vanishing outside of finite intervals. The obvious approximation method gives

$$2\pi F(x) = \lim_{a\to\infty} \int_{-a}^{a} e^{-ixt} f(t)\,dt$$

as the definition of the Fourier transform of $f(x)$. Convergence is in the metric of $L^2(-\infty, +\infty)$.

A fundamental theorem of Fourier analysis states that every function in $L^2(-\infty, +\infty)$ can be written as the Fourier transform $F(x)$ of an element $f(x)$ of $L^2(-\infty, +\infty)$. An equally fundamental theorem, due to Paley and Wiener, characterizes those functions in $L^2(-\infty, +\infty)$ which are Fourier transforms of functions which vanish outside of a given interval $(-a, a)$. If $f(x)$ vanishes outside of the interval $(-a, a)$, its Fourier transform $F(x)$ is (equal almost everywhere to) the restriction of an entire function $F(z)$ to the real axis. It is easily verified that $F(z)$ is of exponential type at most a. Paley and Wiener show that any entire function of exponential type at most a which is square integrable on the real axis is the Fourier transform of a function which vanishes outside of the interval $(-a, a)$. If a is given, the set of all such entire functions is a Hilbert space in the metric of $L^2(-\infty, +\infty)$. This Paley-Wiener space has interesting special properties.

If $F(z)$ is an entire function of exponential type at most a which is square integrable on the real axis and if

$$2\pi F(z) = \int_{-\infty}^{+\infty} e^{-izt} f(t)\,dt$$

is its Fourier representation, we have in particular

$$2\pi F(n\pi/a) = \int_{-a}^{a} f(t) e^{-\pi int/a} dt$$

for integer n. Since the functions $(e^{-\pi inx/a})$ are a complete orthogonal set in $L^2(-a, a)$, we can conclude that

$$\sum_{-\infty}^{+\infty} |2\pi F(n\pi/a)|^2 = 2a \int_{-a}^{a} |f(t)|^2 dt.$$

By Plancherel's formula we obtain the identity

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt = (\pi/a) \sum_{-\infty}^{+\infty} |F(n\pi/a)|^2,$$

which is meaningful without any knowledge of Fourier analysis. Although Paley-Wiener spaces were originally obtained by Fourier analysis, it is easier in many ways to derive the theory of the Fourier transformation from the theory of these spaces.

THEOREM 16. The Paley-Wiener space of entire functions $F(z)$ of exponential type at most a which are square integrable on the real axis is a Hilbert space in the norm

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)|^2 dt.$$

The function $[\sin(az - a\bar{w})]/[\pi(z - \bar{w})]$ belongs to the space for every complex number w, and the identity

$$F(w) = \int_{-\infty}^{+\infty} F(t) \frac{\sin(at - aw)}{\pi(t - w)} dt$$

holds for every element $F(z)$ of the space.

Proof of Theorem 16. If an entire function $F(z)$ belongs to the Paley-Wiener space, then

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1}[1 + |F(t)|^2] dt < \infty.$$

By Jensen's inequality,

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\log[1 + |F(t)|^2] dt}{1 + t^2} \leq \log \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1 + |F(t)|^2}{1 + t^2} dt < \infty.$$

It follows that $\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log^+ |F(t)| dt < \infty$. Since $F(z)$ is of exponential type, it follows from Theorem 11 that $F(z)$ is of bounded type in the upper half-plane. By Theorem 10 the mean type of $F(z)$ in the upper half-plane is at most a. By Cauchy's formula,

$$2\pi i e^{iaz} F(z) = \int_{-\infty}^{+\infty} (t - z)^{-1} e^{iat} F(t) dt$$

for $y > 0$ and

$$0 = \int_{-\infty}^{+\infty} (t - z)^{-1} e^{iat} F(t) dt$$

for $y < 0$. Since the same formulas hold with $F(z)$ replaced by $F^*(z)$, it follows that

$$2\pi i F(z) = \int_{-\infty}^{+\infty} (t - z)^{-1}[e^{iat}e^{-iaz} - e^{-iat}e^{iaz}] F(t) dt$$

when z is not real. The formula can be written

$$F(z) = \int_{-\infty}^{+\infty} \frac{\sin(at - az)}{\pi(t - z)} F(t) dt.$$

It is valid also in the limit of real z by the Lebesgue dominated convergence theorem. But it is easily verified that the function $[\sin(az - a\bar{w})]/[\pi(z - \bar{w})]$ belongs to the Paley-Wiener space for every complex number w. We can therefore apply the Schwarz inequality in the last identity to obtain

$$|F(z)|^2 \leq \|F\|^2 \frac{\sin(az - a\bar{z})}{\pi(z - \bar{z})}$$

for every element $F(z)$ of the Paley-Wiener space.

Completeness follows from this inequality. If $(F_n(z))$ is a Cauchy sequence in the space, then

$$|F_n(z) - F_k(z)|^2 \leq \|F_n - F_k\|^2 \frac{\sin(az - a\bar{z})}{\pi(z - \bar{z})}$$

for all complex z. The sequence $(F_n(z))$ therefore converges uniformly on bounded sets. So the limit function $F(z)$ is entire and

$$\int_{-\infty}^{+\infty} |F(t)|^2 dt \leq \lim_{n\to\infty} \int_{-\infty}^{+\infty} |F_n(t)|^2 dt < \infty$$

by Fatou's theorem. Since

$$|F(z)|^2 \leq \lim_{n\to\infty} \|F_n\|^2 \frac{\sin(az - a\bar{z})}{\pi(z - \bar{z})},$$

the limit function $F(z)$ is of exponential type at most a. By Fatou's theorem

$$\int_{-\infty}^{+\infty} |F(t) - F_k(t)|^2 dt \leq \lim_{n\to\infty} \int_{-\infty}^{+\infty} |F_n(t) - F_k(t)|^2 dt.$$

Since we assume that the sequence $(F_n(z))$ is Cauchy, it converges to $F(z)$ in the metric of the Paley-Wiener space.

17. CHARACTERIZATION OF FINITE FOURIER TRANSFORMS

We use this result to prove the Paley-Wiener theorem.

THEOREM 17. If $f(x)$ belongs to $L^2(-\infty, +\infty)$ and vanishes outside of a finite interval $(-a, a)$, then

$$2\pi F(z) = \int_{-\infty}^{+\infty} e^{-itz} f(t)\, dt$$

is an entire function of exponential type at most a such that

$$2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt.$$

Every entire function of exponential type at most a which is square integrable on the real axis is of this form.

Proof of Theorem 17. If $f(x) = e^{ix\bar{w}}$ for $-a \leq x \leq a$, then

$$2\pi \frac{\sin (az - a\bar{w})}{\pi(z - \bar{w})} = \int_{-a}^{a} e^{-itz} e^{it\bar{w}} dt$$

belongs to the Paley-Wiener space. By Theorem 16, the identity

$$2\pi \int_{-\infty}^{+\infty} \frac{\sin (at - a\bar{w}_1)}{\pi(t - \bar{w}_1)} \frac{\sin (at - aw_2)}{\pi(t - w_2)} dt = \int_{-a}^{a} e^{-it\bar{w}_1} e^{itw_2} dt$$

holds for all complex numbers w_1 and w_2. If $f(x)$ is a finite linear combination of functions $e^{ix\bar{w}}$, w complex, we can therefore conclude that

$$2\pi F(z) = \int_{-a}^{a} e^{-itz} f(t)\, dt$$

belongs to the Paley-Wiener space and that

$$2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_{-a}^{a} |f(t)|^2 dt.$$

The same conclusion follows by continuity when $f(x)$ belongs to the closed span of the functions $e^{ix\bar{w}}$ in $L^2(-a, a)$. Let $\mathcal{M}(a)$ be the set of functions $f(x)$ in $L^2(-\infty, +\infty)$ which vanish outside of the interval $(-a, a)$ and whose restrictions to the interval $(-a, a)$ belong to the closed span of the functions $e^{ix\bar{w}}$. The Fourier transformation takes $\mathcal{M}(a)$ onto a closed subspace of the Paley-Wiener space of type a. We show that this subspace is the full space by showing that its orthogonal complement contains no nonzero element. If $F(z)$ belongs to the orthogonal complement, then it is orthogonal to $[\sin (az - a\bar{w})]/[\pi(z - \bar{w})]$ for every complex number w. By Theorem 16, $F(z)$ vanishes identically. To prove the theorem we must show that $\mathcal{M}(a)$ contains any element of $L^2(-\infty, +\infty)$ which vanishes outside of $(-a, a)$.

We show that $\mathcal{M}(a)$ is contained in $\mathcal{M}(b)$ when $a < b$. If $f(x)$ belongs to $\mathcal{M}(a)$ and if $g(x)$ is the projection of $f(x)$ in $\mathcal{M}(b)$, then

$$2\pi G(z) = \int_{-b}^{b} e^{-itz} g(t)\, dt = \int_{-b}^{b} e^{-itz} f(t)\, dt = 2\pi F(z)$$

and

$$\int_{-b}^{b} |f(t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |G(t)|^2 dt = \int_{-b}^{b} |g(t)|^2 dt.$$

Since $\int_{-b}^{b} [f(t) - g(t)]\bar{g}(t)\, dt = 0$, it follows that $\int_{-b}^{b} |f(t) - g(t)|^2 dt = 0$ and that $g(x) = f(x)$ almost everywhere.

We next compute the projection of $\mathcal{M}(b)$ in $\mathcal{M}(a)$ when $a < b$. If $f(x)$

is the element of $\mathcal{M}(b)$ which is equal to $e^{ix\bar{w}}$ when $-b \leq x \leq b$, then its Fourier transform is

$$F(z) = \frac{\sin(bz - b\bar{w})}{\pi(z - \bar{w})}.$$

By Theorem 16, this is the element of the Paley-Wiener space of type b such that

$$G(w) = \left\langle G(t), \frac{\sin(bt - b\bar{w})}{\pi(t - \bar{w})} \right\rangle$$

holds for every element $G(z)$ of the Paley-Wiener space of type b. The same formula holds with b replaced by a in the Paley-Wiener space of type a. It follows that $\dfrac{\sin(az - a\bar{w})}{\pi(z - \bar{w})}$ is the projection of $\dfrac{\sin(bz - b\bar{w})}{\pi(z - \bar{w})}$ in the Paley-Wiener space of type a. But $\dfrac{\sin(az - a\bar{w})}{\pi(z - \bar{w})}$ is the Fourier transform of the element of $\mathcal{M}(a)$ which is equal to $e^{ix\bar{w}}$ for $-a \leq x \leq a$. Thus if $f(x)$ is the element of $\mathcal{M}(b)$ which is equal to $e^{ix\bar{w}}$ for $-b \leq x \leq b$, and if $g(x)$ is the projection of $f(x)$ in $\mathcal{M}(a)$, then $g(x) = f(x)$ for $-a \leq x \leq a$. The same conclusion holds for every element of $\mathcal{M}(b)$ since the closed span of such special functions is the full space.

We show that $\mathcal{M}(b)$ contains every function in $L^2(-\infty, +\infty)$ which vanishes outside of $(-b, b)$ by showing that there is no nonzero function $f(x)$ in $L^2(-\infty, +\infty)$ which vanishes outside of $(-b, b)$ and which is orthogonal to $\mathcal{M}(b)$. If $0 < a < b$, the function which is equal to $e^{ix\bar{w}}$ for $-a \leq x \leq a$ and which is 0 otherwise belongs to $\mathcal{M}(b)$ for every complex number w. Since $f(x)$ is orthogonal to all such functions

$$\int_0^a [f(t)e^{it\bar{w}} + f(-t)e^{-it\bar{w}}]dt = \int_{-a}^a f(t)e^{it\bar{w}}dt = 0.$$

It follows that $f(x)e^{ix\bar{w}} + f(-x)e^{-ix\bar{w}}$ is orthogonal to all step functions in $L^2(0, b)$. Since the step functions are dense in $L^2(0, b)$, $f(x)e^{ix\bar{w}} + f(-x)e^{-ix\bar{w}}$ vanishes for almost all x. By the arbitrariness of w, $f(x)$ vanishes almost everywhere.

18. L^2 FOURIER TRANSFORMATION

The theory of the Fourier transformation in $L^2(-\infty, +\infty)$ follows from the Paley-Wiener theorem.

THEOREM 18. If $f(x)$ belongs to $L^2(-\infty, +\infty)$, then

$$2\pi F(x) = \lim_{a \to \infty} \int_{-a}^a e^{-ixt} f(t)\, dt$$

exists in the metric of $L^2(-\infty, +\infty)$ and

$$2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt.$$

Every element of $L^2(-\infty, +\infty)$ is a Fourier transform.

Proof of Theorem 18. If $2\pi F_a(z) = \int_{-a}^{a} e^{-izt} f(t)\,dt$, then $F_a(z)$ belongs to the Paley-Wiener space of type a and

$$2\pi \int_{-\infty}^{+\infty} |F_a(t)|^2 dt = \int_{-a}^{a} |f(t)|^2 dt.$$

If $a < b$, then

$$2\pi \int_{-\infty}^{+\infty} |F_b(t) - F_a(t)|^2 dt = \int_{a}^{b} |f(t)|^2 dt + \int_{-b}^{-a} |f(t)|^2 dt.$$

Since $f(x)$ belongs to $L^2(-\infty, +\infty)$ and since $L^2(-\infty, +\infty)$ is a Hilbert space, $F(x) = \lim F_a(x)$ exists in the metric of $L^2(-\infty, +\infty)$ as $a \to \infty$ and

$$\begin{aligned} 2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt &= \lim_{a\to\infty} 2\pi \int_{-\infty}^{+\infty} |F_a(t)|^2 dt \\ &= \lim_{a\to\infty} \int_{-a}^{a} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt. \end{aligned}$$

The set of elements of $L^2(-\infty, +\infty)$ which are Fourier transforms is a closed vector subspace of the full space. To show that it is the full space, we must show that there is no nonzero element $F(x)$ of $L^2(-\infty, +\infty)$ which is orthogonal to all Fourier transforms. Since $(1 - e^{iaz}e^{-ia\bar{w}})/(z - \bar{w})$ belongs to the Paley-Wiener space of type a for every complex number w,

$$\int_{-\infty}^{+\infty} F(t) \frac{1 - e^{-iat}e^{iaw}}{t - w}\, dt = 0.$$

When w is in the upper half-plane, we obtain

$$\int_{-\infty}^{+\infty} \frac{F(t)\,dt}{t - w} = 0$$

in the limit as $a \to \infty$. Since $(1 - e^{-iaz}e^{ia\bar{w}})/(z - \bar{w})$ belongs to the Paley-Wiener space for every complex number w, a similar argument will obtain the last identity when w is in the lower half-plane. It follows that

$$\frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{F(t)\,dt}{(t - x)^2 + y^2} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{F(t)\,dt}{t - z} - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{F(t)\,dt}{t - \bar{z}} = 0$$

when $y > 0$. But $F(x)$ can be written as a linear combination of nonnegative functions which belong to $L^2(-\infty, +\infty)$. By the Stieltjes inversion formula, $\int_{a}^{b} F(t)\,dt = 0$ for all except countably many a and b. It follows that $F(x)$ vanishes almost everywhere.

19. FUNCTION VALUES AS INNER PRODUCTS

A generalization of Fourier analysis is obtained when the Paley-Wiener spaces are replaced by more general Hilbert spaces of entire functions. A Hilbert space of entire functions is associated with any entire function $E(z)$ which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. The space $\mathcal{H}(E)$ associated with such a function $E(z)$ is the set of all entire functions $F(z)$ such that

$$\|F\|_E^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

and such that both ratios $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. The Paley-Wiener space of type a is the space $\mathcal{H}(E)$ in the case $E(z) = e^{-iaz}$.

The space $\mathcal{H}(E)$ is a vector space over the complex numbers. An inner product is defined in the space by

$$\langle F(t), G(t) \rangle_E = \int_{-\infty}^{+\infty} F(t)\bar{G}(t)/|E(t)|^2 dt.$$

In working with the space, we will write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z. Explicitly these functions are

$$A(z) = \tfrac{1}{2}[E(z) + E^*(z)] \quad \text{and} \quad B(z) = \tfrac{1}{2}i[E(z) - E^*(z)].$$

We omit the subscript E from norms and inner products whenever we can do so without ambiguity. We show that any space $\mathcal{H}(E)$ contains nonzero elements.

PROBLEM 40. If $\mathcal{H}(E)$ is a given space, show that

$$K(w, w) = [B(w)\bar{A}(w) - A(w)\bar{B}(w)]/[\pi(w - \bar{w})]$$

is a continuous function of w.

THEOREM 19. Let $E(z)$ be a given entire function which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. Then

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})]$$

belongs to $\mathcal{H}(E)$ as a function of z for every complex number w and

$$F(w) = \langle F(t), K(w, t) \rangle$$

for every $F(z)$ in $\mathcal{H}(E)$.

Proof of Theorem 19. Since $2\pi i(\bar{w} - z)K(w, z) = E(z)\bar{E}(w) - E^*(z)E(\bar{w})$, where $E^*(z)/E(z)$ is bounded by 1 in the upper half-plane, the function $2\pi i(\bar{w} - z)K(w, z)/E(z)$ is bounded in the half-plane. Since $2\pi i(\bar{w} - z)$ is of bounded type in the half-plane, the quotient $K(w, z)/E(z)$ is of bounded type in the half-plane. The mean type of the quotient is nonpositive because a bounded function has nonpositive mean type and because a nonzero polynomial has zero mean type. For the same reasons, $K^*(w, z)/E(z) = K(\bar{w}, z)/E(z)$ is of bounded type and of nonpositive mean type for $y > 0$. When w is not real,

$$\int_{-\infty}^{+\infty} \left| \frac{K(w, t)}{E(t)} \right|^2 dt = \int_{-\infty}^{+\infty} \left| \frac{E(t)\bar{E}(w) - E^*(t)E(\bar{w})}{2\pi i(\bar{w} - t)E(t)} \right|^2 dt < \infty$$

because

$$\int_{-\infty}^{+\infty} \left| \frac{E(t)\bar{E}(w)}{2\pi i(\bar{w} - t)E(t)} \right|^2 dt < \infty$$

and

$$\int_{-\infty}^{+\infty} \left| \frac{E^*(t)E(\bar{w})}{2\pi i(\bar{w} - t)E(t)} \right|^2 dt < \infty.$$

If w is real the integral is finite because the integrand is a continuous function of t in the interval $(w - 1, w + 1)$ and because the last two integrals converge when this interval is omitted. It follows that $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z for every w.

If $F(z)$ belongs to $\mathcal{H}(E)$, $F(z)/E(z)$ is analytic in the upper half-plane and continuous in the closed half-plane except for possible singularities at the real zeros of $E(z)$. If $E(z)$ has a zero of order $r > 0$ at a real point h, then $E(z) = (z - h)^r G(z)$ for some entire function $G(z)$ which has a nonzero value at h. Since

$$\|F(t)\|^2 = \int_{-\infty}^{+\infty} |(t - h)^{-r} F(t)/G(t)|^2 dt < \infty$$

and $\int_{h-1}^{h+1} (t - h)^{-2k} dt = \infty$ for every positive integer k, $F(z)$ must have a zero of order r or more at h. The ratio $F(z)/E(z)$ is therefore continuous at h and has no singularities on the real axis. By Theorem 12,

$$F(z)/E(z) = (2\pi i)^{-1} \int_{-\infty}^{+\infty} (t - z)^{-1} F(t)/E(t)\, dt,$$

$$0 = \int_{-\infty}^{+\infty} (t - \bar{z})^{-1} F(t)/E(t)\, dt$$

for $y > 0$. The formulas hold also when $F(z)$ is replaced by $F^*(z)$. The four formulas so obtained imply that $F(w) = \langle F(t), K(w, t)\rangle$ for all nonreal w.

If w is real, choose a sequence (w_n) of nonreal numbers such that $w = \lim w_n$. Then

$$K(w, x)/E(x) = \lim K(w_n, x)/E(x)$$

uniformly on every bounded subset of the real axis. If $-\infty < a < b < \infty$,

$$\int_a^b |K(w, t) - K(w_n, t)|^2 \, |E(t)|^{-2} dt = \lim_{k\to\infty} \int_a^b |K(w_k, t) - K(w_n, t)|^2 \, |E(t)|^{-2} dt.$$

Since

$$\begin{aligned}\int_{-\infty}^{+\infty} |K(w_k, t) - K(w_n, t)|^2 \, |E(t)|^{-2} dt \\ &= \langle K(w_k, t) - K(w_n, t), K(w_k, t) - K(w_n, t)\rangle \\ &= K(w_k, w_k) - K(w_k, w_n) - K(w_n, w_k) + K(w_n, w_n),\end{aligned}$$

it follows from Problem 40 that

$$\begin{aligned}\int_a^b |K(w, t) - K(w_n, t)|^2 \, |E(t)|^{-2} dt \\ &\leq K(w, w) - K(w, w_n) - K(w_n, w) + K(w_n, w_n).\end{aligned}$$

Since a and b are arbitrary, $K(w, z) = \lim K(w_n, z)$ in the metric of $\mathcal{H}(E)$. It follows that

$$F(w) = \lim F(w_n) = \lim \langle F(t), K(w_n, t)\rangle = \langle F(t), K(w, t)\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$.

PROBLEM 41. If (z_n) is a sequence of nonzero numbers such that $\lim 1/z_n = 0$, show that the Weierstrass product

$$P(z) = \prod_1^\infty (1 - z/z_n) \exp\left[(z/z_n) + \tfrac{1}{2}(z/z_n)^2 + \cdots + \frac{1}{n}(z/z_n)^n\right]$$

converges uniformly on every bounded subset of the plane.

PROBLEM 42. In Problem 41 show that

$$\log\left[1 + |P(z) - 1|\right] \leq \sum_{n=1}^\infty |z/z_n|^{n+1}$$

for all complex z.

PROBLEM 43. Let (x_n) be a sequence of real numbers which has no finite limit point. Show that there exists an entire function $S(z)$, which is real for real z, and which has the sequence (x_n) as its sequence of zeros.

PROBLEM 44. If $\mathcal{H}(E)$ is a given space, show that $E(z) = S(z)E_0(z)$ where $\mathcal{H}(E_0)$ exists, $E_0(z)$ has no real zeros, and $S(z)$ is an entire function which is real for real z. Show that $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E_0)$ onto $\mathcal{H}(E)$.

PROBLEM 45. Let $\mathcal{H}(E)$ be a given space. If w is a nonreal number, show that $F(z)/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and vanishes at w. Show that the same conclusion holds for a real number w if, and only if, $E(w) \neq 0$.

20. ALTERNATIVE DEFINITION OF THE SPACE $\mathcal{H}(E)$

An alternative definition of the space $\mathcal{H}(E)$ uses an explicit estimate in the complex plane.

THEOREM 20. A necessary and sufficient condition that an entire function $F(z)$ belong to $\mathcal{H}(E)$ is that

$$\|F(t)\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

and that $|F(z)|^2 \leq \|F(t)\|^2 K(z, z)$ for all complex z.

Proof of Theorem 20. The necessity follows on applying the Schwarz inequality in Theorem 19. For the sufficiency we need only show that $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. Bounded type is obtained by Theorem 11. The hypotheses of the theorem are satisfied on the real axis by the proof of Theorem 16 since $F(x)/E(x)$ is square summable. The growth hypothesis in the upper half-plane follows from the fact that

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \log^+ |K(re^{i\theta}, re^{i\theta}) E(re^{i\theta})^{-2}| \sin\theta \, d\theta = 0.$$

This is true because

$$K(re^{i\theta}, re^{i\theta}) \leq |E(re^{i\theta})|^2/(4\pi r \sin\theta)$$

for $0 < \theta < \pi$, where

$$\lim_{r\to\infty} r^{-1} \int_0^\pi \log^+ |1/(4\pi r \sin\theta)| \sin\theta \, d\theta = 0.$$

Nonpositive mean type is obtained from the same estimate by Theorem 10.

21. COMPLETENESS OF THE SPACE $\mathcal{H}(E)$

The explicit estimate of Theorem 20 is used in proving completeness of the space.

THEOREM 21. The space $\mathcal{H}(E)$ is a Hilbert space.

Proof of Theorem 21. Consider any Cauchy sequence $(F_n(z))$ in the metric of $\mathcal{H}(E)$. Since

$$|F_k(w) - F_n(w)|^2 \leq \|F_k(t) - F_n(t)\|^2 K(w, w)$$

for all complex w, the sequence $(F_n(w))$ is a Cauchy sequence of numbers for any fixed w. By the completeness of the number system, $F(w) = \lim F_n(w)$

exists. Since $K(w, w)$ is a continuous function of w by Problem 40, it remains bounded on any bounded set. The convergence is therefore uniform on bounded sets and the limit is entire. If (a, b) is any finite interval,

$$\int_a^b |[F(t) - F_n(t)]/E(t)|^2 dt = \lim_{k\to\infty} \int_a^b |[F_k(t) - F_n(t)]/E(t)|^2 dt \le \lim_{k\to\infty} \|F_k(t) - F_n(t)\|^2$$

where the limit on the right exists because the sequence is Cauchy. Since a and b are arbitrary,

$$\int_{-\infty}^{+\infty} |[F(t) - F_n(t)]/E(t)|^2 dt \le \lim_{k\to\infty} \|F_k(t) - F_n(t)\|^2.$$

Since $|F_k(w) - F_n(w)|^2 \le \|F_k(t) - F_n(t)\|^2 K(w, w)$ for every w,

$$|F(w) - F_n(w)|^2 \le \lim_{k\to\infty} \|F_k(t) - F_n(t)\|^2 K(w, w)$$

for all complex w. By the proof of Theorem 20, $F(z) - F_n(z)$ belongs to $\mathcal{H}(E)$. Since $F_n(z)$ belongs to $\mathcal{H}(E)$ and since $\mathcal{H}(E)$ is a vector space, $F(z)$ belongs to $\mathcal{H}(E)$. Since $\|F(t) - F_n(t)\| \le \lim_{k\to\infty} \|F_k(t) - F_n(t)\|$ and since the given sequence is Cauchy, it converges to $F(z)$ in the metric of $\mathcal{H}(E)$. This completes the proof of the theorem.

PROBLEM 46. If $\mathcal{H}(E)$ is a given space, show that there is at most one real number α, modulo π, such that $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$.

PROBLEM 47. Let $f(z)$ be a function which is analytic in the complex plane except for isolated singularities at points (t_n) on the real axis. Suppose that $f^*(z) = f(z)$ and that Re $-if(z) > 0$ for $y > 0$. Show that there exist positive numbers p_n and a nonnegative number p such that

$$[f(z) - \bar{f}(w)]/(z - \bar{w}) = p + \sum p_n(t_n - z)^{-1}(t_n - \bar{w})^{-1}$$

when z and w are not real. Show that

$$p_n = \lim_{z\to t_n} (t_n - z)f(z)$$

for every n.

PROBLEM 48. If $E(z)$ is a given entire function which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$, show that there exists a continuous function $\varphi(x)$ of real x such that $E(x)$ exp $[i\varphi(x)]$ is real for all values of x. If $\varphi(x)$ is any such function, show that

$$\varphi'(x) = \pi K(x, x)|E(x)|^{-2} > 0$$

for all real x. Such a function is said to be a phase function associated with $E(z)$.

PROBLEM 49. Let $\mathcal{H}(E)$ be a given space and let $\varphi(x)$ be a phase function associated with $E(z)$. Show that

$$\bar{E}(c)^{-1} K(c, z) = \{E(z) - E^*(z) \exp[-2i\varphi(c)]\}/[2\pi i(c - z)]$$

belongs to $\mathcal{H}(E)$ for every real number c and that

$$E(c)^{-1} F(c) = \langle F(t), \bar{E}(c)^{-1} K(c, t)\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$. Show that

$$\langle \bar{E}(a)^{-1} K(a, t), \bar{E}(b)^{-1} K(b, t)\rangle = \{1 - \exp[2i\varphi(b) - 2i\varphi(a)]\}/[2\pi i(a - b)]$$

if a and b are distinct real numbers.

22. ORTHOGONAL SETS IN THE SPACE $\mathcal{H}(E)$

Phase functions are used to construct orthogonal sets in $\mathcal{H}(E)$ which yield a remarkable formula for norms in the space.

THEOREM 22. Let $\mathcal{H}(E)$ be a given space and let $\varphi(x)$ be a phase function associated with $E(z)$. If α is a given real number, the functions $\{\bar{E}(t_n)^{-1} \times K(t_n, z)\}$, t_n real and $\varphi(t_n) \equiv \alpha$ modulo π, are an orthogonal set in $\mathcal{H}(E)$. The only elements of $\mathcal{H}(E)$ which are orthogonal to $\bar{E}(t_n)^{-1}K(t_n, z)$ for every n are constant multiples of $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$. If this function does not belong to $\mathcal{H}(E)$, then

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \sum |F(t_n)/E(t_n)|^2 \pi/\varphi'(t_n)$$

for every $F(z)$ in $\mathcal{H}(E)$.

Proof of Theorem 22. Orthogonality follows from Problem 49. Explicit proof of the theorem is restricted to the case $\alpha = 0$. The general case then follows because $E(z)$ can be replaced by $e^{i\alpha}E(z)$ without change of the corresponding space. Let $f(z) = -A(z)/B(z)$. Since $A(z)$ and $B(z)$ are real for real z, $f^*(z) = f(z)$. Since $K(z, z) > 0$ when z is not real, Re $-if(z) > 0$ when $y > 0$. The singularities of $f(z)$ are the real points where $B(z)$ has a zero of higher multiplicity than $A(z)$. These are the points (t_n) where $\varphi(t_n) \equiv 0$ modulo π. By Problem 47, there exist positive numbers (p_n) and a nonnegative number p such that

$$[f(z) - \bar{f}(w)]/(z - \bar{w}) = p + \sum p_n(t_n - z)^{-1}(t_n - \bar{w})^{-1}$$

when z and w are not real. The numbers (p_n) are given by

$$p_n = \lim_{z \to t_n} (z - t_n)A(z)/B(z) = A(t_n)/B'(t_n).$$

Explicitly the formula reads

$$K(w, z) = \frac{p}{\pi} B(z)\bar{B}(w) + \sum \frac{A(t_n)}{\pi B'(t_n)} \frac{B(z)}{z - t_n} \frac{\bar{B}(w)}{\bar{w} - t_n}.$$

Written in this way the formula is valid for all complex z and w. We now show that convergence takes place in the metric of $\mathcal{H}(E)$.

Since $B(z)/(z - t_n) = \pi\bar{E}(t_n)^{-1}K(t_n, z)$, the functions are an orthogonal sequence in $\mathcal{H}(E)$. Note that

$$\begin{aligned}\|B(t)/(t - t_n)\|^2 &= \pi^2\langle\bar{E}(t_n)^{-1}K(t_n, t), \bar{E}(t_n)^{-1}K(t_n, t)\rangle \\ &= \pi^2 K(t_n, t_n)\,|E(t_n)|^{-2} = \pi B'(t_n)/A(t_n).\end{aligned}$$

To obtain convergence of the orthogonal series in the metric of $\mathcal{H}(E)$, we need only show that

$$\sum \left\| \frac{A(t_n)}{\pi B'(t_n)} \frac{B(t)}{t - t_n} \frac{\bar{B}(w)}{\bar{w} - t_n} \right\|^2 < \infty.$$

This is true because

$$\sum \frac{A(t_n)}{\pi B'(t_n)} \frac{B(w)}{w - t_n} \frac{\bar{B}(w)}{\bar{w} - t_n} \le K(w, w).$$

In particular the sum of the orthogonal series belongs to $\mathcal{H}(E)$. Since $K(w, z)$ belongs to $\mathcal{H}(E)$, it follows that $pB(z)\bar{B}(w)$ belongs to $\mathcal{H}(E)$. If $F(z)$ is an element of $\mathcal{H}(E)$ which is orthogonal to $\bar{E}(t_n)^{-1}K(t_n, z)$ for every n, then

$$\begin{aligned}F(w) &= \langle F(t), K(w, t)\rangle = \langle F(t), (p/\pi)B(t)\bar{B}(w)\rangle \\ &= B(w)\langle F(t), (p/\pi)B(t)\rangle.\end{aligned}$$

By the arbitrariness of w, $F(z)$ is a constant multiple of $B(z)$. If $B(z)$ does not belong to $\mathcal{H}(E)$, then $F(z)$ vanishes identically and the orthogonal set is complete. In this case we obtain for every $F(z)$ in $\mathcal{H}(E)$,

$$\begin{aligned}\|F(t)\|^2 &= \sum |\langle F(t), \bar{E}(t_n)^{-1}K(t_n, t)\rangle|^2/\|\bar{E}(t_n)^{-1}K(t_n, t)\|^2 \\ &= \sum |F(t_n)/E(t_n)|^2\, \pi/\varphi'(t_n)\end{aligned}$$

by the properties of orthogonal sets.

PROBLEM 50. Show that any space $\mathcal{H}(E)$ has the following properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z - \bar{w})/(z - w)$ is in the space and has the same norm as $F(z)$.

(H2) For every nonreal number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space and it always has the same norm as $F(z)$.

PROBLEM 51. If $\mathcal{H}(E)$ is a given space, show that the function $L(w, z) = 2\pi i(\bar{w} - z)K(w, z)$ satisfies the identity

$$L(w, z) = L(\alpha, z)L(\alpha, \alpha)^{-1}L(w, \alpha) + L(\bar{\alpha}, z)L(\bar{\alpha}, \bar{\alpha})^{-1}L(w, \bar{\alpha})$$

for every nonreal number α.

PROBLEM 52. Let $K(w, z)$ be an entire function of z defined for every nonreal number w, such that $\bar{K}(z, w) = K(w, z)$ and $K(w, w) > 0$. Assume that $L(w, z) = 2\pi i(\bar{w} - z)K(w, z)$ satisfies the identity of Problem 51 for some nonreal number α. Show that

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z - \bar{w})]$$

for some entire functions $A(z)$ and $B(z)$ such that $B(z)A^*(z) = A(z)B^*(z)$.

PROBLEM 53. Assume that $K(\bar{w}, z) = \bar{K}(w, \bar{z})$ for some nonreal w in Problem 52. Show that $A(z)$ and $B(z)$ can be chosen real for real z and that the function $E(z) = A(z) - iB(z)$ then satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$.

23. CHARACTERIZATION OF THE SPACE $\mathcal{H}(E)$

The axioms (H1), (H2), and (H3) of Problem 50 characterize the space $\mathcal{H}(E)$.

THEOREM 23. A Hilbert space $\mathcal{H}$, whose elements are entire functions, which satisfies (H1), (H2), and (H3), and which contains a nonzero element, is equal isometrically to some space $\mathcal{H}(E)$.

Proof of Theorem 23. Because of (H2) there exists a unique element $K(w, z)$ of $\mathcal{H}$, when w is not real, such that $F(w) = \langle F(t), K(w, t)\rangle$ for every $F(z)$ in $\mathcal{H}$. We prove the theorem by showing that $K(w, z)$ is of the known form for a space $\mathcal{H}(E)$. If α is a nonreal number, the inequality $K(\alpha, \alpha) = \langle K(\alpha, t), K(\alpha, t)\rangle \geq 0$ holds by the positivity of an inner product. The inequality is strict unless $K(\alpha, z)$ vanishes identically, which implies that every element of $\mathcal{H}$ vanishes at α. Since

$$F(z)(z - \bar{\alpha})/(z - \alpha) = F(z) + (\alpha - \bar{\alpha})F(z)/(z - \alpha),$$

the axiom (H1) implies that $F(z)/(z - \alpha)$ belongs to $\mathcal{H}$ whenever $F(z)$ belongs to $\mathcal{H}$ and has a zero at α. If every element of $\mathcal{H}$ vanishes at α, it follows inductively that $F(z)/(z - \alpha)^n$ belongs to $\mathcal{H}$ and vanishes at α for every $n = 1, 2, 3, \cdots$. Since $F(z)$ is an entire function, it must then vanish identically. The hypothesis that $\mathcal{H}$ contains a nonzero element therefore implies that $K(\alpha, \alpha) > 0$.

Consider the function $K(w, z) - K(\alpha, z)K(\alpha, \alpha)^{-1}K(w, \alpha)$, which belongs to $\mathcal{H}$ as a function of z for every nonreal number w. By (H1),

$$[K(w, z) - K(\alpha, z)K(\alpha, \alpha)^{-1}K(w, \alpha)](z - \bar{\alpha})/(z - \alpha)$$

belongs to $\mathcal{H}$. The function vanishes at $\bar{\alpha}$. If $F(z)$ belongs to $\mathcal{H}$ and vanishes at $\bar{\alpha}$, then because of (H1),

$$\begin{aligned}\langle F(t), [K(w, t) &- K(\alpha, t)K(\alpha, \alpha)^{-1}K(w, \alpha)](t - \bar{\alpha})/(t - \alpha)\rangle \\ &= \langle F(t)(t - \alpha)/(t - \bar{\alpha}), K(w, t) - K(\alpha, t)K(\alpha, \alpha)^{-1}K(w, \alpha)\rangle \\ &= F(w)(w - \alpha)/(w - \bar{\alpha}) \\ &= \langle F(t), [K(w, t) - K(\bar{\alpha}, t)K(\bar{\alpha}, \bar{\alpha})^{-1}K(w, \bar{\alpha})](\bar{w} - \bar{\alpha})/(\bar{w} - \alpha)\rangle.\end{aligned}$$

Since $[K(w, z) - K(\bar{\alpha}, z)K(\bar{\alpha}, \bar{\alpha})^{-1}K(w, \bar{\alpha})](\bar{w} - \bar{\alpha})/(\bar{w} - \alpha)$ vanishes at $\bar{\alpha}$ and since $F(z)$ is an arbitrary element of $\mathcal{H}$ which vanishes at $\bar{\alpha}$, we can conclude that

$$\begin{aligned}[K(w, z) &- K(\alpha, z)K(\alpha, \alpha)^{-1}K(w, \alpha)](z - \bar{\alpha})/(z - \alpha) \\ &= [K(w, z) - K(\bar{\alpha}, z)K(\bar{\alpha}, \bar{\alpha})^{-1}K(w, \bar{\alpha})](\bar{w} - \bar{\alpha})/(\bar{w} - \alpha).\end{aligned}$$

This identity is equivalent to the identity of Problem 51.

Now apply the axiom (H3). It implies that $\bar{K}(w, \bar{z})$ belongs to $\mathcal{H}$ for every nonreal number w. If $F(z)$ is in $\mathcal{H}$, then

$$\langle F(t), \bar{K}(w, \bar{t})\rangle = \langle F^*(t), K(w, t)\rangle^- = F(\bar{w}) = \langle F(t), K(\bar{w}, t)\rangle.$$

By the arbitrariness of $F(z)$, $\bar{K}(w, \bar{z}) = K(\bar{w}, z)$. Problem 53 now introduces a space $\mathcal{H}(E)$. We show that it is equal isometrically to $\mathcal{H}$.

The function $K(w, z)$ belongs to both spaces for every nonreal number w. The inner product of two such functions is the same in $\mathcal{H}$ as it is in $\mathcal{H}(E)$. A finite linear combination of such functions therefore has the same norm in $\mathcal{H}$ as it does in $\mathcal{H}(E)$. There is no nonzero element of $\mathcal{H}$ which is orthogonal to all such functions since it then vanishes for all nonreal values of w, and hence, by continuity, for all w. So if $F(z)$ is in $\mathcal{H}$, there is a sequence $(F_n(z))$ of finite linear combinations of such special functions such that $F(z) = \lim F_n(z)$ in the metric of $\mathcal{H}$. Since the sequence is Cauchy in the metric of $\mathcal{H}$ and since the approximating functions have the same norms and inner products in $\mathcal{H}(E)$ as they do in $\mathcal{H}$, the sequence is Cauchy in the metric of $\mathcal{H}(E)$. Since $\mathcal{H}(E)$ is a Hilbert space, $G(z) = \lim F_n(z)$ exists in the metric of $\mathcal{H}(E)$. For every nonreal number w,

$$\begin{aligned}G(w) &= \langle G(t), K(w, t)\rangle_E = \lim \langle F_n(t), K(w, t)\rangle_E \\ &= \lim \langle F_n(t), K(w, t)\rangle = \langle F(t), K(w, t)\rangle = F(w).\end{aligned}$$

By the arbitrariness of w, $G(z) = F(z)$. Also

$$\|G(t)\|_E = \lim \|F_n(t)\|_E = \lim \|F_n(t)\| = \|F(t)\|.$$

So $\mathcal{H}$ is contained isometrically in $\mathcal{H}(E)$. A similar argument will show that $\mathcal{H}(E)$ is contained isometrically in $\mathcal{H}$. The theorem follows.

PROBLEM 54. Let $\mathcal{H}$ be a Hilbert space of entire functions which satisfies (H1) and (H2) and which contains a nonzero element. Show that there exists a space $\mathcal{H}(E)$ and an entire function $U(z)$ such that $U(z)U^*(z) = 1$ and such that $F(z) \to U(z)F(z)$ is an isometric transformation of $\mathcal{H}$ onto $\mathcal{H}(E)$.

24. UNIQUENESS OF SPACES WITH GIVEN PHASE FUNCTIONS

A space $\mathcal{H}(E)$ is essentially uniquely determined by any phase function associated with $E(z)$.

THEOREM 24. Let $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ be given spaces such that $E_1(z)$ has no real zeros, and let $\varphi_1(x)$ and $\varphi_2(x)$ be corresponding phase functions. If $\varphi_1(x) = \varphi_2(x)$ whenever $\varphi_1(x) \equiv 0$ modulo $\frac{1}{2}\pi$ or $\varphi_2(x) \equiv 0$ modulo $\frac{1}{2}\pi$, then $\tan \varphi_2(x)/\tan \varphi_1(x)$ is a positive constant. There exists an entire function $S(z)$, which is real for z and which has only real zeros, such that $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E_1)$ onto $\mathcal{H}(E_2)$.

Proof of Theorem 24. The zeros of $B_1(z)/A_1(z)$ are real and simple and are the points x where $\varphi_1(x) \equiv 0$ modulo π. The zeros of $A_1(z)/B_1(z)$ are real and simple and are the points x where $\varphi_1(x) \equiv \frac{1}{2}\pi$ modulo π. Since $iA_1(z)/B_1(z)$ is analytic and has a nonnegative real part in the upper half-plane, it is of bounded type in the half-plane by Problem 20. For the same reasons, $iA_2(z)/B_2(z)$ is of bounded type in the half-plane. The zeros of $B_2(z)/A_2(z)$ are real and simple and are the points x where $\varphi_2(x) \equiv 0$ modulo π. The zeros of $A_2(z)/B_2(z)$ are real and simple and are the points x where $\varphi_2(x) \equiv \frac{1}{2}\pi$ modulo π. The hypotheses now imply that

$$k(z) = [B_2(z)/A_2(z)]/[B_1(z)/A_1(z)]$$

is an entire function which has no zeros. Since $k^*(z) = k(z)$ and since $k(z)$ is of bounded type in the upper half-plane, it is of Pólya class by Problem 34. (Compare it with the function 1 which is of Pólya class.) By Lemma 2,

$$k(z) = k(0) \exp(-az^2 - ibz)$$

where $a \geq 0$ and Re $b \geq 0$. Since $k(z)$ is of bounded type in the upper half-plane, $a = 0$ and b is real. Since $k(z)$ is real for real z, $k(z) = k$ is a constant.

Since $iA_2(z)/B_2(z)$ and $iA_1(z)/B_1(z)$ have nonnegative real parts in the upper half-plane, k is positive. Since $\tan\varphi_1(x) = B_1(x)/A_1(x)$ and $\tan\varphi_2(x) = B_2(x)/A_2(x)$, $\tan\varphi_2(x)/\tan\varphi_1(x) = k$ is a positive constant. Write $k = c^2$ where $c > 0$. Since we assume that $E_1(z)$ has no real zeros,

$$S(z) = c^{-1}B_2(z)/B_1(z) = cA_2(z)/A_1(z)$$

is an entire function, and it is real for real z. It has only real zeros and satisfies the identity

$$K_2(w, z) = S(z)K_1(w, z)\bar{S}(w).$$

If $F(z)$ is a finite linear combination of functions $K_1(w; z)\bar{S}(w)$, the identity implies that $S(z)F(z)$ belongs to $\mathcal{H}(E_2)$ and has the same norm as $F(z)$ in $\mathcal{H}(E_1)$. Since such special functions are dense in $\mathcal{H}(E_1)$, $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E_1)$ into $\mathcal{H}(E_2)$. Since the range of the transformation is a closed subspace of $\mathcal{H}(E_2)$ which contains $K_2(w, z)$ whenever $S(w) \neq 0$, it contains every element of $\mathcal{H}(E_2)$.

PROBLEM 55. Let $F(z)$ and $G(z)$ be polynomials which are real for real z and have only real simple zeros. Assume that there exists a continuous, increasing function $\psi(x)$ of real x such that the zeros of $G(z)$ are the points x where $\psi(x) \equiv 0$ modulo π and the zeros of $F(z)$ are the points x where $\psi(x) \equiv \frac{1}{2}\pi$ modulo π. Show that $G'(x)F(x) - F'(x)G(x)$ is of constant sign on the set where $\psi(x) \equiv 0$ modulo $\frac{1}{2}\pi$.

PROBLEM 56. If the degree of $G(z)$ does not exceed the degree of $F(z)$ in Problem 55, show that there exists a real number h such that

$$\frac{G(z)}{F(z)} = h + \sum_{F(t)=0} \frac{G(t)}{F'(t)(z-t)}.$$

PROBLEM 57. If $F(z)$ and $G(z)$ are not both constants in Problem 55, show that either $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$.

PROBLEM 58. If s and t are positive numbers such that $s < t$, show that

$$\left|\frac{1 - z/t}{1 - z/s} - 1\right| \leq \exp\left[\frac{1/s - 1/t}{|1/s - 1/z|}\right] - 1.$$

PROBLEM 59. Let (s_n) and (t_n) be unbounded, increasing sequences of positive numbers such that

$$s_n < t_n < s_{n+1} < t_{n+1}$$

for every n. Show that

$$\prod_{n=1}^{\infty} \frac{1 - z/t_n}{1 - z/s_n}$$

converges if $z \neq s_n$ for every n. If $\rho(z) = \min |1/z - 1/s_n|$, show that the convergence is uniform in any set on which $\rho(z)$ is bounded away from zero.

PROBLEM 60. Let $\psi(x)$ be a continuous, increasing function of real x which has 0 as a value. Show that there exists a function $f(z)$ with these properties:

(1) The function is analytic in the complex plane except for isolated singularities on the real axis, $f^*(z) = f(z)$, and $\text{Re} -if(z) > 0$ for $y > 0$.

(2) The zeros of $f(z)$ are real and simple and are the points x where $\psi(x) \equiv 0$ modulo π.

(3) The zeros of $1/f(z)$ are real and simple and are the points x where $\psi(x) \equiv \frac{1}{2}\pi$ modulo π.

PROBLEM 61. In Problem 60 show that there exists an entire function $A(z)$, which is real for real z and which has only real simple zeros, and whose zeros are the points x where $\psi(x) \equiv \frac{1}{2}\pi$ modulo π. Show that $B(z) = A(z)f(z)$ is an entire function which is real for real z. Show that $E(z) = A(z) - iB(z)$ is an entire function which has no real zeros and which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. Show that there exists a phase function $\varphi(x)$ associated with $E(z)$ such that $\varphi(x) = \psi(x)$ whenever $\varphi(x) \equiv 0$ modulo $\frac{1}{2}\pi$ or $\psi(x) \equiv 0$ modulo $\frac{1}{2}\pi$.

PROBLEM 62. Let $F(z)$ and $G(z)$ be entire functions which are real for real z and which have only real simple zeros. Assume that there exists a continuous, increasing function $\psi(x)$ of real x such that the zeros of $G(z)$ are the points x where $\psi(x) \equiv 0$ modulo π and the zeros of $F(z)$ are the points x where $\psi(x) \equiv \frac{1}{2}\pi$ modulo π. If $F(z)$ or $G(z)$ has a zero and if $G(z)/F(z)$ is of bounded type for $y > 0$, show that either $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$.

PROBLEM 63. Let $E(z)$ be an entire function of Pólya class which has no real zeros and which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. Let $\varphi(x)$ be a phase function associated with $E(z)$. Show that there exists a number $p \geq 0$ such that

$$\frac{\partial}{\partial y} \log |E(x + iy)| = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\varphi(t)}{(t - x)^2 + y^2}$$

for $y > 0$.

PROBLEM 64. Show that $E(z)$ in Problem 61 can be chosen of Pólya class if

$$\int_{-\infty}^{+\infty} \frac{d\psi(t)}{1+t^2} < \infty.$$

25. FUNCTIONS ASSOCIATED WITH $\mathcal{H}(E)$

In work with a space $\mathcal{H}(E)$ we must consider functions which are associated with the space but which do not necessarily belong to it. Examples of such functions are $A(z)$ and $B(z)$. We now identify the relevant class of functions.

THEOREM 25. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be an entire function which has a nonzero value at a point α. A necessary and sufficient condition that $[F(z)S(\alpha) - S(z)F(\alpha)]/(z - \alpha)$ belong to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ is that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |S(t)/E(t)|^2 dt < \infty,$$

that $S(z)/E(z)$ and $S^*(z)/E(z)$ be of bounded type in the upper half-plane, and that these ratios be of nonpositive mean type.

Note that these conditions do not depend on α.

Proof of Theorem 25. If difference quotients belong to $\mathcal{H}(E)$, then $[F(z)S(\alpha) - S(z)F(\alpha)]/[(z-\alpha)E(z)]$ and $[F^*(z)\bar{S}(\alpha) - S^*(z)\bar{F}(\alpha)]/[(z-\bar{\alpha})E(z)]$ are of bounded type and of nonpositive mean type in the upper half-plane whenever $F(z)$ belongs to $\mathcal{H}(E)$. Since a nonzero polynomial has zero mean type, $[F(z)S(\alpha) - S(z)F(\alpha)]/E(z)$ and $[F^*(z)\bar{S}(\alpha) - S^*(z)\bar{F}(\alpha)]/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. Since $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type and of nonpositive mean type, so are $F(\alpha)S(z)/E(z)$ and $\bar{F}(\alpha)S^*(z)/E(z)$. If there exists an element $F(z)$ of $\mathcal{H}(E)$ having a nonzero value at α, we can conclude that $S(z)/E(z)$ and $S^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. Such an element must exist, for otherwise $F(z)/(z-\alpha)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and the proof of Theorem 23 will show that $\mathcal{H}(E)$ contains no nonzero element. Since $[F(z)S(\alpha) - S(z)F(\alpha)]/(z-\alpha)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$,

$$\int_{-\infty}^{+\infty} |[F(t)S(\alpha) - S(t)F(\alpha)]/[(t-\alpha)E(t)]|^2 dt < \infty$$

for every $F(z)$ in $\mathcal{H}(E)$. Since $(z-\alpha)/(z-i)$ is bounded on the real axis,

it follows that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |[F(t)S(\alpha) - S(t)F(\alpha)]/E(t)|^2 dt < \infty.$$

Since $\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$, we obtain

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |S(t)/E(t)|^2 dt < \infty$$

when $F(\alpha) \neq 0$. The necessity follows. The sufficiency is obtained by reversing steps in the proof of necessity.

PROBLEM 65. Let $f(z)$ be a function which has an absolutely convergent representation

$$f(z) = \int_{-\infty}^{+\infty} (t - z)^{-1} h(t) d\mu(t)$$

for $y > 0$, where $h(x)$ is a Borel measurable function of real x and $\mu(x)$ is a nondecreasing function of real x. Show that $f(z)$ is analytic and of bounded type in the upper half-plane and that it has nonpositive mean type. *Hint:* Write $f(z)$ as a linear combination of functions which are analytic and have nonnegative real parts in the upper half-plane.

PROBLEM 66. Let $E(z)$ and $S(z)$ be entire functions, let (a, b) be a finite interval, and let

$$M(z) = \max |[E(t)S(z) - S(t)E(z)]/(t - z)|$$

for $a \leq t \leq b$. Show that $M(z)$ remains bounded on every bounded set.

PROBLEM 67. Let $\mathcal{H}(E)$ be a given space and let $S(z) = A(z)u + B(z)v$ where u and v are numbers, not both zero, such that $\bar{u}v = \bar{v}u$. Show that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and that the identity

$$\begin{aligned} 0 = \langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\ - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), G(t)S(\beta)\rangle \\ + (\alpha - \bar{\beta})\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$ and all complex numbers α and β.

PROBLEM 68. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be an entire function such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Assume that the identity of Problem 67 holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$ and for all complex numbers α and β. Show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = \bar{v}u$.

PROBLEM 69. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be a nonzero entire function such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Assume that there exists a nondecreasing function $\mu(x)$ of real x such that $S(z)$ is μ-equivalent to zero and such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = \bar{v}u$. Show that $\mu(x)$ is a step function whose points of increase are zeros of $S(z)$ and that $\mu(t+) - \mu(t-) = K(t, t)^{-1}$ at each such zero. Show that $\mathcal{H}(E)$ fills $L^2(\mu)$.

26. FUNCTIONS SATISFYING AN ESTIMATE ON THE IMAGINARY AXIS

The functions associated with a space $\mathcal{H}(E)$ can be identified from different conditions which require a precise estimate on the imaginary axis.

THEOREM 26. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be an entire function such that $S(z)/E(z)$ and $S^*(z)/E(z)$ are of bounded type in the upper half-plane. Assume that $E(z)$ has no real zeros and that $\mu(x)$ is a given nondecreasing function of real x such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Assume that there exists no nonzero entire function $Q(z)$ which is μ-equivalent to zero such that $[F(z)Q(w) - Q(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. If

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |S(t)|^2 d\mu(t) < \infty,$$

if

$$\limsup_{y \to +\infty} |S(iy)/E(iy)| < \infty,$$

and if

$$\limsup_{y \to +\infty} |S(-iy)/E(iy)| < \infty,$$

then $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$.

Proof of Theorem 26. If $F(z)$ belongs to $\mathcal{H}(E)$, we must show that $[F(z)S(w) - S(z)F(w)]/(z - w)$ is orthogonal to every element $h(x)$ of $L^2(\mu)$ which is orthogonal to $\mathcal{H}(E)$. Let $h(x)$ be held fixed. If w is a given number, we show that there exists a number $L(w)$ such that the identity

$$F(w)L(w) = \int_{-\infty}^{+\infty} \frac{F(t)S(w) - S(t)F(w)}{t - w} \bar{h}(t)\, d\mu(t)$$

holds for every $F(z)$ in $\mathcal{H}(E)$. If w lies on or above the real axis, define $L(w)$ by

$$E(w)L(w) = \int_{-\infty}^{+\infty} \frac{E(t)S(w) - S(t)E(w)}{t - w} \bar{h}(t)\, d\mu(t).$$

This is possible because $E(z)$ has a nonzero value at w by hypothesis and because the hypotheses imply that

$$\int_{-\infty}^{+\infty} \left| \frac{E(t)S(w) - S(t)E(w)}{t - w} \right|^2 d\mu(t) < \infty.$$

The integral in the definition of $L(w)$ is absolutely convergent by the Schwarz inequality in $L^2(\mu)$. If $F(z)$ is in $\mathcal{H}(E)$,

$$\begin{aligned} F(w)[E(z)S(w) - S(z)E(w)]/(z - w) \\ = E(w)[F(z)S(w) - S(z)F(w)]/(z - w) \\ + S(w)[E(z)F(w) - F(z)E(w)]/(z - w) \end{aligned}$$

where the last term belongs to $\mathcal{H}(E)$. The desired identity follows since $h(x)$ is orthogonal to $\mathcal{H}(E)$ and since $E(z)$ has a nonzero value at w. The same argument with $E(z)$ replaced by $E^*(z)$ will show that

$$E^*(w)L(w) = \int_{-\infty}^{+\infty} \frac{E^*(t)S(w) - S(t)E^*(w)}{t - w} \bar{h}(t)\,d\mu(t)$$

for all such w. This formula is used to define $L(w)$ when w is in the lower half-plane. The previous argument will show that the desired identity holds for all complex w.

When z is in the upper half-plane,

$$L(z) = \frac{S(z)}{E(z)} \int_{-\infty}^{+\infty} \frac{E(t)\bar{h}(t)\,d\mu(t)}{t - z} - \int_{-\infty}^{+\infty} \frac{S(t)\bar{h}(t)\,d\mu(t)}{t - z}$$

where each integral represents a function which is analytic and of bounded type in the upper half-plane by Problem 65. Since we assume that $S(z)/E(z)$ is of bounded type in the upper half-plane, $L(z)$ is analytic and of bounded type in the upper half-plane. Since we assume that $S^*(z)/E(z)$ is of bounded type in the upper half-plane, a similar argument will show that $L^*(z)$ is of bounded type in the upper half-plane.

We show that $L(z)$ is an entire function. If (a, b) is any finite interval, write

$$\begin{aligned} E(z)L(z) = S(z) \int_{-\infty}^{a} \frac{E(t)\bar{h}(t)\,d\mu(t)}{t - z} - E(z) \int_{-\infty}^{a} \frac{S(t)\bar{h}(t)\,d\mu(t)}{t - z} \\ + \int_a^b \frac{E(t)S(z) - S(t)E(z)}{t - z} \bar{h}(t)\,d\mu(t) \\ + S(z) \int_b^{\infty} \frac{E(t)\bar{h}(t)\,d\mu(t)}{t - z} - E(z) \int_b^{\infty} \frac{S(t)\bar{h}(t)\,d\mu(t)}{t - z} \end{aligned}$$

where the first two integrals represent functions analytic in the half-plane $x > a$ and the last two integrals represent functions analytic in the half-plane

$x < b$. The middle integral is a limit of Stieltjes sums

$$\sum_{k=1}^{r} \frac{E(t_k)S(z) - S(t_k)E(z)}{t_k - z} \int_{t_{k-1}}^{t_k} \bar{h}(t)\,d\mu(t)$$

taken over the partitions $a = t_0 < t_1 \cdots < t_r = b$ of the interval (a, b). If $M(z)$ is defined as in Problem 66,

$$\left| \sum_{k=1}^{r} \frac{E(t_k)S(z) - S(t_k)E(z)}{t_k - z} \int_{t_{k-1}}^{t_k} \bar{h}(t)\,d\mu(t) \right| \leq M(z) \int_a^b |h(t)|\,d\mu(t).$$

Since $M(z)$ remains bounded on every bounded set, it follows that

$$\int_a^b \frac{E(t)S(z) - S(t)E(z)}{t - z}\, \bar{h}(t)\,d\mu(t)$$

is an entire function. We have now shown that $E(z)L(z)$ is analytic in any vertical strip of points $x + iy$ such that $a < x < b$. By the arbitrariness of a and b, $E(z)L(z)$ is an entire function. A similar argument will show that $E^*(z)L(z)$ is an entire function. Since $E(z)$ has no zeros on or above the real axis, it follows that $L(z)$ is an entire function.

By the Schwarz inequality,

$$\left| \int_{-\infty}^{+\infty} \frac{E(t)\bar{h}(t)\,d\mu(t)}{t - z} \right|^2 \leq \int_{-\infty}^{+\infty} \frac{|E(t)|^2 d\mu(t)}{(t - x)^2 + y^2} \int_{-\infty}^{+\infty} |h(s)|^2 d\mu(s).$$

By the Lebesgue dominated convergence theorem,

$$\lim_{y \to +\infty} \int_{-\infty}^{+\infty} \frac{E(t)\bar{h}(t)\,d\mu(t)}{t - iy} = 0.$$

A similar argument will show that

$$\lim_{y \to +\infty} \int_{-\infty}^{+\infty} \frac{S(t)\bar{h}(t)\,d\mu(t)}{t - iy} = 0.$$

Since we assume that

$$\limsup_{y \to +\infty} |S(iy)/E(iy)| < \infty,$$

it follows that

$$\lim_{y \to +\infty} |L(iy)| = 0.$$

Since we assume that

$$\limsup_{y \to +\infty} |S(-iy)/E(iy)| < \infty,$$

a similar argument will show that

$$\lim_{y \to +\infty} |L(-iy)| = 0.$$

By Problem 39, $L(z)$ vanishes identically.

We have shown that

$$0 = \int_{-\infty}^{+\infty} \frac{S(t)E(w) - E(t)S(w)}{t - w} \bar{h}(t)d\mu(t)$$

for all complex w whenever $h(x)$ is an element of $L^2(\mu)$ which is orthogonal to $\mathcal{H}(E)$. By the arbitrariness of $h(x)$, $[S(z)E(w) - E(z)S(w)]/(z - w)$ coincides in $L^2(\mu)$ with an element of $\mathcal{H}(E)$. For any fixed w, $S(z)$ is μ-equivalent to an entire function $T(z)$ such that $[T(z)E(w) - E(z)T(w)]/(z - w)$ belongs to $\mathcal{H}(E)$. The entire function $P(z) = S(z) - T(z)$ is μ-equivalent to zero and $[P(z)E(w) - E(z)P(w)]/(z - w)$ coincides in $L^2(\mu)$ with an element of $\mathcal{H}(E)$. This element must be of the form $[Q(z)E(w) - E(z)Q(w)]/(z - w)$ for some entire function $Q(z)$ which is μ-equivalent to zero. By hypothesis such a function $Q(z)$ vanishes identically. It follows that $P(w) = 0$. By the arbitrariness of w, $P(z) = 0$ and $S(z) = T(z)$. The theorem follows.

PROBLEM 70. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be an entire function such that $S(z)/E(z)$ and $S^*(z)/E(z)$ are of bounded type in the upper half-plane. Assume that $E(z)$ has no real zeros and that $\mu(x)$ is a given nondecreasing function of real x such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Assume that there exists a nonzero entire function $Q(z)$ which is μ-equivalent to zero such that $[F(z)Q(w) - Q(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. If

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |S(t)|^2 d\mu(t) < \infty,$$

if

$$\limsup_{y \to +\infty} |S(iy)/Q(iy)| < \infty,$$

and if

$$\limsup_{y \to +\infty} |S(-iy)/Q(iy)| < \infty,$$

show that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$.

PROBLEM 71. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros, and let $S(z)$ be an entire function which is real for real z and has no zeros, such that $S(z)/E(z)$ is of bounded type in the upper half-plane. Let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. If

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |S(t)|^2 d\mu(t) < \infty,$$

show that $E(z)/S(z)$ is of Pólya class and that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$.

PROBLEM 72. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)$ has no real zeros. Let $S(z)$ be an entire function which has no zeros. If $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$, show that it belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$.

PROBLEM 73. Let A, B, C, D be complex numbers such that $AD - BC \neq 0$. Show that the transformation $z \to (Az + B)/(Cz + D)$ maps the upper half-plane into itself if, and only if, $i(C\bar{A} - A\bar{C}) \geq 0$, $i(D\bar{B} - B\bar{D}) \geq 0$, and $\text{Re}\,(A\bar{D} - B\bar{C}) \geq |AD - BC|$.

PROBLEM 74. Let $\mathcal{H}(E)$ be a given space and let α be a real number such that $S(z) = e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ does not belong to $\mathcal{H}(E)$. Show that $F(z) \to [F(z)S(w) - S(z)F(w)]/(z - w)$ is an everywhere defined and bounded transformation in $\mathcal{H}(E)$ for every complex number w. For each fixed $F(z)$ in $\mathcal{H}(E)$, show that $[F(z)S(w) - S(z)F(w)]/(z - w)$ depends continuously on w in the metric of $\mathcal{H}(E)$. *Hint:* Use Theorem 22.

PROBLEM 75. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be any entire function such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Show that $[F(z)S(w) - S(z)F(w)]/(z - w)$ depends continuously on $F(z)$ in the metric of $\mathcal{H}(E)$ for every fixed w. Show also that $[F(z)S(w) - S(z)F(w)]/(z - w)$ depends continuously on w in the metric of $\mathcal{H}(E)$ for each fixed $F(z)$. Let $R(w)$ be the transformation $F(z) \to [F(z)S(w) - S(z)F(w)]/(z - w)$ in $\mathcal{H}(E)$. Show that

$$(\alpha - \beta)R(\alpha)R(\beta) = R(\alpha)S(\beta) - R(\beta)S(\alpha)$$

for all complex numbers α and β.

PROBLEM 76. In Problem 75 show that

$$\begin{aligned} 0 = \langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\ - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), G(t)S(\beta)\rangle \\ + (\alpha - \bar{\beta})\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \end{aligned}$$

whenever $F(z)$ belongs to $\mathcal{H}(E)$ and vanishes at α, and $G(z)$ belong to $\mathcal{H}(E)$ and vanishes at β.

PROBLEM 77. If $\varphi(x)$ is a continuous, increasing function of real x, if $f(x)$ is a nonnegative, continuous function of x, and if $g(\theta) = \Sigma f(x)$ taken over all real numbers x such that $\varphi(x) \equiv \theta$ modulo π, show that

$$\int_{-\infty}^{+\infty} f(t)\,d\varphi(t) = \int_0^\pi g(\theta)\,d\theta.$$

PROBLEM 78. Let A, B, C, D be complex numbers such that $AD - BC \neq 0$, $i(A\bar{B} - BA) \geq 0$, $i(C\bar{D} - D\bar{C}) \geq 0$, and $\operatorname{Re}(A\bar{D} - B\bar{C}) \geq |AD - BC|$. Show that

$$\pi \operatorname{Re} \frac{D + iC}{A - iB} = \int_0^\pi \operatorname{Re} \frac{e^{i\theta}(D + iC) + e^{-i\theta}(D - iC)}{e^{i\theta}(A - iB) - e^{-i\theta}(A + iB)}\, d\theta.$$

27. CHARACTERIZATION OF FUNCTIONS ASSOCIATED WITH $\mathcal{H}(E)$

The functions associated with $\mathcal{H}(E)$ by Theorem 25 have another characterization in terms of matrices of entire functions of the form

$$M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

THEOREM 27. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be a given entire function which does not vanish identically. A necessary and sufficient condition that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belong to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ is that there exist entire functions $C(z)$ and $D(z)$, which are real for real z, such that

$$A(z)D(z) - B(z)C(z) = S(z)S^*(z),$$

$$\operatorname{Re}[A(z)\bar{D}(z) - B(z)\bar{C}(z)] \geq \tfrac{1}{2}|S(z)|^2 + \tfrac{1}{2}|S^*(z)|^2$$

for all complex z, and such that $[D(z) + iC(z)]/E(z)$ has no real singularities. In this case the functions can be chosen so that

$$\lim_{y \to +\infty} \operatorname{Re} y^{-1}[D(iy) + iC(iy)]/E(iy) = 0.$$

The function $D(z) + iC(z)$ is then uniquely determined within an added imaginary multiple of $E(z)$. If $C(z)$ and $D(z)$ are linearly independent, a space $\mathcal{H}(D + iC)$ exists and there exists a partially isometric transformation $F(z) \to \tilde{F}(z)$ of $\mathcal{H}(E) = \mathcal{H}(A - iB)$ onto $\mathcal{H}(D + iC)$ such that

$$\begin{aligned}
\pi F(\alpha)\tilde{G}(\beta)^- - \pi\tilde{F}(\alpha)G(\beta)^- \\
&= \langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle_E \\
&\quad - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), G(t)S(\beta)\rangle_E \\
+ (\alpha - \bar{\beta})\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle_E
\end{aligned}$$

for all $F(z)$ and $G(z)$ in $\mathcal{H}(E)$ and all complex numbers α and β. If $F(z)$ is in $\mathcal{H}(E)$ and if $G(z) = [F(z)S(w) - S(z)F(w)]/(z - w)$ for some number w, then $\tilde{G}(z) = [\tilde{F}(z)S(w) - S(z)\tilde{F}(w)]/(z - w)$. The function

$$[S(z)\bar{S}(w) - A(z)\bar{D}(w) + B(z)\bar{C}(w)]/[\pi(z - \bar{w})]$$

belongs to $\mathcal{H}(E)$ for every w and

$$\tilde{F}(w) = \langle F(t), [S(t)\bar{S}(w) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]/[\pi(t - \bar{w})]\rangle_E$$

for every $F(z)$ in $\mathcal{H}(E)$.

Proof of Theorem 27, *the sufficiency*. Since

$$|S(z)S^*(z)| \leq \tfrac{1}{2}|S(z)|^2 + \tfrac{1}{2}|S^*(z)|^2$$

for all complex z, the hypotheses imply that

$$|A(z)\bar{D}(z) - B(z)\bar{C}(z)|^2 \geq |A(z)D(z) - B(z)C(z)|^2.$$

On expansion the inequality simplifies to

$$i[A(z)\bar{B}(z) - B(z)\bar{A}(z)]i[C(z)\bar{D}(z) - D(z)\bar{C}(z)] \geq 0.$$

Since $i[A(z)\bar{B}(z) - B(z)\bar{A}(z)] > 0$ for $y > 0$ we obtain $i[C(z)\bar{D}(z) - D(z)\bar{C}(z)] \geq 0$ for $y > 0$. By Problem 73 the transformation

$$w \to [D(z)w + B(z)]/[C(z)w + A(z)]$$

maps the upper half-plane into itself when z is in the upper half-plane and $S(z)S^*(z)$ has a nonzero value. It follows that the inverse transformation

$$w \to [A(z)w - B(z)]/[-C(z)w + D(z)]$$

maps the lower half-plane into itself. Reversing the sign of the dependent and independent variables, we find that the transformation

$$w \to [A(z)w + B(z)]/[C(z)w + D(z)]$$

maps the upper half-plane into itself. By Problem 73, $i[C(z)\bar{A}(z) - A(z)\bar{C}(z)] \geq 0$ and $i[D(z)\bar{B}(z) - B(z)\bar{D}(z)] \geq 0$ for $y > 0$. Since we assume that

$$\text{Re}\,[D(z)\bar{A}(z) - C(z)B(z)] \geq \tfrac{1}{2}|S(z)|^2 + \tfrac{1}{2}|S^*(z)|^2,$$

it follows that

$$\text{Re}\,[D(z) + iC(z)][\bar{A}(z) + i\bar{B}(z)] \geq \tfrac{1}{2}|S(z)|^2 + \tfrac{1}{2}|S^*(z)|^2$$

for $y > 0$, or equivalently,

$$\text{Re}\,[D(z) + iC(z)]/[A(z) - iB(z)] \geq \tfrac{1}{2}|S(z)/E(z)|^2 + \tfrac{1}{2}|S^*(z)/E(z)|^2.$$

The hypotheses imply that the function on the left is continuous in the closed half-plane and that equality holds on the real axis. By Problems 2 and 3, there is a nonnegative number p such that

$$\operatorname{Re}\frac{D(z)+iC(z)}{A(z)-iB(z)} = py + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|S(t)/E(t)|^2dt}{(t-x)^2+y^2}$$

for $y > 0$. It follows that

$$\int_{-\infty}^{+\infty}(1+t^2)^{-1}\,|S(t)/E(t)|^2dt < \infty$$

and that

$$\frac{1}{2}\left|\frac{S(z)}{E(z)}\right|^2 + \frac{1}{2}\left|\frac{S^*(z)}{E(z)}\right|^2 \leq py + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|S(t)/E(t)|^2dt}{(t-x)^2+y^2}$$

for $y > 0$. By Problem 20, $[D(z)+iC(z)]/[A(z)-iB(z)]$ is of bounded type in the upper half-plane. Since the function has no zeros in the half-plane, it is the square of a function which is of bounded type in the half plane. Since $S(z)/E(z)$ and $S^*(z)/E(z)$ are then dominated in the upper half-plane by a function of bounded type in the half-plane, they are of bounded type in the half-plane. Nonpositive mean type is obtained by the second part of Theorem 10. The required estimate on the imaginary axis is obtained by the Lebesgue dominated convergence theorem.

Proof of Theorem 27, *the necessity.* To obtain the desired functions $C(z)$ and $D(z)$, we must determine the identity which generalizes the results of Problem 76 when $F(\alpha)$ and $G(\beta)$ are not zero. To do this we choose a point on the real axis where $S(z)$, and hence also $E(z)$, has a nonzero value. We assume for definiteness that the point is the origin, but a similar argument can be given with respect to any other real reference point.

We first define a linear functional $F(z) \to \tilde{F}(0)$ on $\mathcal{H}(E)$ in such a way that

$$\pi F(0)\tilde{G}(0)^- - \pi\tilde{F}(0)G(0)^- = \langle F(t)S(0), [G(t)S(0) - S(t)G(0)]/t\rangle - \langle [F(t)S(0) - S(t)F(0)]/t, G(t)S(0)\rangle$$

for all $F(z)$ and $G(z)$ in $\mathcal{H}(E)$. For the existence of such a functional, let $T(z) = A(z)u + B(z)v$ where u and v are real numbers chosen so that $A(0)u + B(0)v = 1$. By Problem 67, the identity

$$0 = \langle F(t), [G(t) - T(t)G(0)]/t\rangle - \langle [F(t) - T(t)F(0)]/t, G(t)\rangle$$

holds for all $F(z)$ and $G(z)$ in $\mathcal{H}(E)$. The desired identity is now obtained on defining

$$\pi\tilde{F}(0) = S(0)\langle F(t), [S(t) - T(t)S(0)]/t\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$.

We now define a linear functional $F(z) \to \tilde{F}(w)$ on $\mathcal{H}(E)$ for every w. Consider the function

$$G(z) = F(z)S(w) + w[F(z)S(w) - S(z)F(w)]/(z - w),$$

which belongs to $\mathcal{H}(E)$ by hypothesis, and notice that $G(0) = S(0)F(w)$. Define $\tilde{F}(w)$ so that $\tilde{G}(0) = S(0)\tilde{F}(w)$. By Problem 75, $F(z) \to \tilde{F}(w)$ is a continuous linear functional on $\mathcal{H}(E)$ for every w and $\tilde{F}(w)$ is a continuous function of w for every $F(z)$. If $F(z)$ is in $\mathcal{H}(E)$ and if $G(z) = [F(z)S(\alpha) - S(z)F(\alpha)]/(z - \alpha)$ for some number α, then it follows from the identity of Problem 75 that $\tilde{G}(w) = [\tilde{F}(w)S(\alpha) - S(w)\tilde{F}(\alpha)]/(w - \alpha)$ for all complex w. Since $[\tilde{F}(z)S(\alpha) - S(z)\tilde{F}(\alpha)]/(z - \alpha)$ must then be a continuous function of z for every α, $\tilde{F}(z)$ is an entire function. If h is a real zero of $E(z)$ of multiplicity r, then $S(z)$ and every element of $\mathcal{H}(E)$ has a zero of multiplicity at least r at h. It follows that $\tilde{F}(z)$ has a zero of multiplicity at least r at each such point. The values of $\tilde{F}(z)$ are used to generalize the identity of Problem 76.

If $F(z)$ and $G(z)$ are in $\mathcal{H}(E)$, then

$$\{[zF(z)S(\alpha) - S(z)\alpha F(\alpha)]/(z - \alpha) - S(z)F(\alpha)\}/z = [F(z)S(\alpha) - S(z)F(\alpha)]/(z - \alpha),$$

$$\{[zG(z)S(\beta) - S(z)\beta G(\beta)]/(z - \beta) - S(z)G(\beta)\}/z = [G(z)S(\beta) - S(z)G(\beta)]/(z - \beta)$$

for all complex numbers α and β. The definitions of $\tilde{F}(\alpha)$ and $\tilde{G}(\beta)$ and the defining property of $\tilde{F}(0)$ and $\tilde{G}(0)$ yield

$$\begin{aligned}
&\langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\
&\quad - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), G(t)S(\beta)\rangle \\
&\quad + (\alpha - \bar{\beta})\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\
&= \langle [tF(t)S(\alpha) - S(t)\alpha F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\
&\quad - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), [tG(t)S(\beta) - S(t)\beta G(\beta)]/(t - \beta)\rangle \\
&\qquad = \pi F(\alpha)\tilde{G}(\beta)^- - \pi\tilde{F}(\alpha)G(\beta)^-.
\end{aligned}$$

This is the general identity for difference quotients.

Since $F(z) \to \tilde{F}(w)$ is a continuous linear functional on $\mathcal{H}(E)$ for every w, there is a unique element $L(w, z)$ of $\mathcal{H}(E)$ such that

$$\tilde{F}(w) = \langle F(t), L(w, t)\rangle$$

for every $F(z)$ in $\mathcal{H}(E)$. The required functions $C(z)$ and $D(z)$ are obtained by finding the form of $L(w, z)$, which is restricted by the identity for difference quotients. If $F(z) = K(\alpha, z)$ for some number α, then

$$\tilde{F}(w) = \langle K(\alpha, t), L(w, t)\rangle = \langle L(w, t), K(\alpha, t)\rangle^- = L(w, \alpha)^-.$$

By the arbitrariness of w, $\tilde{F}(z) = L(z, \alpha)^-$ whenever $F(z) = K(\alpha, z)$ for some number α. Consider the special identity for difference quotients,

$$\langle F(t)S(w), [G(t)S(\bar{w}) - S(t)G(\bar{w})]/(t - \bar{w})\rangle - \langle [F(t)S(w) - S(t)F(w)]/(t - w), G(t)S(\bar{w})\rangle = \pi F(w)\tilde{G}(\bar{w})^- - \pi\tilde{F}(w)G(\bar{w})^-$$

and substitute $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for fixed numbers α and β. The resulting identity simplifies to

$$[B(w)\bar{A}(\alpha) - A(w)\bar{B}(\alpha)]P(\bar{w}, \beta) - [B(\beta)A(w) - A(\beta)B(w)]\bar{P}(w, \alpha) = -S(w)S^*(w)[B(\beta)\bar{A}(\alpha) - A(\beta)\bar{B}(\alpha)]$$

in terms of $P(w, z) = \pi(z - \bar{w})L(w, z) - S(z)\bar{S}(w)$.

Consider each side of the identity as a function of β for fixed α and w. The identity implies that $P(w, z)$ is a linear combination of $A(z)$ and $B(z)$ for each fixed w. Since $A(z)$ and $B(z)$ are linearly independent, there exist unique numbers $C(w)$ and $D(w)$ such that

$$P(w, z) = -A(z)\bar{D}(w) + B(z)\bar{C}(w).$$

The identity now states that $C(w) = \bar{C}(\bar{w})$, that $D(w) = \bar{D}(\bar{w})$, and that $A(w)D(w) - B(w)C(w) = S(w)S^*(w)$. In this notation the form of $L(w, z)$ is

$$L(w, z) = [S(z)\bar{S}(w) - A(z)\bar{D}(w) + B(z)\bar{C}(w)]/[\pi(z - \bar{w})].$$

If $F(z) = K(\alpha, z)$ for some fixed number α, $\tilde{F}(z) = \bar{L}(z, \alpha)$ is an entire function of z. It follows that $C(z)$ and $D(z)$ are entire functions.

To complete the proof we need to determine $\tilde{F}(z) = Q(\alpha, z)$ when $F(z) = L(\alpha, z)$ for some given number α. To do so let $G(z) = K(\beta, z)$ for a fixed β and substitute in the special identity for difference quotients. The result is

$$S(w)[\tilde{G}(\alpha)^- S^*(w) - \bar{S}(\alpha)\tilde{G}(w)^-]/(\bar{\alpha} - \bar{w}) - S^*(w)[F(\beta)S(w) - S(\beta)F(w)]/(\beta - w) = \pi F(w)\tilde{G}(\bar{w})^- - \pi\tilde{F}(w)G(\bar{w})^-.$$

Substitute the definitions of $F(z)$ and $G(z)$ and simplify. A short calculation yields

$$Q(\alpha, w) = [D(w)\bar{C}(\alpha) - C(w)\bar{D}(\alpha)]/[\pi(w - \bar{\alpha})].$$

So if $F(z) = L(\alpha, z)$ for some number α, then

$$\tilde{F}(z) = [D(z)\bar{C}(\alpha) - C(z)\bar{D}(\alpha)]/[\pi(z - \bar{\alpha})].$$

If h is a real zero of $E(z)$ of order r, then $[D(z)\bar{C}(\alpha) - C(z)\bar{D}(\alpha)]/[\pi(z - \bar{\alpha})]$ must have a zero of order at least r at h for every choice of α. By the arbitrariness of α, $C(z)$ and $D(z)$ each have a zero of order at least r at h. It follows that $C(z)/E(z)$ and $D(z)/E(z)$ have no singularities on the real axis.

The required inequalities for $C(z)$ and $D(z)$ are obtained by the positivity of an inner product. If w is a fixed number and if u and v are complex coefficients, then $K(w, z)u + L(w, z)v$ belongs to $\mathcal{H}(E)$ and its self-product is

$$\begin{aligned} &u\bar{u}[B(w)\bar{A}(w) - A(w)\bar{B}(w)]/[\pi(w - \bar{w})] \\ &\quad - u\bar{v}[S(w)\bar{S}(w) - D(w)\bar{A}(w) + C(w)\bar{B}(w)]/[\pi(w - \bar{w})] \\ &\quad + v\bar{u}[S(w)\bar{S}(w) - A(w)\bar{D}(w) + B(w)\bar{C}(w)]/[\pi(w - \bar{w})] \\ &\quad\quad + v\bar{v}[D(w)\bar{C}(w) - C(w)\bar{D}(w)]/[\pi(w - \bar{w})]. \end{aligned}$$

Since the expression is nonnegative for all u and v, we obtain

$$[D(w)\bar{C}(w) - C(w)\bar{D}(w)]/(w - \bar{w}) \geq 0,$$

$$\begin{aligned} &|[S(w)\bar{S}(w) - A(w)\bar{D}(w) + B(w)\bar{C}(w)]/(w - \bar{w})|^2 \\ &\quad \leq [B(w)\bar{A}(w) - A(w)\bar{B}(w)]/(w - \bar{w})\ [D(w)\bar{C}(w) - C(w)\bar{D}(w)]/(w - \bar{w}). \end{aligned}$$

The last inequality reduces to

$$\text{Re}\,[A(w)\bar{D}(w) - B(w)\bar{C}(w)] \geq \tfrac{1}{2}\,|S(w)|^2 + \tfrac{1}{2}\,|S^*(w)|^2.$$

If $F(z)$ is in $\mathcal{H}(E)$,

$$\begin{aligned} F(w)D(w) - \tilde{F}(w)B(w) &= \langle F(t), [B(t)\bar{A}(w) - A(t)\bar{B}(w)]\bar{D}(w)/[\pi(t - \bar{w})]\rangle \\ &\quad - \langle F(t), [S(t)\bar{S}(w) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]\bar{B}(w)/[\pi(t - \bar{w})]\rangle \\ &= S(w)\langle F(t), [B(t)S(\bar{w}) - S(t)B(\bar{w})]/[\pi(t - \bar{w})]\rangle \end{aligned}$$

for all complex numbers w. Apply the identity with $w = \alpha$ and with

$$\begin{aligned} F(z) &= [B(z)\bar{A}(\beta) - A(z)\bar{B}(\beta)]\bar{D}(\beta)/[\pi(z - \bar{\beta})] \\ &\quad - [S(z)\bar{S}(\beta) - A(z)\bar{D}(\beta) + B(z)\bar{C}(\beta)]\bar{B}(\beta)/[\pi(z - \bar{\beta})] \\ &= \bar{S}(\beta)[B(z)S(\bar{\beta}) - S(z)B(\bar{\beta})]/[\pi(z - \bar{\beta})], \end{aligned}$$

in which case

$$\begin{aligned} \tilde{F}(z) &= -[S(z)\bar{S}(\beta) - D(z)\bar{A}(\beta) + C(z)\bar{B}(\beta)]\bar{D}(\beta)/[\pi(z - \bar{\beta})] \\ &\quad - [D(z)\bar{C}(\beta) - C(z)\bar{D}(\beta)]\bar{B}(\beta)/[\pi(z - \bar{\beta})] \\ &= \bar{S}(\beta)[D(z)S(\bar{\beta}) - S(z)D(\bar{\beta})]/[\pi(z - \bar{\beta})] \end{aligned}$$

and

$$F(z)D(z) - \tilde{F}(z)B(z) = S(z)\bar{S}(\beta)[B(z)D(\bar{\beta}) - D(z)B(\bar{\beta})]/[\pi(z - \bar{\beta})].$$

It follows that

$$[B(\alpha)D(\bar{\beta}) - D(\alpha)B(\bar{\beta})]/[\pi(\alpha - \bar{\beta})]$$
$$= \langle [B(t)S(\bar{\beta}) - S(t)B(\bar{\beta})]/[\pi(t - \bar{\beta})], [B(t)S(\bar{\alpha}) - S(t)B(\bar{\alpha})]/[\pi(t - \bar{\alpha})]\rangle$$

if $S(\alpha)$ and $S(\beta)$ are not zero. The formula follows by continuity (Problem 75) for all complex numbers α and β. Let $\varphi(x)$ be a phase function associated with $E(z)$. If $B(z)$ does not belong to $\mathcal{H}(E)$, Theorem 22 can be used to calculate the inner product. The identity then implies that

$$\mathrm{Re}\, \frac{iD(z)}{B(z)} = \sum \frac{1}{\varphi'(t)} \left| \frac{S(t)}{E(t)} \right|^2 \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$, where summation is over the real numbers t such that $\varphi(t) \equiv 0$ modulo π. If α is any real number such that $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ does not belong to $\mathcal{H}(E)$, the same argument applies with $E(z)$ replaced by $e^{i\alpha}E(z)$, $A(z)$ replaced by $A(z) \cos \alpha + B(z) \sin \alpha$, $B(z)$ replaced by $B(z) \cos \alpha - A(z) \sin \alpha$, $D(z)$ replaced by $D(z) \cos \alpha - C(z) \sin \alpha$, and $C(z)$ replaced by $C(z) \cos \alpha + D(z) \sin \alpha$. The formula obtained in this case reads

$$\mathrm{Re}\, \frac{e^{i\alpha}[D(z) + iC(z)] + e^{-i\alpha}[D(z) - iC(z)]}{e^{i\alpha}[A(z) - iB(z)] - e^{-i\alpha}[A(z) + iB(z)]} = \sum \frac{1}{\varphi'(t)} \left| \frac{S(t)}{E(t)} \right|^2 \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$, where the summation is over the real points t such that $\varphi(t) \equiv \alpha$ modulo π.

Since $\varphi(x)$ is an increasing function of x which has a positive derivative everywhere and since $\varphi'(x)$ is a continuous function of x,

$$\frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 \frac{dt}{(t-x)^2 + y^2} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\varphi'(t)} \left| \frac{S(t)}{E(t)} \right|^2 \frac{d\varphi(t)}{(t-x)^2 + y^2}$$

for $y > 0$. By Problems 77 and 78 we can conclude that

$$\mathrm{Re}\, \frac{D(z) + iC(z)}{A(z) - iB(z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 \frac{dt}{(t-x)^2 + y^2}$$

for $y > 0$. By the Lebesgue dominated convergence theorem,

$$\lim_{y \to +\infty} \mathrm{Re}\, y^{-1}[D(iy) + iC(iy)]/E(iy) = 0.$$

This completes the proof of necessity. The essential uniqueness of $C(z)$ and $D(z)$ follows from the last formula. If $C_1(z)$ and $D_1(z)$ satisfy the hypotheses of the theorem,

$$\mathrm{Re}\, \frac{D_1(z) + iC_1(z)}{A(z) - iB(z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 \frac{dt}{(t-x)^2 + y^2} = \mathrm{Re}\, \frac{D(z) + iC(z)}{A(z) - iB(z)}$$

for $y > 0$. Thus

$$[D_1(z) + iC_1(z)]/E(z) - [D(z) + iC(z)]/E(z)$$

is a function which is analytic in the upper half-plane and whose real part vanishes identically. It is therefore a constant. If this constant is denoted ih, then h is real and

$$D_1(z) + iC_1(z) = D(z) + iC(z) + ihE(z),$$

so that $D_1(z) = D(z) + hB(z)$ and $C_1(z) = C(z) + hA(z)$.

Since $A(z)D(z) - B(z)C(z) = S(z)S^*(z)$ does not vanish identically, $C(z)$ and $D(z)$ cannot both vanish identically and $D(z) + iC(z)$ is not identically zero. Since $[D(z) + iC(z)]/E(z)$ has a nonnegative real part in the upper half-plane, it has no zeros in the half-plane by the maximum principle. It follows that $D(z) + iC(z)$ has no zeros in the upper half-plane. Since

$$[D(w)\bar{C}(w) - C(w)\bar{D}(w)]/(w - \bar{w}) = \pi\langle L(w, t), L(w, t)\rangle \geq 0$$

for all complex w,

$$|D(x - iy) + iC(x - iy)| \leq |D(x + iy) + iC(x + iy)|$$

for $y > 0$. By Problem 14, the inequality is strict for $y > 0$ if $C(z)$ and $D(z)$ are linearly independent.

If $F(z)$ is an element of $\mathcal{H}(E)$ which is a finite linear combination of special functions

$$[S(z)\bar{S}(w) - A(z)\bar{D}(w) + B(z)\bar{C}(w)]/[\pi(z - \bar{w})],$$

then $\tilde{F}(z)$ is a finite linear combination of the functions

$$[D(z)\bar{C}(w) - C(z)\bar{D}(w)]/[\pi(z - \bar{w})].$$

It follows that $\tilde{F}(z)$ belongs to $\mathcal{H}(D + iC)$. An obvious calculation will show that $\tilde{F}(z)$ has the same norm in $\mathcal{H}(D + iC)$ as $F(z)$ has in $\mathcal{H}(E)$. If on the other hand $F(z)$ is an element of $\mathcal{H}(E)$ which belongs to the closed span of such special functions, then the same conclusion follows by continuity: $\tilde{F}(z)$ belongs to $\mathcal{H}(D + iC)$ and has the same norm there as $F(z)$ has in $\mathcal{H}(E)$. If $F(z)$ is orthogonal to such special functions, $\tilde{F}(z)$ vanishes identically. Thus the transformation $F(z) \rightarrow \tilde{F}(z)$ is a partial isometry of $\mathcal{H}(E)$ into $\mathcal{H}(D + iC)$. Since the range of the transformation is a closed subspace of $\mathcal{H}(D + iC)$ which contains $[D(z)\bar{C}(w) - C(z)\bar{D}(w)]/[\pi(z - \bar{w})]$ for all complex w, it is all of $\mathcal{H}(D + iC)$.

If $D_1(z) = D(z) + hB(z)$ and $C_1(z) = C(z) + hA(z)$ for some real number h, a similar argument will show that a space $\mathcal{H}(D_1 + iC_1)$ exists if $C_1(z)$ and $D_1(z)$ are linearly independent, and that the transformation $F(z) \to \tilde{F}(z) + hF(z)$ is a partial isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(D_1 + iC_1)$.

PROBLEM 79. If $C(z)$ and $D(z)$ are linearly dependent in Theorem 27, show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = \bar{v}u$.

28. CONSTRUCTION OF THE SPACE $\mathcal{H}_S(M)$

The proof of Theorem 27 leads to the construction of a new Hilbert space whose elements are pairs of entire functions. The following matrix notation is used to simplify work with pairs of functions. A bar is used to denote the conjugate transpose of a rectangular matrix. Thus

$$\begin{pmatrix} a \\ b \end{pmatrix}^- = (\bar{a} \quad \bar{b}), \qquad (a \quad b)^- = \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix},$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^- = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix}.$$

A square matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is said to be nonnegative if the number $\begin{pmatrix} a \\ b \end{pmatrix}^- \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$ is nonnegative for all choices of numbers a and b. In this case we write $M \geq 0$. Let

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

PROBLEM 80. Show that a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonnegative if, and only if, $A \geq 0$, $D \geq 0$, $C = \bar{B}$, and $BC \leq AD$.

PROBLEM 81. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix of numbers having a nonzero determinant, and let S and T be numbers such that $\det M = AD - BC = ST$. Show that the matrix $(MI\bar{M} - SI\bar{S})/i$ is nonnegative if, and only if, $i(A\bar{B} - B\bar{A}) \geq 0$, $i(C\bar{D} - D\bar{C}) \geq 0$, and $\operatorname{Re}(A\bar{D} - B\bar{C}) \geq \frac{1}{2}|S|^2 + \frac{1}{2}|T|^2$.

PROBLEM 82. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a matrix of numbers having a nonzero determinant. If $i(A\bar{B} - B\bar{A}) \geq 0$, $i(C\bar{D} - D\bar{C}) \geq 0$, and $\operatorname{Re}(A\bar{D} - B\bar{C}) \geq |AD - BC|$, show that $i(C\bar{A} - A\bar{C}) \geq 0$ and that $i(D\bar{B} - B\bar{D}) \geq 0$.

THEOREM 28. Let $M(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}$ be a matrix of entire functions which are real for real z, such that $\det M(z) = S(z)S^*(z)$ for some entire function $S(z)$ and such that

$$[M(z)I\bar{M}(z) - S(z)I\bar{S}(z)]/(z - \bar{z}) \geq 0$$

for all complex z. Assume that $A(z) - iB(z)$ has no zeros in the upper half-plane and that $[D(z) + iC(z)]/[A(z) - iB(z)]$ has no real singularities. Then there exists a unique Hilbert space $\mathcal{H}_S(M)$, whose elements are pairs $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of entire functions, such that

$$\frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to the space for all complex numbers u, v, and w, and such that

$$\begin{pmatrix} u \\ v \end{pmatrix}^{-}\begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{2\pi(t - \bar{w})}\begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}_S(M)$. The pair

$$\begin{pmatrix} [F_+(z)S(w) - S(z)F_+(w)]/(z - w) \\ [F_-(z)S(w) - S(z)F_-(w)]/(z - w) \end{pmatrix}$$

belongs to $\mathcal{H}_S(M)$ whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathcal{H}_S(M)$, and the identity

$$\begin{aligned} 2\pi\begin{pmatrix} G_+(\beta) \\ G_-(\beta) \end{pmatrix}^{-} I \begin{pmatrix} F_+(\alpha) \\ F_-(\alpha) \end{pmatrix} \\ = \left\langle \begin{pmatrix} F_+(t)S(\alpha) \\ F_-(t)S(\alpha) \end{pmatrix}, \begin{pmatrix} [G_+(t)S(\beta) - S(t)G_+(\beta)]/(t - \beta) \\ [G_-(t)S(\beta) - S(t)G_-(\beta)]/(t - \beta) \end{pmatrix} \right\rangle \\ - \left\langle \begin{pmatrix} [F_+(t)S(\alpha) - S(t)F_+(\alpha)]/(t - \alpha) \\ [F_-(t)S(\alpha) - S(t)F_-(\alpha)]/(t - \alpha) \end{pmatrix}, \begin{pmatrix} G_+(t)S(\beta) \\ G_-(t)S(\beta) \end{pmatrix} \right\rangle \\ + (\alpha - \bar{\beta})\left\langle \begin{pmatrix} [F_+(t)S(\alpha) - S(t)F_+(\alpha)]/(t - \alpha) \\ [F_-(t)S(\alpha) - S(t)F_-(\alpha)]/(t - \alpha) \end{pmatrix}, \right. \\ \left. \begin{pmatrix} [G_+(t)S(\beta) - S(t)G_+(\beta)]/(t - \beta) \\ [G_-(t)S(\beta) - S(t)G_-(\beta)]/(t - \beta) \end{pmatrix} \right\rangle \end{aligned}$$

holds for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ of $\mathcal{H}_S(M)$ and all complex numbers α and β.

Proof of Theorem 28. By Problem 81, the hypotheses imply that

$$[B(z)\bar{A}(z) - A(z)\bar{B}(z)]/(z - \bar{z}) \geq 0$$

for all complex z. It follows that

$$|A(x - iy) - iB(x - iy)| \leq |A(x + iy) - iB(x + iy)|$$

for $y > 0$. Since we assume that $A(z) - iB(z)$ has no zeros in the upper half-plane, the inequality is strict if $A(z)$ and $B(z)$ are linearly independent. In this case a space $\mathcal{H}(E)$ exists, $E(z) = A(z) - iB(z)$. By Problems 81 and 82,

$$\begin{aligned}\operatorname{Re}\,[D(z) + iC(z)]/[A(z) - iB(z)]& \\ = |E(z)|^{-2} \operatorname{Re}\,[A(z)\bar{D}(z) &- B(z)\bar{C}(z)] \\ + \tfrac{1}{2}\,|E(z)|^{-2}[iC(z)\bar{A}(z) &- iA(z)\bar{C}(z)] \\ + \tfrac{1}{2}\,|E(z)|^{-2}[iD(z)\bar{B}(z) &- iB(z)\bar{D}(z)] \geq 0\end{aligned}$$

for $y > 0$. On the other hand we assume that $[D(z) + iC(z)]/[A(z) - iB(z)]$ has no real singularities and

$$\operatorname{Re}\,[D(z) + iC(z)]/[A(z) - iB(z)] = |S(z)/E(z)|^2$$

for all real z. By Problems 2 and 3, there exists a number $p \geq 0$ such that

$$\operatorname{Re} \frac{D(z) + iC(z)}{A(z) - iB(z)} = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{dt}{(t - x)^2 + y^2}$$

for $y > 0$. By the Lebesgue dominated convergence theorem,

$$p = \lim_{y \to +\infty} \operatorname{Re} y^{-1}[D(iy) + iC(iy)]/[A(iy) - iB(iy)].$$

If a space $\mathcal{H}(E)$ exists and if $p = 0$, then $[S(z)\bar{S}(w) - A(z)\bar{D}(w) + B(z)\bar{C}(w)]/[\pi(z - \bar{w})]$ belongs to $\mathcal{H}(E)$ as a function of z for every w. By the proof of Theorem 27, there exists an entire function $\tilde{F}(z)$, associated with every $F(z)$ in $\mathcal{H}(E)$, such that

$$\tilde{F}(w) = \langle F(t), [S(t)\bar{S}(w) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]/[\pi(t - \bar{w})]\rangle_E$$

for all complex w. Let $\mathcal{H}$ be the set of pairs $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ of such entire functions. Then $\mathcal{H}$ is a Hilbert space in the norm

$$\left\| \begin{pmatrix} F(t) \\ \tilde{F}(t) \end{pmatrix} \right\|^2 = 2\,\|F(t)\|_E^2.$$

If $G(z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z-\bar{w})]$ in $\mathcal{H}(E)$, then

$$\tilde{G}(z) = -[S(z)\bar{S}(w) - D(z)\bar{A}(w) + C(z)\bar{B}(w)]/[\pi(z-\bar{w})]$$

and

$$\begin{pmatrix} G(z) \\ \tilde{G}(z) \end{pmatrix} = \frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{\pi(z-\bar{w})}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

belongs to $\mathcal{H}$. If $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ is in $\mathcal{H}$, then

$$\left\langle \begin{pmatrix} F(t) \\ \tilde{F}(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{\pi(t-\bar{w})}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

$$= 2\langle F(t), [B(t)\bar{A}(w) - A(t)\bar{B}(w)]/[\pi(t-\bar{w})]\rangle_E$$

$$= 2F(w).$$

On the other hand, if $G(z) = [S(z)\bar{S}(w) - A(z)\bar{D}(w) + B(z)\bar{C}(w)]/[\pi(z-\bar{w})]$ in $\mathcal{H}(E)$, then $\tilde{G}(z) = [D(z)\bar{C}(w) - C(z)\bar{D}(w)]/[\pi(z-\bar{w})]$, and

$$\begin{pmatrix} G(z) \\ \tilde{G}(z) \end{pmatrix} = \frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{\pi(z-\bar{w})}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

belongs to $\mathcal{H}$. If $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ is in $\mathcal{H}$, then

$$\left\langle \begin{pmatrix} F(t) \\ \tilde{F}(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{\pi(t-\bar{w})}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

$$= 2\langle F(t), [S(t)\bar{S}(w) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]/[\pi(t-\bar{w})]\rangle_E$$

$$= 2\tilde{F}(w).$$

By linearity

$$\frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{2\pi(z-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathcal{H}$ for all complex numbers u, v, and w, and

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F(w) \\ \tilde{F}(w) \end{pmatrix} = \left\langle \begin{pmatrix} F(t) \\ \tilde{F}(t) \end{pmatrix}, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{2\pi(t-\bar{w})}\begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

for all elements $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ of $\mathcal{H}$. The space $\mathcal{H}$ is therefore the required space $\mathcal{H}_S(M)$ in this case. By Theorem 27,

$$\begin{pmatrix} [F(z)S(w) - S(z)F(w)]/(z-w) \\ [\tilde{F}(z)S(w) - S(z)\tilde{F}(w)]/(z-w) \end{pmatrix}$$

belongs to $\mathcal{H}_S(M)$ whenever $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ belongs to $\mathcal{H}_S(M)$ for every w. If $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ and $\begin{pmatrix} G(z) \\ \tilde{G}(z) \end{pmatrix}$ belong to $\mathcal{H}_S(M)$ and if α and β are complex numbers, then

$$\left\langle \begin{pmatrix} F(t)S(\alpha) \\ \tilde{F}(t)S(\alpha) \end{pmatrix}, \begin{pmatrix} [G(t)S(\beta) - S(t)G(\beta)]/(t-\beta) \\ [\tilde{G}(t)S(\beta) - S(t)\tilde{G}(\beta)]/(t-\beta) \end{pmatrix} \right\rangle$$

$$- \left\langle \begin{pmatrix} [F(t)S(\alpha) - S(t)F(\alpha)]/(t-\alpha) \\ [\tilde{F}(t)S(\alpha) - S(t)\tilde{F}(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} G(t)S(\beta) \\ \tilde{G}(t)S(\beta) \end{pmatrix} \right\rangle$$

$$+ (\alpha - \bar{\beta}) \left\langle \begin{pmatrix} [F(t)S(\alpha) - S(t)F(\alpha)]/(t-\alpha) \\ [\tilde{F}(t)S(\alpha) - S(t)\tilde{F}(\alpha)]/(t-\alpha) \end{pmatrix}, \begin{pmatrix} [G(t)S(\beta) - S(t)G(\beta)]/(t-\beta) \\ [\tilde{G}(t)S(\beta) - S(t)\tilde{G}(\beta)]/(t-\beta \end{pmatrix} \right\rangle$$

$$= 2\langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t-\beta)\rangle$$

$$-2\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t-\alpha), G(t)S(\beta)\rangle$$

$$+ (\alpha - \bar{\beta})\langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t-\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t-\beta)\rangle$$

$$= 2\pi F(\alpha)\tilde{G}(\beta)^- - 2\pi\tilde{F}(\alpha)G(\beta)^-$$

$$= 2\pi \begin{pmatrix} G(\beta) \\ \tilde{G}(\beta) \end{pmatrix}^- I \begin{pmatrix} F(\alpha) \\ \tilde{F}(\alpha) \end{pmatrix}$$

by Theorem 27.

If a space $\mathcal{H}(E)$ exists and if $p > 0$, the proof of sufficiency for Theorem 27 will show that $[F(z)S(w) - S(z)F(w)]/(z-w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. By the necessity for Theorem 27, there exist entire functions $C_1(z)$ and $D_1(z)$, which are real for real z, such that

$$A(z)D_1(z) - B(z)C_1(z) = S(z)S^*(z),$$

$$\operatorname{Re}\,[A(z)\bar{D}_1(z) - B(z)\bar{C}_1(z)] \geq \tfrac{1}{2}\,|S(z)|^2 + \tfrac{1}{2}\,|S^*(z)|^2$$

for all complex z, $[D_1(z) + iC_1(z)]/E(z)$ has no real singularities, and

$$\lim_{y\to+\infty} \operatorname{Re} y^{-1}[D_1(iy) + iC_1(iy)]/E(iy) = 0.$$

By the proof of sufficiency for Theorem 27,

$$\operatorname{Re} \frac{D_1(z) + iC_1(z)}{A(z) - iB(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{dt}{(t-x)^2 + y^2}$$

$$= \operatorname{Re}\left\{ipz + \frac{D(z) + iC(z)}{A(z) - iB(z)}\right\}$$

for $y > 0$. It follows that

$$\frac{D_1(z) + iC_1(z)}{A(z) - iB(z)} - ipz - \frac{D(z) + iC(z)}{A(z) - iB(z)} = ih$$

for some real constant h. Then

$$C_2(z) = C_1(z) - hA(z) \quad \text{and} \quad D_2(z) = D_1(z) - hB(z)$$

are entire functions which are real for real z,

$$A(z)D_2(z) - B(z)C_2(z) = S(z)S^*(z),$$

$$\operatorname{Re}\,[A(z)\bar{D}_2(z) - B(z)\bar{C}_2(z)] \geq \tfrac{1}{2}\,|S(z)|^2 + \tfrac{1}{2}\,|S^*(z)|^2$$

for all complex z, $[D_2(z) + iC_2(z)]/E(z)$ has no real singularities, and

$$\lim_{y\to+\infty} \operatorname{Re} y^{-1}[D_2(iy) + iC_2(iy)]/E(iy) = 0.$$

Since $C(z) = C_2(z) - pzA(z)$ and $D(z) = D_2(z) - pzB(z)$,

$$M(z) = \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} M_2(z) = \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} \begin{pmatrix} A_2(z) & B_2(z) \\ C_2(z) & D_2(z) \end{pmatrix}$$

and we obtain the identity

$$\frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}$$

$$= \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} \frac{M_2(z)I\bar{M}_2(w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})} \begin{pmatrix} 1 & -p\bar{w} \\ 0 & 1 \end{pmatrix}$$

$$+ S(z)\begin{pmatrix} 0 & 0 \\ 0 & p/(2\pi) \end{pmatrix}\bar{S}(w).$$

Since there is no nonzero element $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}_S(M_2)$ such that $F_+(z) = 0$, there is no nonzero element $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}_S(M_2)$ such that $\begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is a constant multiple of $\begin{pmatrix} 0 \\ S(z) \end{pmatrix}$. The required space $\mathcal{H}_S(M)$ is now the set of all pairs

$$\lambda\begin{pmatrix} 0 \\ S(z) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

where λ is a constant and $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathcal{H}_S(M_2)$, the norm of such an element being defined by

$$\left\| \lambda \begin{pmatrix} 0 \\ S(t) \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -pt & 1 \end{pmatrix} \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2 = |\lambda|^2 2\pi/p + \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2_{\mathcal{H}_S(M_2)}.$$

The required properties of the space are obtained in a routine way from the known properties of $\mathcal{H}_S(M_2)$.

If $A(z)$ and $B(z)$ are linearly dependent, then one function, say $A(z)$, does not vanish identically and we can write $B(z) = hA(z)$ for some real number h. Since

$$\begin{aligned} \operatorname{Re} \{\bar{A}(z)[D(z) - hC(z)]\} &= \operatorname{Re} [\bar{A}(z)D(z) - \bar{B}(z)C(z)] \\ &\geq \tfrac{1}{2} |S(z)|^2 + \tfrac{1}{2} |S^*(z)|^2 \\ &\geq |S(z)S^*(z)| \\ &\geq |\bar{A}(z)[D(z) - hC(z)]|, \end{aligned}$$

$[D(z) - hC(z)]/A(z)$ is a real valued analytic function. It is therefore a constant. Since $|S(z)/A(z)|$ is a constant, $S(z)$ is a constant multiple of $A(z)$. We know that there exists a number $p \geq 0$ such that

$$\operatorname{Re} \frac{D(z) + iC(z)}{A(z) - iB(z)} = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 \frac{dt}{(t - x)^2 + y^2}$$

for $y > 0$. Since $S(z)/E(z)$ is a constant, we find that the real part of $[D(z) + iC(z)]/[A(z) - iB(z)] + ipz$ is a constant. It follows that this analytic function is a constant, and we obtain the identity

$$\frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})} = S(z) \begin{pmatrix} 0 & 0 \\ 0 & p/(2\pi) \end{pmatrix} \bar{S}(w).$$

If $p = 0$, the required space $\mathcal{H}_S(M)$ contains no nonzero element. If $p > 0$, the space consists of pairs $\lambda \begin{pmatrix} 0 \\ S(z) \end{pmatrix}$ in the norm $\left\| \lambda \begin{pmatrix} 0 \\ S(t) \end{pmatrix} \right\|^2 = |\lambda|^2\, 2\pi/p$. This constructs the space $\mathcal{H}_S(M)$ in all cases. Uniqueness is proved as in SSPS Lemma 11.

PROBLEM 83. If $\mathcal{H}(E)$ is a given space, show that the hypotheses of Theorem 28 are satisfied with $S(z) = E(z)$, $C(z) = -B(z)$, and $D(z) = A(z)$. Show that the transformation $F(z) \to \dfrac{1}{\sqrt{2}} \begin{pmatrix} F(z) \\ iF(z) \end{pmatrix}$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}_S(M)$.

PROBLEM 84. If $\mathcal{H}_S(M)$ is a given space, show that a space $\mathcal{H}_{S^*}(M)$ exists and that the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \rightarrow \begin{pmatrix} F_+^*(z) \\ F_-^*(z) \end{pmatrix}$$

takes $\mathcal{H}_S(M)$ isometrically onto $\mathcal{H}_{S^*}(M)$.

PROBLEM 85. Show that an element $S(z)$ of a space $\mathcal{H}(E)$ is of the form $S(z) = A(z)u + B(z)v$ for some numbers u and v if, and only if,

$$[K(w, z)S(w) - K(w, w)S(z)]/(z - w) = [K(\bar{w}, z)S(\bar{w}) - K(\bar{w}, \bar{w})S(z)]/(z - \bar{w})$$

for all complex z and w. If $S(z)$ is of this form, show that $\bar{u}v = \bar{v}u$.

29. DOMAIN OF MULTIPLICATION BY z IN $\mathcal{H}(E)$

If $\mathcal{H}(E)$ is a given space, multiplication by z in $\mathcal{H}(E)$ is the transformation defined by $F(z) \rightarrow zF(z)$ whenever $F(z)$ and $zF(z)$ are in $\mathcal{H}(E)$. In general, multiplication by z is not a densely defined transformation in $\mathcal{H}(E)$, but the domain of the transformation is never very far from dense.

THEOREM 29. A necessary and sufficient condition that an element $S(z)$ of a space $\mathcal{H}(E)$ be orthogonal to the domain of multiplication by z in $\mathcal{H}(E)$ is that $S(z) = A(z)u + B(z)v$ for some numbers u and v.

Proof of Theorem 29, *the sufficiency.* If $F(z)$ is in the domain of multiplication by z in $\mathcal{H}(E)$ and if w is not real, then

$$\begin{aligned}
&\langle (t - w)F(t), S(t)\rangle K(w, w) \\
&\quad = \langle (t - w)F(t), S(t)K(w, w) - K(w, t)S(w)\rangle \\
&\quad = \langle (t - w)F(t), [S(t)K(\bar{w}, \bar{w}) - K(\bar{w}, t)S(\bar{w})](t - w)/(t - \bar{w})\rangle \\
&\quad = \langle (t - \bar{w})F(t), S(t)K(\bar{w}, \bar{w}) - K(\bar{w}, t)S(\bar{w})\rangle \\
&\quad = \langle (t - \bar{w})F(t), S(t)\rangle K(\bar{w}, \bar{w})
\end{aligned}$$

by (H1) and Problem 85. Since $K(w, w) = K(\bar{w}, \bar{w})$ is not zero when w is not real, $\langle (t - w)F(t), S(t)\rangle = \langle (t - \bar{w})F(t), S(t)\rangle$. Since w is not real, it follows that $S(z)$ is orthogonal to $F(z)$.

Proof of Theorem 29, *the necessity.* If $F(z)$ is in the domain of multiplication by z in $\mathcal{H}(E)$, then it is orthogonal to $S(z)$ and

$$\langle (t - w)F(t), S(t)\rangle K(w, w) = \langle (t - \bar{w})F(t), S(t)\rangle K(\bar{w}, \bar{w})$$

for nonreal w. It follows that

$$\langle (t-w)F(t), S(t)K(w,w) - K(w,t)S(w)\rangle$$
$$= \langle (t-\bar{w})F(t), S(t)K(\bar{w},\bar{w}) - K(\bar{w},t)S(\bar{w})\rangle.$$

Because of (H1),

$$\langle (t-w)F(t), S(t)K(w,w) - K(w,t)S(w)\rangle$$
$$= \langle (t-w)F(t), [S(t)K(\bar{w},\bar{w}) - K(\bar{w},t)S(\bar{w})](t-w)/(t-\bar{w})\rangle.$$

It follows from (H1) that every element of $\mathcal{H}(E)$ which vanishes at w is of the form $(z-w)F(z)$ for some element $F(z)$ of $\mathcal{H}(E)$. Since

$$[S(z)K(w,w) - K(w,z)S(w)]$$
$$- [S(z)K(\bar{w},\bar{w}) - K(\bar{w},z)S(\bar{w})](z-w)/(z-\bar{w})$$

vanishes at w and is orthogonal to every function which vanishes at w, it vanishes identically. The necessity now follows from Problem 85.

PROBLEM 86. Show that a space $\mathcal{H}(E(b))$ has dimension 1 if, and only if,

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))\begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z\end{pmatrix}$$

where $A(a,z)$ and $B(a,z)$ are linearly dependent entire functions which are real for real z, and where α, β, γ are real numbers, not all zero, such that $\alpha \geq 0$, $\gamma \geq 0$, and $\alpha\gamma = \beta^2$. Show that

$$\alpha = \pi u\bar{u}, \qquad \beta = \pi u\bar{v} = \pi v\bar{u}, \qquad \gamma = \pi v\bar{v}$$

for some numbers u and v such that

$$S(z) = A(a,z)u + B(a,z)v = A(b,z)u + B(b,z)v$$

is an element of norm 1 in $\mathcal{H}(E(b))$.

PROBLEM 87. Let $\mathcal{H}(E(b))$ be a given space which has dimension greater than 1 and in which multiplication by z is not densely defined. Show that

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))\begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z\end{pmatrix}$$

for some space $\mathcal{H}(E(a))$ which is contained isometrically in $\mathcal{H}(E(b))$ and for some numbers α, β, γ, not all zero such that $\alpha \geq 0$, $\gamma \geq 0$, and $\alpha\gamma = \beta^2$. Show that

$$\alpha = \pi u\bar{u}, \qquad \beta = \pi u\bar{v} = \pi v\bar{u}, \qquad \gamma = \pi v\bar{v}$$

for some numbers u and v such that

$$S(z) = A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$$

is an element of norm 1 in $\mathcal{H}(E(b))$ which spans the orthogonal complement of $\mathcal{H}(E(a))$.

PROBLEM 88. Show that multiplication by z is not densely defined in a space $\mathcal{H}(E)$ if the space has finite dimension. Show that a space $\mathcal{H}(E)$ has finite dimension r if, and only if, $E(z) = S(z)E_0(z)$ where $S(z)$ is an entire function which is real for real z and $E_0(z)$ is a polynomial of degree r which has no real zeros.

30. MEASURES AND $\mathcal{L}(\varphi)$ SPACES

In a space $\mathcal{H}(E)$ there are many ways of computing norms by integration on the real axis. Some of these are given by Theorem 22. We now study general measures associated with a space $\mathcal{H}(E)$.

THEOREM 30. Let $\mathcal{H}(E)$ be a given space and let $S(z)$ be an entire function, not identically zero, such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Let $C(z)$ and $D(z)$ be entire functions which are real for real z, such that

$$A(z)D(z) - B(z)C(z) = S(z)S^*(z),$$

$$\operatorname{Re} [A(z)\bar{D}(z) - B(z)\bar{C}(z)] \geq \tfrac{1}{2} |S(z)|^2 + \tfrac{1}{2} |S^*(z)|^2$$

for all complex z, $[D(z) + iC(z)]/E(z)$ has no real singularities and

$$\lim_{y\to+\infty} \operatorname{Re} y^{-1}[D(iy) + iC(iy)]/E(iy) = 0.$$

If $F(z)$ is in $\mathcal{H}(E)$, let $\tilde{F}(z)$ be the unique entire function such that

$$\tilde{F}(w) = \langle F(t), [S(t)\bar{S}(w) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]/[\pi(t - \bar{w})]\rangle$$

for all complex w. Let $\mu(x)$ be a nondecreasing function of real x. If

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/S(t)|^2 d\mu(t)$$

for every $F(z)$ in $\mathcal{H}(E)$, then there exists a function $\varphi(z)$, analytic for $y > 0$, such that

$$\operatorname{Re} \varphi(x + iy) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$, and such that

$$F(z) \to [i\varphi(z)F(z) + \tilde{F}(z)]/S(z)$$

is an isometric transformation of $\mathcal{H}(E)$ into $\mathfrak{L}(\varphi)$.

Proof of Theorem 30. If w is not real and is not a zero of $S(z)$, we show that there exists a number $\varphi(w)$ such that

$$\frac{i\varphi(w)F(w) + \tilde{F}(w)}{S(w)} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F(t)}{S(t)} \frac{d\mu(t)}{t - w}$$

for every $F(z)$ in $\mathcal{H}(E)$. The hypotheses imply that the integral is absolutely convergent whenever $F(z)$ belongs to $\mathcal{H}(E)$. The existence of a number $\varphi(w)$ with this property follows once we show that the identity

$$\frac{\tilde{F}(w)}{S(w)} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F(t)}{S(t)} \frac{d\mu(t)}{t - w}$$

holds whenever $F(z)$ belongs to $\mathcal{H}(E)$ and vanishes at w. If $G(z)$ is an element of $\mathcal{H}(E)$ which has a nonzero value at $\bar{w}$, the required identity is equivalent to

$$\pi \frac{\tilde{F}(w)}{S(w)} G^*(w) = \int_{-\infty}^{+\infty} \frac{F(t)\bar{G}(t)S^*(w)}{t - w} \frac{d\mu(t)}{|S(t)|^2} - \int_{-\infty}^{+\infty} F(t) \frac{\bar{G}(t)S^*(w) - \bar{S}(t)G^*(w)}{t - w} \frac{d\mu(t)}{|S(t)|^2}.$$

The hypotheses imply that this formula can be written

$$\pi\tilde{F}(w)G^*(w) = \langle F(t)S(w)/(t - w), G(t)S(\bar{w})\rangle - \langle F(t)S(w), [G(t)S(\bar{w}) - S(t)G(\bar{w})]/(t - \bar{w})\rangle.$$

It is now recognized as a special case of the identity for difference quotients, Theorem 27. This completes the proof of existence of $\varphi(w)$ when w is not a zero of $S(z)$. The existence of $\varphi(w)$ follows by continuity for all nonreal values of w. We now solve for $\varphi(w)$ in terms of $\mu(x)$.

Let α be a nonreal number, $\alpha \neq w$, and replace $F(z)$ by $[F(z)S(\alpha) - S(z)F(\alpha)]/(z - \alpha)$ in the definition of $\varphi(w)$. A short calculation gives the identity

$$\frac{\varphi(\alpha) - \varphi(w)}{\alpha - w} = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - \alpha)(t - w)}.$$

Let $\psi(w)$ be the function of nonreal w such that

$$\frac{i\psi(w)F(w) + \hat{F}(w)}{S^*(w)} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{F(t)}{S^*(t)} \frac{d\mu(t)}{t - w}$$

for every $F(z)$ in $\mathcal{H}(E)$ where $\hat{F}(z)$ is the entire function defined by

$$\hat{F}(w) = \langle F(t), [S^*(t)S(\bar{w}) - A(t)\bar{D}(w) + B(t)\bar{C}(w)]/[\pi(t - \bar{w})]\rangle.$$

An obvious calculation will show that $\psi(w) = \varphi(w)$. Since $\hat{F}(z)$ is the conjugate of $\tilde{F}(z)$ by Problem 84, we obtain

$$\frac{i\varphi(w)F^*(w) + \tilde{F}(\bar{w})^-}{S^*(w)} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\bar{F}(t)}{\bar{S}(t)}\frac{d\mu(t)}{t - w}.$$

Replacing w by its conjugate and then conjugating each side of the equation, we obtain $\varphi^*(w) = -\varphi(w)$. It follows that

$$\frac{\varphi(z) + \bar{\varphi}(w)}{z - \bar{w}} = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})}$$

when z and w are not real. When $z = w$, we have

$$\text{Re}\,\varphi(x + iy) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}.$$

A space $\mathcal{L}(\varphi)$ exists by Theorem 5 and

$$F(z) \to [i\varphi(z)F(z) + \tilde{F}(z)]/S(z)$$

is an isometric transformation of $\mathcal{H}(E)$ into $\mathcal{L}(\varphi)$.

PROBLEM 89. Let $\mathcal{H}(E)$ be a given space and let $\varphi(x)$ be a choice of phase function associated with $E(z)$. Show that there exists a number $p = p(\alpha) \geq 0$ for every real number α such that

$$\text{Re}\,\frac{e^{i\alpha}E(z) + e^{-i\alpha}E^*(z)}{e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)} = py + \sum \frac{1}{\varphi'(t)}\frac{y}{(t - x)^2 + y^2}$$

for $y > 0$, where summation is over all real numbers t such that $\varphi(t) \equiv \alpha$ modulo π. Show that $p > 0$ if, and only if, $e^{i\alpha}E(z) - e^{-i\alpha}E^*(z)$ belongs to $\mathcal{H}(E)$.

31. $\mathcal{L}(\varphi)$ SPACES ASSOCIATED WITH $\mathcal{H}(E)$

By the proof of Theorem 28, the conclusion of Theorem 30 states that the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}\,\frac{i\varphi(z)F_+(z) + F_-(z)}{S(z)}$$

is an isometry of $\mathcal{H}_S(M)$ into $\mathfrak{L}(\varphi)$. This implies a relation between $\varphi(z)$ and $M(z)$.

THEOREM 31. If $\mathfrak{L}(\varphi)$ and $\mathcal{H}_S(M)$ are given spaces and if

$$T: \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}\, \frac{i\varphi(z)F_+(z) + F_-(z)}{S(z)}$$

is a transformation of $\mathcal{H}_S(M)$ into $\mathfrak{L}(\varphi)$ which is bounded by 1, then

$$\varphi(z) = \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)}$$

for some function $W(z)$ which is analytic and bounded by 1 in the upper half-plane.

Proof of Theorem 31. The adjoint T^* of T is a transformation of $\mathfrak{L}(\varphi)$ into $\mathcal{H}_S(M)$ which is bounded by 1. If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathcal{H}_S(M)$ and if w is in the upper half-plane,

$$\left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \sqrt{2}\, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{2\pi(t - \bar{w})} \begin{pmatrix} -i\bar{\varphi}(w)/\bar{S}(w) \\ 1/\bar{S}(w) \end{pmatrix} \right\rangle$$
$$= \langle \sqrt{2}[i\varphi(t)F_+(t) + F_-(t)]/S(t), [\varphi(t) + \bar{\varphi}(w)]/[\pi i(\bar{w} - t)] \rangle.$$

It follows that

$$T^*: \frac{\varphi(z) + \bar{\varphi}(w)}{\pi i(\bar{w} - z)} \to \sqrt{2}\, \frac{M(z)I\bar{M}(w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})} \begin{pmatrix} -i\bar{\varphi}(w)/\bar{S}(w) \\ 1/\bar{S}(w) \end{pmatrix}.$$

Since T^* is bounded by 1,

$$\frac{\varphi(w) + \bar{\varphi}(w)}{\pi i(\bar{w} - w)} = \left\| \frac{\varphi(t) + \bar{\varphi}(w)}{\pi i(\bar{w} - t)} \right\|^2$$
$$\geq \left\| \sqrt{2}\, \frac{M(t)I\bar{M}(w) - S(t)I\bar{S}(w)}{2\pi(t - \bar{w})} \begin{pmatrix} -i\bar{\varphi}(w)/\bar{S}(w) \\ 1/\bar{S}(w) \end{pmatrix} \right\|^2$$
$$\geq (i\bar{\varphi}(w)/S(w) \quad 1/S(w))\, \frac{M(w)I\bar{M}(w) - S(w)I\bar{S}(w)}{\pi(w - \bar{w})} \begin{pmatrix} -i\bar{\varphi}(w)/\bar{S}(w) \\ 1/\bar{S}(w) \end{pmatrix}.$$

The inequality reduces to

$$(i\varphi(w) \quad 1)\, \frac{M(w)I\bar{M}(w)}{i} \begin{pmatrix} -i\bar{\varphi}(w) \\ 1 \end{pmatrix} \leq 0$$

and can be written

$$|\varphi(w)[A(w) + iB(w)] + [D(w) - iC(w)]| \\ \geq |\varphi(w)[A(w) - iB(w)] - [D(w) + iC(w)]|.$$

Since Re $[D(w) - iC(w)]/[A(w) + iB(w)] \geq 1$ for real w and since Re $\varphi(w) \geq 0$ in the upper half-plane, $\varphi(z)[A(z) + iB(z)] + [D(z) - iC(z)]$ does not vanish identically in the upper half-plane. The function

$$W(z) = \frac{\varphi(z)[A(z) - iB(z)] - [D(z) + iC(z)]}{\varphi(z)[A(z) + iB(z)] + [D(z) - iC(z)]}$$

is analytic and bounded by 1 in the upper half-plane. The theorem follows on solving for $\varphi(z)$.

PROBLEM 90. Let $\mathcal{H}(E)$ be a given space and let $\mu(x)$ be a nondecreasing function of real x such that

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 d\mu(t)$$

for every $F(z)$ in $\mathcal{H}(E)$. Show that there exists a function $W(z)$, analytic and bounded by 1 in the upper half-plane, such that

$$\mathrm{Re}\, \frac{E(z) + E^*(z)W(z)}{E(z) - E^*(z)W(z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$. If $\mu(x)$ is constant in an interval (a, b), show that $W(z)$ is analytic across (a, b) if defined in the lower half-plane by $W^*(z)W(z) = 1$.

PROBLEM 91. Let $W(z)$ be a function which is analytic and bounded by 1 in the upper half-plane and which is analytic across an interval (a, b) of the real axis when defined in the lower half-plane by $W^*(z)W(z) = 1$. Show that $W(z) = \exp[2i\psi(x)]$ for $a < x < b$ where $\psi(x)$ is a nondecreasing, differentiable function of x.

PROBLEM 92. In Problem 90, let $\varphi(x)$ be a phase function associated with $E(z)$. Show that $\varphi(b) - \varphi(a) \leq \pi$ and that the inequality is strict unless $W(z)$ is a constant of absolute value 1.

PROBLEM 93. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$. Let $\varphi(a, x)$ and $\varphi(b, x)$ be phase functions associated with $E(a, z)$ and $E(b, z)$. Show that

$$\varphi(a, t) - \varphi(a, s) \leq \varphi(b, t) - \varphi(b, s)$$

whenever $-\infty < s < t < \infty$ and $\varphi(b, t) - \varphi(b, s) \equiv 0$ modulo π.

PROBLEM 94. Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that

$$\operatorname{Re} f(x + iy) = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$ where $p \geq 0$ and $\mu(x)$ is a nondecreasing function of real x which is constant in an interval (a, b). Let $z = x + iy$ where $y > 0$ and $a < x < b$. Show that

$$\operatorname{Re} f(x + iy) \leq \frac{(c - x)^2 + h^2}{(c - x)^2 + y^2} \frac{y}{h} \operatorname{Re} f(c + ih)$$

for $0 < y < h$, where $c = a$ if $x \leq \frac{1}{2}(a + b)$ and $c = b$ if $x \geq \frac{1}{2}(a + b)$.

PROBLEM 95. Let $f(z) = \lim f_n(z)$ where $f(z)$ and each $f_n(z)$ is analytic and has a nonnegative real part in the upper half-plane. Assume that

$$\operatorname{Re} f_n(x + iy) = p_n y + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu_n(t)}{(t - x)^2 + y^2}$$

for $y > 0$ for every n, where $p_n \geq 0$ and $\mu_n(x)$ is a nondecreasing function of x which is constant in an interval (a, b). Show that

$$\operatorname{Re} f(x + iy) = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$, where $p \geq 0$ and $\mu(x)$ is a nondecreasing function of x which is constant in (a, b).

PROBLEM 96. If $W(z)$ is a function which is analytic and bounded by 1 in the upper half-plane, show that there exists a sequence $(W_n(z))$ of finite Blaschke products such that $W(z) = \lim W_n(z)$ for $y > 0$. (A Blaschke product is a product as in Problem 23 multiplied by a constant of absolute value 1.) *Hint*: See SSPS Theorem 21.

PROBLEM 97. If $\mathcal{H}(E)$ is a given space and if $W(z)$ is a given function which is analytic and bounded by 1 in the upper half-plane, show that there exists a sequence $(P_n(z))$ of polynomials of Pólya class with these properties: If $E_n(z) = E(z)P_n(z)$, then $B_n(z)$ does not belong to $\mathcal{H}(E_n)$ and

$$\frac{E(z) + E^*(z)W(z)}{E(z) - E^*(z)W(z)} = \lim_{n \to \infty} \frac{iA_n(z)}{B_n(z)}$$

for $y > 0$. Show that $F(z) \to P_n(z)F(z)$ is an isometric transformation of $\mathcal{H}(E)$ into $\mathcal{H}(E_n)$ for every n.

32. MEASURES ASSOCIATED WITH $\mathcal{H}(E)$

We now construct the measures associated with a given space $\mathcal{H}(E)$.

THEOREM 32. Let $\mathcal{H}(E)$ be a given space and let $W(z)$ be a given function which is analytic and bounded by 1 in the upper half-plane. There exists a number $p(E, E) \geq 0$ and a nondecreasing function $\mu(x)$ of real x such that

$$\operatorname{Re} \frac{E(z) + E^*(z)W(z)}{E(z) - E^*(z)W(z)} = p(E, E)y + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. Let $S(z)$ be an entire function such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Let $C(z)$ and $D(z)$ be entire functions which are real for real z, such that

$$A(z)D(z) - B(z)C(z) = S(z)S^*(z),$$

$$\operatorname{Re}\,[A(z)\bar{D}(z) - B(z)\bar{C}(z)] \geq \tfrac{1}{2}\,|S(z)|^2 + \tfrac{1}{2}\,|S^*(z)|^2$$

for all complex z, $[D(z) + iC(z)]/E(z)$ has no real singularities, and

$$\lim_{y \to +\infty} y^{-1}[D(iy) + iC(iy)]/E(iy) = 0.$$

Then there exists a number $p(S, S) \geq 0$ such that

$$\operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)}$$
$$= p(S, S)y + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$, and $p(S, S) = 0$ if $S(z)$ belongs to $\mathcal{H}(E)$. If $F(z)$ belongs to $\mathcal{H}(E)$, then

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 d\mu(t) \leq \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

and equality holds if, and only if, $p(S, S) = 0$ when $S(z) = zF(z)$. Equality holds for every $F(z)$ in $\mathcal{H}(E)$ if, and only if, $p(E, E) = 0$, which is always the case if the domain of multiplication by z is dense in $\mathcal{H}(E)$.

Proof of Theorem 32. Let $(P_n(z))$ be a sequence of polynomials of Pólya class as in Problem 97. If $E_n(z) = E(z)P_n(z)$, then $B_n(z)$ does not belong to $\mathcal{H}(E_n)$ and

$$[E(z) + E^*(z)W(z)]/[E(z) - E^*(z)W(z)] = \lim iA_n(z)/B_n(z)$$

for $y > 0$. By Theorem 22,

$$\int_{-\infty}^{+\infty} |F(t)/E_n(t)|^2 dt = \sum |F(t)/E_n(t)|^2 \, \pi/\varphi_n'(t)$$

for every $F(z)$ in $\mathcal{H}(E_n)$, where the summation is over the real numbers t such that $\varphi_n(t) \equiv 0$ modulo π. By Problem 97, $F(z) \to P_n(z)F(z)$ is an isometric transformation of $\mathcal{H}(E)$ into $\mathcal{H}(E_n)$. It follows that

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \sum |F(t)/E(t)|^2 \, \pi/\varphi_n'(t)$$

for every $F(z)$ in $\mathcal{H}(E)$. Since

$$\operatorname{Re} i \frac{A_n(z)}{B_n(z)} = \sum \frac{1}{\varphi_n'(t)} \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$ by Problem 89,

$$\operatorname{Re} \frac{E(z) + E^*(z)W(z)}{E(z) - E^*(z)W(z)} = \lim_{n\to\infty} \sum \frac{1}{\varphi_n'(t)} \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$. Let $\mu_n(x)$ be a nondecreasing step function whose points of increase are the points where $\varphi_n(x) \equiv 0$ modulo π and which increases by $\pi/\varphi_n'(x)$ at each such point. Choose the functions so that $\mu_n(0) = 0$ for every n. As in the proof of Theorem 4, $(\mu_n(x))$ is a bounded sequence of numbers for each real x. By the Helly selection principle, we can suppose the sequence of polynomials chosen so that $\mu(x) = \lim \mu_n(x)$ exists for all real x.

If $S(z)$ satisfies the hypotheses of the theorem, then

$$\operatorname{Re} \frac{D(z) + iC(z)}{A(z) - iB(z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 \frac{dt}{(t-x)^2 + y^2}$$

for $y > 0$ by the proof of sufficiency for Theorem 27. If $S_n(z) = P_n(z)S(z)$, then by Theorem 25 $[G(z)S_n(w) - S_n(z)G(w)]/(z - w)$ belongs to $\mathcal{H}(E_n)$ whenever $G(z)$ belongs to $\mathcal{H}(E_n)$. By Theorem 27 there exist entire functions $C_n(z)$ and $D_n(z)$, which are real for real z, such that

$$A_n(z)D_n(z) - B_n(z)C_n(z) = S_n(z)S_n^*(z),$$

$$\operatorname{Re}\,[A_n(z)\bar{D}_n(z) - B_n(z)\bar{C}_n(z)] \geq \tfrac{1}{2}|S_n(z)|^2 + \tfrac{1}{2}|S_n^*(z)|^2$$

for all complex z. Choose them as in the proof of the theorem so that

$$\operatorname{Re} \frac{D_n(z) + iC_n(z)}{A_n(z) - iB_n(z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \left| \frac{S_n(t)}{E_n(t)} \right|^2 \frac{dt}{(t-x)^2 + y^2}$$

for $y > 0$. It follows that

$$[D_n(z) + iC_n(z)]/[A_n(z) - iB_n(z)] = [D(z) + iC(z)]/[A(z) - iB(z)] + ih$$

for some real constant h, and hence that

$$[D_n(z) - hB_n(z)] + i[C_n(z) - hA_n(z)] = P_n(z)[D(z) + iC(z)].$$

Since we can replace $C_n(z)$ by $C_n(z) - hA_n(z)$ and $D_n(z)$ by $D_n(z) - hB_n(z)$ without altering the defining properties of these functions, we can suppose them chosen so that $D_n(z) + iC_n(z) = P_n(z)[D(z) + iC(z)]$. By the proof of Theorem 27,

$$\operatorname{Re} i\,\frac{D_n(z)}{B_n(z)} = \sum \frac{1}{\varphi_n'(t)} \left|\frac{S_n(t)}{E_n(t)}\right|^2 \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$. In other words,

$$\operatorname{Re} \frac{[D(z) + iC(z)]P_n(z) + [D(z) - iC(z)]P_n^*(z)}{[A(z) - iB(z)]P_n(z) - [A(z) + iB(z)]P_n^*(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu_n(t)}{(t-x)^2 + y^2}$$

for $y > 0$. Since $S(x)/E(x)$ is a continuous function of real x and since $\mu(x) = \lim \mu_n(x)$ for all real x,

$$\frac{y}{\pi}\int_a^b \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu(t)}{(t-x)^2 + y^2} = \lim_{n\to\infty} \frac{y}{\pi}\int_a^b \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu_n(t)}{(t-x)^2 + y^2}$$

for $y > 0$. By the arbitrariness of a and b,

$$\frac{y}{\pi}\int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu(t)}{(t-x)^2 + y^2} \leq \operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)}$$

for $y > 0$. Whenever $a < b$ are finite,

$$\operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)} - \frac{y}{\pi}\int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu(t)}{(t-x)^2 + y^2}$$
$$\leq \lim_{n\to\infty} \left[\frac{y}{\pi}\int_{-\infty}^{a} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu_n(t)}{(t-x)^2 + y^2} + \frac{y}{\pi}\int_b^{\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu_n(t)}{(t-x)^2 + y^2}\right].$$

By Problem 95 and the arbitrariness of a and b, there is a number $p(S, S) \geq 0$ such that the expression on the left is equal to $p(S, S)y$. In other words

$$\operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)} = p(S, S)y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \left|\frac{S(t)}{E(t)}\right|^2 \frac{d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$. The definition of $p(S, S)$ is such that

$$p(S, S) = \lim_{n\to\infty} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu_n(t)}{(t-x)^2 + y^2} - \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$.

If $S(z)$ and $T(z)$ are entire functions such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ and $[F(z)T(w) - T(z)F(w)]/(z - w)$ belong to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$, define

$$4p(S, T) = p(S + T, S + T) - p(S - T, S - T) + ip(S + iT, S + iT) - ip(S - iT, S - iT).$$

Then $p(S, T)$ is linear in S for each fixed T, $p(T, S) = p(S, T)^-$, and $p(S, S) \geq 0$ for all $S(z)$. In other words, $p(S, T)$ has all the properties of an inner product except that it is not strictly positive. The Schwarz inequality and the triangle inequality are valid for such products.

If $S(z)$ belongs to $\mathcal{H}(E)$, then

$$\begin{aligned}\operatorname{Re} \frac{[D(z) + iC(z)]P_n(z) + [D(z) - iC(z)]P_n^*(z)}{[A(z) - iB(z)]P_n(z) - [A(z) + iB(z)]P_n^*(z)} &= \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu_n(t)}{(t-x)^2 + y^2} \\ &\leq 1/(\pi y) \int_{-\infty}^{+\infty} |S(t)/E(t)|^2 d\mu_n(t) \\ &\leq 1/(\pi y) \int_{-\infty}^{+\infty} |S(t)/E(t)|^2 dt.\end{aligned}$$

Since n is arbitrary,

$$\operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W(z)}{[A(z) - iB(z)] - [A(z) + iB(z)]W(z)} \leq \frac{1}{\pi y} \int_{-\infty}^{+\infty} \left| \frac{S(t)}{E(t)} \right|^2 dt$$

and hence

$$p(S, S)y \leq 1/(\pi y) \int_{-\infty}^{+\infty} |S(t)/E(t)|^2 dt.$$

Since y is arbitrary, $p(S, S) = 0$ in this case. By the Schwarz inequality, $p(S, T) = 0$ whenever $S(z)$ or $T(z)$ belongs to $\mathcal{H}(E)$.

If $S(z) = zG(z)$ where $G(z)$ belongs to $\mathcal{H}(E)$, then

$$[F(z)S(w) - S(z)F(w)]/(z - w) = w[F(z)G(w) - G(z)F(w)]/(z - w) - G(z)F(w)$$

belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. If $y > 0$,

$$\int_{-\infty}^{+\infty} |G(t)/E(t)|^2 d\mu(t) = \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu(t)}{t^2 + y^2} + y^2 \int_{-\infty}^{+\infty} \frac{|G(t)/E(t)|^2 d\mu(t)}{t^2 + y^2}$$

$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{|S(t)/E(t)|^2 d\mu_n(t)}{t^2 + y^2} - \pi p(S, S)$$

$$+ \lim_{n\to\infty} y^2 \int_{-\infty}^{+\infty} \frac{|G(t)/E(t)|^2 d\mu_n(t)}{t^2 + y^2}$$

$$= \lim_{n\to\infty} \int_{-\infty}^{+\infty} |G(t)/E(t)|^2 d\mu_n(t) - \pi p(S, S)$$

$$= \int_{-\infty}^{+\infty} |G(t)/E(t)|^2 dt - \pi p(S, S).$$

From this we see that

$$\int_{-\infty}^{+\infty} |G(t)/E(t)|^2 d\mu(t) \leq \int_{-\infty}^{+\infty} |G(t)/E(t)|^2 dt$$

for every element $G(z)$ of $\mathcal{H}(E)$. Equality holds if, and only if, $p(S, S) = 0$ when $S(z) = zG(z)$. In particular, equality holds whenever $G(z)$ belongs to the domain of multiplication by z in $\mathcal{H}(E)$. Equality follows when $G(z)$ is in the closure of the domain of multiplication by z.

If $p(E, E) = 0$, then $p(E^*, E^*) = 0$ and $p(S, S) = 0$ whenever $S(z)$ is a linear combination of $A(z)$ and $B(z)$. Since

$$\pi z K(w, z) = B(z)\bar{A}(w) - A(z)\bar{B}(w) + \pi \bar{w} K(w, z),$$

we obtain $p(S, S) = 0$ by the triangle inequality whenever $S(z) = zK(w, z)$ for some number w. It follows that

$$\int_{-\infty}^{+\infty} F(t)\bar{K}(w, t)\, |E(t)|^{-2} d\mu(t) = \langle F(t), K(w, t)\rangle = F(w)$$

for every $F(z)$ in $\mathcal{H}(E)$. The identity implies that

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 d\mu(t) = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt$$

whenever $F(z)$ is a finite linear combination of functions of the form $K(w, z)$. Since such combinations are dense in $\mathcal{H}(E)$, the same formula holds for every $F(z)$ in $\mathcal{H}(E)$.

33. ISOMETRIC INCLUSIONS OF SPACES $\mathcal{H}(E)$

The space $\mathcal{H}_S(M)$ is denoted $\mathcal{H}(M)$ when $S(z) = 1$. The theory of such spaces is used to determine the isometric inclusions of $\mathcal{H}(E)$ spaces.

THEOREM 33. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. Then there exists a matrix $M(a, b, z)$ of entire functions such that a space $\mathcal{H}(M(a, b))$ exists and such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

The transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[A(a, z)F_+(z) + B(a, z)F_-(z)]$$

takes $\mathcal{H}(M(a, b))$ isometrically onto the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$.

Proof of Theorem 33. The function $S(z) = E(a, z)$ has the property that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$. Since $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$, it follows from the proof of Theorem 25 that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. By Theorem 27 there exists a transformation $F(z) \to \tilde{F}(z)$ which assigns an entire function to each element of $\mathcal{H}(E(b))$ in such a way that the identity

$$\begin{aligned} \pi F(\alpha)\tilde{G}(\beta)^- - \pi\tilde{F}(\alpha)G(\beta)^- &= \langle F(t)S(\alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \\ &\quad - \langle [F(t)S(\alpha) - S(t)F(\alpha)]/(t - \alpha), G(t)S(\beta)\rangle \\ + (\alpha - \bar{\beta})\langle [F(t)S(\alpha) &- S(t)F(\alpha)]/(t - \alpha), [G(t)S(\beta) - S(t)G(\beta)]/(t - \beta)\rangle \end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E(b))$ and all complex numbers α and β. If $F(z)$ is in $\mathcal{H}(E(b))$ and if $G(z) = [F(z)S(w) - S(z)F(w)]/(z - w)$ for some number w, then $\tilde{G}(z) = [\tilde{F}(z)S(w) - S(z)\tilde{F}(w)]/(z - w)$. It is easily seen that the transformation having these properties is unique within an added real multiple of the identity transformation. By Problem 83 the transformation $F(z) \to iF(z)$ has these properties when it is restricted to $\mathcal{H}(E(a))$. Therefore there exists a real constant h such that $\tilde{F}(z) = iF(z) + hF(z)$ for every $F(z)$ in $\mathcal{H}(E(a))$. Since we can add a real multiple of the identity transformation to the transformation $F(z) \to \tilde{F}(z)$ without altering its defining property, we can choose it so that $\tilde{F}(z) = iF(z)$ whenever $F(z)$ is in $\mathcal{H}(E(a))$.

By Theorem 27 there exist entire functions $C(b, z)$ and $D(b, z)$, which are real for real z, such that

$$A(b, z)D(b, z) - B(b, z)C(b, z) = S(z)S^*(z),$$

$$\operatorname{Re}[A(b, z)\bar{D}(b, z) - B(b, z)\bar{C}(b, z)] \geq \tfrac{1}{2}|S(z)|^2 + \tfrac{1}{2}|S^*(z)|^2$$

for all complex z, $[D(b, z) + iC(b, z)]/E(b, z)$ has no real singularities,

$$\lim_{y\to+\infty} \operatorname{Re} y^{-1}[D(b, iy) + iC(b, iy)]/E(b, iy) = 0,$$

and

$$\tilde{F}(w) = \langle F(t), [S(t)\bar{S}(w) - A(b, t)\bar{D}(b, w) + B(b, t)\bar{C}(b, w)]/[\pi(t - \bar{w})]\rangle$$

for every $F(z)$ in $\mathcal{H}(E(b))$ and all complex numbers w. A space $\mathcal{H}_S(M(b))$ exists by Theorem 28. By the proof of the theorem $F(z) \to \frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ \tilde{F}(z)\end{pmatrix}$ is an isometric transformation of $\mathcal{H}(E(b))$ onto $\mathcal{H}_S(M(b))$. By Problem 83, $F(z) \to \frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ iF(z)\end{pmatrix}$ is an isometric transformation of $\mathcal{H}(E(a))$ onto $\mathcal{H}_S(M(a))$,

$$M(a, z) = \begin{pmatrix} A(a, z) & B(a, z) \\ -B(a, z) & A(a, z)\end{pmatrix}.$$

It follows that $\mathcal{H}_S(M(a))$ is contained isometrically in $\mathcal{H}_S(M(b))$.

Let $M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z)\end{pmatrix}$ be the matrix of analytic functions defined by $M(b, z) = M(a, z)M(a, b, z)$ at points where

$$\det M(a, z) = S(z)S^*(z)$$

has a nonzero value. Since the entries of $M(a, z)$ and $M(b, z)$ are real for real z, the entries of $M(a, b, z)$ are real for real z. Since

$$\det M(a, z) = S(z)S^*(z) = \det M(b, z),$$

we have $\det M(a, b, z) = 1$. Since $E(b, z)/E(a, z)$ has no real singularities by hypothesis, and since $C(b, z)/E(b, z)$ and $D(b, z)/E(b, z)$ have no real singularities by construction, the functions

$$A(a, b, z) + iC(a, b, z) = [A(b, z) + iC(b, z)]/E(a, z),$$

$$D(a, b, z) - iB(a, b, z) = [D(b, z) - iB(b, z)]/E(a, z)$$

have no singularities on or above the real axis. For all complex numbers $u, v, w,$

$$\frac{M(b, z)I\bar{M}(b, w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}\begin{pmatrix} u \\ v\end{pmatrix} - \frac{M(a, z)I\bar{M}(a, w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}\begin{pmatrix} u \\ v\end{pmatrix}$$
$$= \frac{M(b, z)I\bar{M}(b, w) - M(a, z)I\bar{M}(a, w)}{2\pi(z - \bar{w})}\begin{pmatrix} u \\ v\end{pmatrix}$$

belongs to $\mathcal{H}_S(M(b))$ and is orthogonal to $\mathcal{H}_S(M(a))$. If $\begin{pmatrix} F(z) \\ \tilde{F}(z) \end{pmatrix}$ is in $\mathcal{H}_S(M(b))$ and is orthogonal to $\mathcal{H}_S(M(a))$, then

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F(w) \\ \tilde{F}(w) \end{pmatrix} = \left\langle \begin{pmatrix} F(t) \\ \tilde{F}(t) \end{pmatrix}, \frac{M(b, t)I\bar{M}(b, w) - M(a, t)I\bar{M}(a, w)}{2\pi(t - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle.$$

In particular we obtain the inequality

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \frac{M(b, w)I\bar{M}(b, w) - M(a, w)I\bar{M}(a, w)}{2\pi(w - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix}$$
$$= \left\| \frac{M(b, t)I\bar{M}(b, w) - M(a, t)I\bar{M}(a, w)}{2\pi(t - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 \geq 0,$$

which implies that

$$[M(a, b, w)I\bar{M}(a, b, w) - I]/[2\pi(w - \bar{w})] \geq 0$$

whenever $M(a, b, w)$ is defined. By Problems 81 and 82, the matrix inequality implies that

$$|A(a, b, w) - iC(a, b, w)| \leq |A(a, b, w) + iC(a, b, w)|,$$
$$|D(a, b, w) + iB(a, b, w)| \leq |D(a, b, w) - iB(a, b, w)|$$

for w in the upper half-plane. Since $A(a, b, z) + iC(a, b, z)$ and $D(a, b, z) - iB(a, b, z)$ are known to be analytic in the upper half-plane, and since an analytic function cannot remain bounded in the neighborhood of an isolated singularity, $A(a, b, z) - iC(a, b, z)$ and $D(a, b, z) + iB(a, b, z)$ are analytic in the upper half-plane. Since it is known that these functions have no real singularities, it follows that the entries of $M(a, b, z)$ are entire functions. A space $\mathcal{H}(M(a, b))$ exists by Theorem 28.

If $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is a finite sum of functions

$$\frac{M(a, b, z)I\bar{M}(a, b, w) - I}{2\pi(z - \bar{w})} \bar{M}(a, w) \begin{pmatrix} u \\ v \end{pmatrix}$$

for some numbers u, v, w, then $M(a, z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is a finite sum of functions

$$\frac{M(b, z)I\bar{M}(b, w) - M(a, z)I\bar{M}(a, w)}{2\pi(z - \bar{w})} \begin{pmatrix} u \\ v \end{pmatrix},$$

it belongs to $\mathcal{H}_S(M(b))$, and it is orthogonal to $\mathcal{H}_S(M(a))$. It is easily verified that

$$\left\| M(a, t) \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}_S(M(b))} = \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a,b))}.$$

The same conclusion follows by continuity whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to the closed span of such special elements of $\mathcal{H}(M(a, b))$. But if $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is an element of $\mathcal{H}(M(a, b))$ which is orthogonal to such special functions,

$$\begin{pmatrix} u \\ v \end{pmatrix}^- M(a, w) \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix}$$

$$= \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{M(a, b, t) I \bar{M}(a, b, w) - I}{2\pi(t - \bar{w})} \bar{M}(a, w) \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0.$$

By the arbitrariness of u, v, and w, $M(a, z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ vanishes identically. Since $\det M(a, z) = S(z)S^*(z)$ does not vanish identically, this implies that $F_+(z)$ and $F_-(z)$ vanish identically. It follows that the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to M(a, z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

maps $\mathcal{H}(M(a, b))$ isometrically onto the orthogonal complement of $\mathcal{H}_S(M(a))$ in $\mathcal{H}_S(M(b))$. By the construction of $\mathcal{H}_S(M(b))$, the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[A(a, z)F_+(z) + B(a, z)F_-(z)]$$

takes $\mathcal{H}(M(a, b))$ isometrically onto the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$.

PROBLEM 98. If $\mathcal{H}_S(M(a))$ and $\mathcal{H}(M(a, b))$ are given spaces, let $\mathfrak{L}$ be the set of elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M(a, b))$ such that $M(a, z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathcal{H}_S(M(a))$. Show that $\mathfrak{L}$ is a Hilbert space in the norm

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathfrak{L}}^2 = \left\| M(a, t) \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}_S(M(a))}^2 + \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|_{\mathcal{H}(M(a,b))}^2$$

Show that $\begin{pmatrix} [F_+(z) - F_+(w)]/(z - w) \\ [F_-(z) - F_-(w)]/(z - w) \end{pmatrix}$ belongs to $\mathfrak{L}$ whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathfrak{L}$, for every complex number w. Show that the identity

$$0 = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \begin{pmatrix} [G_+(t) - G_+(\beta)]/(t - \beta) \\ [G_-(t) - G_-(\beta)]/(t - \beta) \end{pmatrix} \right\rangle_{\mathfrak{L}}$$

$$- \left\langle \begin{pmatrix} [F_+(t) - F_+(\alpha)]/(t - \alpha) \\ [F_-(t) - F_-(\alpha)]/(t - \alpha) \end{pmatrix}, \begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\rangle_{\mathfrak{L}}$$

$$+ (\alpha - \bar{\beta}) \left\langle \begin{pmatrix} [F_+(t) - F_+(\alpha)]/(t - \alpha) \\ [F_-(t) - F_-(\alpha)]/(t - \alpha) \end{pmatrix}, \begin{pmatrix} [G_+(t) - G_+(\beta)]/(t - \beta) \\ [G_-(t) - G_-(\beta)]/(t - \beta) \end{pmatrix} \right\rangle_{\mathfrak{L}}$$

holds for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ of $\mathfrak{L}$ and all complex numbers α and β. Show that the linear functionals defined on $\mathfrak{L}$ by

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to F_+(w) \quad \text{and} \quad \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to F_-(w)$$

are continuous for every w.

PROBLEM 99. In Problem 98 show that there exists a matrix

$$\Phi(z) = \begin{pmatrix} P(z) & Q(z) \\ R(z) & T(z) \end{pmatrix}$$

of entire functions such that

$$\frac{\Phi(z) + \bar{\Phi}(w)}{\pi i(\bar{w} - z)} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathfrak{L}$ for all numbers u, v, w, and such that the identity

$$\begin{pmatrix} u \\ v \end{pmatrix}^- \begin{pmatrix} F_+(w) \\ F_-(w) \end{pmatrix} = \left\langle \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix}, \frac{\Phi(t) + \bar{\Phi}(w)}{\pi i(\bar{w} - t)} \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle$$

holds for all elements $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathfrak{L}$. Show that $\Phi(z) = -\bar{\Phi}(\bar{z})$ and that $\operatorname{Re} \Phi(z) = \frac{1}{2}[\Phi(z) + \bar{\Phi}(z)] \geq 0$ for $y > 0$. If u and v are complex numbers, show that $\begin{pmatrix} u \\ v \end{pmatrix}^- \Phi(z) \begin{pmatrix} u \\ v \end{pmatrix}$ is an entire function whose real part is nonnegative in the upper half-plane and zero on the real axis. Show that the entries of $\Phi(z)$ are linear functions of z and that $F_+(z)$ and $F_-(z)$ are constants whenever $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathfrak{L}$. *Hint*: See Theorem 6.

34. A CONVERSE RESULT ON ISOMETRIC INCLUSIONS

THEOREM 34. If $\mathcal{H}(E(a))$ and $\mathcal{H}(M(a, b))$ are given spaces, then there exists a space $\mathcal{H}(E(b))$ such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$

and $E(a, z)/E(b, z)$ has no real zeros. The space $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$ and the inclusion does not increase norms. If there is no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}(M(a, b))$ such that $A(a, z)u + B(a, z)v$ belongs to $\mathcal{H}(E(a))$, then $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and the transformation

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to \sqrt{2}[A(a, z)F_+(z) + B(a, z)F_-(z)]$$

takes $\mathcal{H}(M(a, b))$ isometrically onto the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$.

Proof of Theorem 34. If $S(z) = E(a, z)$ and if

$$M(a, z) = \begin{pmatrix} A(a, z) & B(a, z) \\ -B(a, z) & A(a, z) \end{pmatrix},$$

a space $\mathcal{H}_S(M(a))$ exists by Problem 83, and the transformation

$$F(z) \to \frac{1}{\sqrt{2}} \begin{pmatrix} F(z) \\ iF(z) \end{pmatrix}$$

is an isometry of $\mathcal{H}(E(a))$ onto $\mathcal{H}_S(M(a))$. There is no nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}(M(a, b))$ such that $M(a, z)\begin{pmatrix} u \\ v \end{pmatrix}$ belongs to $\mathcal{H}_S(M(a))$, for this implies that $A(a, z)u + B(a, z)v$ belongs to $\mathcal{H}(E(a))$ and that

$$i[A(a, z)u + B(a, z)v] = -B(a, z)u + A(a, z)v.$$

Since $A(a, z)$ and $B(a, z)$ are linearly independent, $v = iu$. Since $\bar{u}v = \bar{v}u$ by Problem 85, we obtain $u = v = 0$.

By Problems 98 and 99, there is no nonzero element $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ of $\mathcal{H}(M(a, b))$ such that $M(a, z)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ belongs to $\mathcal{H}_S(M(a))$. Let $\mathcal{H}$ be the set of all pairs

$$\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} + M(a, z)\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

of entire functions such that $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is in $\mathcal{H}_S(M(a))$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ is in $\mathcal{H}(M(a, b))$. We can define a norm unambiguously in $\mathcal{H}$ by

$$\left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} + M(a, t)\begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} F_+(t) \\ F_-(t) \end{pmatrix} \right\|^2_{\mathcal{H}_S(M(a))} + \left\| \begin{pmatrix} G_+(t) \\ G_-(t) \end{pmatrix} \right\|^2_{\mathcal{H}(M(a,b))}.$$

If $M(b, z) = M(a, z)M(a, b, z)$, then

$$\frac{M(b, z)I\bar{M}(b, w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}\begin{pmatrix}u\\v\end{pmatrix} = \frac{M(a, z)I\bar{M}(a, w) - S(z)I\bar{S}(w)}{2\pi(z - \bar{w})}\begin{pmatrix}u\\v\end{pmatrix}$$
$$+ M(a, z)\,\frac{M(a, b, z)I\bar{M}(a, b, w) - I}{2\pi(z - \bar{w})}\,\bar{M}(a, w)\begin{pmatrix}u\\v\end{pmatrix}$$

belongs to $\mathcal{H}$ for all complex numbers u, v, and w, and

$$\begin{pmatrix}u\\v\end{pmatrix}^{-}\begin{pmatrix}L_+(w)\\L_-(w)\end{pmatrix} = \left\langle\begin{pmatrix}L_+(t)\\L_-(t)\end{pmatrix}, \frac{M(b, t)I\bar{M}(b, w) - S(t)I\bar{S}(w)}{2\pi(t - \bar{w})}\begin{pmatrix}v\\v\end{pmatrix}\right\rangle$$

for all elements $\begin{pmatrix}L_+(z)\\L_-(z)\end{pmatrix} = \begin{pmatrix}F_+(z)\\F_-(z)\end{pmatrix} + M(a, z)\begin{pmatrix}G_+(z)\\G_-(z)\end{pmatrix}$ of $\mathcal{H}$. Since

$$\begin{pmatrix}1\\0\end{pmatrix}^{-}\frac{M(b, w)I\bar{M}(b, w) - S(w)I\bar{S}(w)}{2\pi(w - \bar{w})}\begin{pmatrix}1\\0\end{pmatrix}$$
$$\geq \begin{pmatrix}1\\0\end{pmatrix}^{-}\frac{M(a, w)I\bar{M}(a, w) - S(w)I\bar{S}(w)}{2\pi(w - \bar{w})}\begin{pmatrix}1\\0\end{pmatrix}$$

for all complex w, we obtain the inequality

$$\frac{B(b, w)\bar{A}(b, w) - A(b, w)\bar{B}(b, w)}{\pi(w - \bar{w})} \geq \frac{B(a, w)\bar{A}(a, w) - A(a, w)\bar{B}(a, w)}{\pi(w - \bar{w})} > 0$$

when w is not real. It follows that $E(b, z) = A(b, z) - iB(b, z)$ satisfies the inequality $|E(b, x - iy)| < |E(b, x + iy)|$ for $y > 0$. A space $\mathcal{H}(E(b))$ therefore exists. Since the entries of $M(a, z)/S(z)$ have no real singularities and since the entries of $M(a, b, z)$ are entire functions, the entries of $M(b, z)/S(z)$ have no real singularities. It follows that $[D(b, z) + iC(b, z)]/E(b, z)$ has no real singularities and that $E(a, z)/E(b, z)$ has no real zeros. A space $\mathcal{H}_S(M(b))$ exists by Theorem 28, and it is equal isometrically to $\mathcal{H}$ by the uniqueness part of the theorem. The space $\mathcal{H}_S(M(a))$ is contained isometrically in $\mathcal{H}_S(M(b))$ by the construction of $\mathcal{H}$.

If $F(z)$ is in $\mathcal{H}(E(a))$, then $\begin{pmatrix}F(z)\\iF(z)\end{pmatrix}$ is in $\mathcal{H}_S(M(a))$ and hence in $\mathcal{H}_S(M(b))$. By the proof of Theorem 28, $F(z)$ belongs to $\mathcal{H}(E(b))$ and

$$\|F(t)\|^2_{\mathcal{H}(E(b))} \leq \tfrac{1}{2}\left\|\begin{pmatrix}F(t)\\iF(t)\end{pmatrix}\right\|^2_{\mathcal{H}_S(M(b))} = \|F(t)\|^2_{\mathcal{H}(E(a))}.$$

So $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$ and the inclusion does not increase norms. By the proof of Theorem 28, the inclusion is isometric if $\begin{pmatrix}0\\S(z)\end{pmatrix}$ does not belong to $\mathcal{H}_S(M(b))$. If it does belong, we can write

$$\begin{pmatrix} 0 \\ S(z) \end{pmatrix} = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} + M(a, z) \begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$$

with $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ in $\mathcal{H}_S(M(a))$ and $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix}$ a nonzero element of $\mathcal{H}(M(a, b))$. If w is a complex number, then $\begin{pmatrix} [G_+(z) - G_+(w)]/(z - w) \\ [G_-(z) - G_-(w)]/(z - w) \end{pmatrix}$ belongs to $\mathcal{H}(M(a, b))$ by Theorem 28, and a calculation will show that

$$M(a, z) \begin{pmatrix} [G_+(z) - G_+(w)]/(z - w) \\ [G_-(z) - G_-(w)]/(z - w) \end{pmatrix}$$

belongs to $\mathcal{H}_S(M(a))$. It follows that this element of $\mathcal{H}(M(a, b))$ vanishes identically. Therefore $\begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ is a nonzero constant in $\mathcal{H}(M(a, b))$ such that $-F_+(z) = A(a, z)u + B(a, z)v$ belongs to $\mathcal{H}(E(a))$.

PROBLEM 100. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. Let $\mathcal{H}(M_1(a, b))$ and $\mathcal{H}(M_2(a, b))$ be spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M_k(a, b, z)$$

for $k = 1, 2$. Show that $M_1(a, b, z) = M_2(a, b, z)$.

PROBLEM 101. If $\mathcal{H}(M(a, c))$ is a given space and if there exists a constant $\begin{pmatrix} u \\ v \end{pmatrix}$ of norm 1 in $\mathcal{H}(M(a, c))$, show that $\bar{u}v = \bar{v}u$ and that a space $\mathcal{H}(M(a, b))$ exists,

$$M(a, b, z) = \begin{pmatrix} 1 - 2\pi u\bar{v}z & 2\pi u\bar{u}z \\ -2\pi v\bar{v}z & 1 + 2\pi u\bar{v}z \end{pmatrix}.$$

Show that $\mathcal{H}(M(a, b))$ is contained isometrically in $\mathcal{H}(M(a, c))$, that $M(a, c, z) = M(a, b, z)M(b, c, z)$ for some space $\mathcal{H}(M(b, c))$, and that $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to M(a, b, z) \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is an isometric transformation of $\mathcal{H}(M(b, c))$ onto the orthogonal complement of $\mathcal{H}(M(a, b))$ in $\mathcal{H}(M(a, c))$.

PROBLEM 102. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(c))$, and $\mathcal{H}(M(a, c))$ be given spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z)$$

and $\mathcal{H}(E(a))$ is not contained isometrically in $\mathcal{H}(E(c))$. If $M(a, c, z) = M(a, b, z)M(b, c, z)$ as in Problem 101, show that there exists a space $\mathcal{H}(E(b))$ such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

and $\mathcal{H}(E(b))$ is contained isometrically in $\mathcal{H}(E(c))$.

PROBLEM 103. Show that the functions $e^{\frac{1}{2}ix}e^{inx}$, n integral, are a complete orthogonal set in $L^2(0, 2\pi)$. If $g(x)$ belongs to $L^2(0, 2\pi)$, if $\int_0^{2\pi} g(t)\,dt = 0$, and if $f(x) = \int_0^x g(t)\,dt$, show that

$$\int_0^{2\pi} g(t)e^{\frac{1}{2}it}e^{int}\,dt = -i(n + \tfrac{1}{2}) \int_0^{2\pi} f(t)e^{\frac{1}{2}it}e^{int}\,dt$$

for every n. Show that

$$\int_0^{2\pi} |f(t)|^2 dt \le 4 \int_0^{2\pi} |g(t)|^2 dt.$$

Hint: For completeness use SSPS Theorem 34.

PROBLEM 104. If $g(x)$ belongs to $L^2(a, b)$ where (a, b) is a finite interval, if $\int_a^b g(t)\,dt = 0$, and if $f(x) = \int_a^x g(t)\,dt$, show that

$$\pi^2 \int_a^b |f(t)|^2 dt \le (b - a)^2 \int_a^b |g(t)|^2 dt.$$

PROBLEM 105. A real valued, continuous function $f(z)$, defined in a region Ω, is said to be subharmonic in the region if

$$f(w) \le 1/(2\pi) \int_0^{2\pi} f(w + ae^{it})\,dt$$

whenever the closed disk $|z - w| \le a$ is contained in the region. If $f(z)$ is subharmonic and has continuous second partial derivatives with respect to x and y, show that

$$\frac{\partial^2}{\partial x^2} f(x + iy) + \frac{\partial^2}{\partial y^2} f(x + iy) = \lim_{a \searrow 0} \frac{4}{a^2} \left\{\frac{1}{2\pi} \int_0^{2\pi} f(z + ae^{it})\,dt - f(z)\right\} \ge 0$$

for all complex z. Show that the maximum of two subharmonic functions is a subharmonic function. Show that $\log^+ |g(z)|$ is a subharmonic function if $g(z)$ is analytic in Ω.

PROBLEM 106. Let $f(z)$ be a subharmonic function defined in the complex plane, and let

$$f_n(z) = n^2 \int_{-1/n}^{1/n} f(z + t)\,|t|\,dt.$$

Show that $f_n(z)$ is a subharmonic function for every $n = 1, 2, 3, \cdots$, and that $f(z) = \lim f_n(z)$ uniformly on every bounded set. Show that

$$\frac{\partial}{\partial x} f_n(x + iy) = n^2 \int_{-1/n}^{1/n} f(x + iy + t) \operatorname{sgn} t \, dt$$

is a continuous function. If $f(x + iy)$ has a continuous partial derivative with respect to x, show that $f_n(x + iy)$ has a continuous second partial derivative with respect to x given by

$$\frac{\partial^2}{\partial x^2} f_n(x + iy) = n^2 \int_{-1/n}^{1/n} \frac{\partial}{\partial x} f(x + iy + t) \operatorname{sgn} t \, dt.$$

If $f(x + iy)$ has a continuous partial derivative with respect to y, show that $f_n(x + iy)$ and $(\partial/\partial x) f_n(x + iy)$ have continuous partial derivatives with respect to y given by

$$\frac{\partial}{\partial y} f_n(x + iy) = n^2 \int_{-1/n}^{1/n} \frac{\partial}{\partial y} f(x + iy + t) \, |t| dt,$$

$$\frac{\partial}{\partial y} \frac{\partial}{\partial x} f_n(x + iy) = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f_n(x + iy)$$

$$= n^2 \int_{-1/n}^{1/n} \frac{\partial}{\partial y} f(x + iy + t) \operatorname{sgn} t \, dt.$$

If $f(x + iy)$ has a continuous partial derivative with respect to y, show that $f_n(x + iy)$ has a continuous second partial derivative with respect to y given by

$$\frac{\partial^2}{\partial y^2} f_n(x + iy) = n^2 \int_{-1/n}^{1/n} \frac{\partial^2}{\partial y^2} f(x + iy + t) \, |t| dt.$$

PROBLEM 107. Let $f(z)$ be a nonnegative subharmonic function, defined in the complex plane, which is periodic of period $2\pi i$. Construct a sequence $(f_n(z))$ of nonnegative subharmonic functions, defined in the complex plane and periodic of period $2\pi i$, with these properties:

(1) $f(z) = \lim f_n(z)$ uniformly on bounded sets.

(2) $f_n(x + iy)$ has continuous second partial derivatives with respect to x and y for every n.

(3) $f_n(u + iv) = 0$ whenever $f(z)$ vanishes in the square $u - 2/n \leq x \leq u + 2/n$, $v - 2/n \leq y \leq v + 2/n$.

PROBLEM 108. Let $f(x)$ be a nonnegative, continuous, convex function defined in a half-line $[a, \infty)$. Show that $f(x)$ is bounded on the half-line if

$$\liminf_{x \to \infty} f(x)/x = 0.$$

PROBLEM 109. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces which are contained isometrically in a space $L^2(\mu)$. Assume that $E(a, z)$ and $E(b, z)$ have no real zeros and that $E(b, z)/E(a, z)$ is of bounded type in the upper half-plane. Show that $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$ if $\mathcal{H}(E(b))$ fills $L^2(\mu)$.

35. ORDERING THEOREM FOR SUBSPACES OF $\mathcal{H}(E)$

The theory of subharmonic functions is used to obtain the ordering theorem for Hilbert spaces of entire functions.

THEOREM 35. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces which are contained isometrically in a space $L^2(\mu)$. If $E(b, z)/E(a, z)$ is of bounded type in the upper half-plane and has no real zeros or singularities, then either $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ or $\mathcal{H}(E(b))$ contains $\mathcal{H}(E(a))$.

The proof depends on Carleman's method of estimating the size of the set on which an entire function of zero exponential type remains bounded.

LEMMA 7. Let $F(z)$ be a given entire function, let

$$2\pi Q(r)^2 = \int_0^{2\pi} [\log^+ |F(re^{i\theta})|]^2 d\theta,$$

and let $2\pi P(r)$ be the sum of the lengths of the θ-intervals on which $|F(re^{i\theta})| > 1$. If $0 < r < s < t$ and if $Q(r) > 0$, then

$$\begin{aligned} Q(s)^2 \int_r^t \exp\left(\int_s^u v^{-1}P(v)^{-1}dv\right)u^{-1}du &\\ \leq Q(r)^2 \int_r^s \exp\left(\int_s^u v^{-1}P(v)^{-1}dv\right)u^{-1}du &\\ + Q(t)^2 \int_s^t \exp\left(\int_s^u v^{-1}P(v)^{-1}dv\right)u^{-1}du.& \end{aligned}$$

LEMMA 8. Let $F_1(z)$ and $F_2(z)$ be entire functions such that

$$\lim_{r\to\infty} r^{-2} \int_0^{2\pi} [\log^+ |F_k(re^{i\theta})|]^2 d\theta = 0$$

for $k = 1, 2$. If

$$\min(|F_1(x+iy)|, |F_2(x+iy)|) \leq |y|^{-1}$$

for all complex z, then either $F_1(z)$ or $F_2(z)$ vanishes identically.

Proof of Lemma 7. By Problem 105, $f(z) = \log^+ |F(\exp z)|$ is subharmonic in the complex plane. The function is nonnegative and periodic of period $2\pi i$. Choose a sequence $(f_n(z))$ of approximating functions as in Problem 107. Let

$$2\pi q_n(x)^2 = \int_0^{2\pi} f_n(x+iy)^2 dy$$

and let $2\pi p_n(x)$ be the sum of the lengths of the y-intervals, $0 < y < 2\pi$, on which $f_n(x+iy) > 0$. We first obtain an inequality for $p_n(x)$ and $q_n(x)$. From the definition of $q_n(x)$ we obtain

$$q_n(x)q_n'(x) = \frac{1}{2\pi}\int_0^{2\pi} f_n(x+iy)\frac{\partial}{\partial x} f_n(x+iy)dy.$$

Differentiation under the integral sign is permissible because the integrand has a continuous partial derivative with respect to x. Differentiating again we obtain

$$q_n'(x)^2 + q_n(x)q_n''(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial x} f_n(x+iy)\frac{\partial}{\partial x} f_n(x+iy)dy + \frac{1}{2\pi}\int_0^{2\pi} f_n(x+iy)\frac{\partial^2}{\partial x^2} f_n(x+iy)dy.$$

By the Schwarz inequality,

$$q_n(x)^2 q_n'(x)^2 \leq q_n(x)^2 \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial x} f_n(x+iy)\frac{\partial}{\partial x} f_n(x+iy)dy.$$

If $q_n(x) > 0$ we can conclude that

$$q_n(x)q_n''(x) \geq \frac{1}{2\pi}\int_0^{2\pi} f_n(x+iy)\frac{\partial^2}{\partial x^2} f_n(x+iy)dy.$$

Since $f_n(x+iy) \geq 0$ and since

$$\frac{\partial^2}{\partial x^2} f_n(x+iy) + \frac{\partial^2}{\partial y^2} f_n(x+iy) \geq 0$$

by Problem 105,

$$q_n(x)q_n''(x) \geq -\frac{1}{2\pi}\int_0^{2\pi} f_n(x+iy)\frac{\partial^2}{\partial y^2} f_n(x+iy)dy.$$

Since $f_n(x)$ is periodic of period $2\pi i$, we can integrate by parts to obtain

$$q_n(x)q_n''(x) \geq \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial y} f_n(x+iy)\frac{\partial}{\partial y} f_n(x+iy)dy.$$

By Problem 104,

$$q_n(x)q_n''(x) \geq \tfrac{1}{4} p_n(x)^{-2} q_n(x)^2.$$

It follows that

$$\begin{aligned}[q_n(x)q_n'(x)]' &= q_n'(x)^2 + q_n(x)q_n''(x) \\ &\geq q_n'(x)^2 + \tfrac{1}{4}p_n(x)^{-2}q_n(x)^2 \\ &\geq p_n(x)^{-1}[q_n(x)q_n'(x)].\end{aligned}$$

This implies that

$$q_n(x)q_n'(x)/\exp\left(\int_b^x p_n(t)^{-1}dt\right)$$

is a nondecreasing function of x. But this expression is the derivative of $\frac{1}{2}q_n(x)^2$ with respect to the increasing function

$$\int_b^x \exp\left(\int_b^u p_n(v)^{-1}dv\right)du.$$

Since the derivative is nondecreasing, $q_n(x)^2$ is a convex function of

$$\int_b^x \exp\left(\int_b^u p_n(v)^{-1}dv\right)du.$$

If $a < b < c$ are numbers which belong to an interval in which $q_n(x) > 0$, then

$$\begin{aligned}q_n(b)^2 \int_a^c \exp\left(\int_b^u p_n(v)^{-1}dv\right)du \\ \leq q_n(a)^2 \int_a^b \exp\left(\int_b^u p_n(v)^{-1}dv\right)du \\ + q_n(c)^2 \int_b^c \exp\left(\int_b^u p_n(v)^{-1}dv\right)du\end{aligned}$$

by convexity. If

$$q(x)^2 = \frac{1}{2\pi}\int_0^{2\pi} f(x+iy)^2 dy,$$

then $q(x) = \lim q_n(x)$ as $n \to \infty$ since $f(x+iy) = \lim f_n(x+iy)$ uniformly on bounded sets. If $2\pi p(x)$ is the sum of the lengths of the y-intervals, $0 < y < 2\pi$, in which $f(x+iy) > 0$, then $p(x) = \lim p_n(x)$ uniformly on finite intervals by Problem 107. In any interval where $q(x) > 0$, we have $q_n(x) > 0$, $p_n(x) > 0$, and

$$\int_b^x \exp\left(\int_b^u p(v)^{-1}dv\right)du = \lim_{n\to\infty}\int_b^x \exp\left(\int_b^u p_n(v)^{-1}dv\right)du.$$

So if $a < b < c$ are numbers which belong to an interval in which $q(x) > 0$, then

$$\begin{aligned}q(b)^2 \int_a^c \exp\left(\int_b^u p(v)^{-1}dv\right)du \\ \leq q(a)^2 \int_a^b \exp\left(\int_b^u p(v)^{-1}dv\right)du \\ + q(c)^2 \int_b^c \exp\left(\int_b^u p(v)^{-1}dv\right)du.\end{aligned}$$

With a change of notation the inequality reads

$$Q(\exp b)^2 \int_a^c \exp\left(\int_b^u P(\exp v)^{-1}dv\right)du$$
$$\leq Q(\exp a)^2 \int_a^b \exp\left(\int_b^u P(\exp v)^{-1}dv\right)du$$
$$+ Q(\exp c)^2 \int_b^c \exp\left(\int_b^u P(\exp v)^{-1}dv\right)du.$$

An equivalent inequality is

$$Q(\exp b)^2 \int_a^c \exp\left(\int_{\exp b}^{\exp u} v^{-1}P(v)^{-1}dv\right)du$$
$$\leq Q(\exp a)^2 \int_a^b \exp\left(\int_{\exp b}^{\exp u} v^{-1}P(v)^{-1}dv\right)du$$
$$+ Q(\exp c)^2 \int_b^c \exp\left(\int_{\exp b}^{\exp u} v^{-1}P(v)^{-1}dv\right)du.$$

This can also be written as

$$Q(\exp b)^2 \int_{\exp a}^{\exp c} \exp\left(\int_b^u v^{-1}P(v)^{-1}dv\right)u^{-1}du$$
$$\leq Q(\exp a)^2 \int_{\exp a}^{\exp b} \exp\left(\int_b^u v^{-1}P(v)^{-1}dv\right)u^{-1}du$$
$$+ Q(\exp c)^2 \int_{\exp b}^{\exp c} \exp\left(\int_b^u v^{-1}P(v)^{-1}dv\right)u^{-1}du.$$

The lemma follows on making a change of variable.

Proof of Lemma 8. Let $P_k(r)$ and $Q_k(r)$ be defined for $F_k(z)$ as in Lemma 7, $k = 1, 2$. The hypotheses imply that either $|F_1(re^{i\theta})| < 1$ or $|F_2(re^{i\theta})| < 1$ when $|\sin\theta| > 1/r$. Since $|\sin\theta| \geq |2\theta/\pi|$ whenever $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$, we obtain

$$P_1(r) + P_2(r) \leq 1 + 1/r.$$

It follows that

$$1/P_1(r) + 1/P_2(r) \geq 4/[P_1(r) + P_2(r)] \geq 4r/(r+1).$$

We use the inequality to show that $Q_1(r)$ or $Q_2(r)$ is bounded.

Argue by contradiction, assuming that both functions are unbounded. Choose a half-line $[a, \infty)$ in which both functions are positive. By Lemma 7 and Problem 108, there exists some number $c > 0$ such that

$$Q_k(r)^2 \geq c \int_a^r \exp\left(\int_a^u v^{-1}P_k(v)^{-1}dv\right)u^{-1}du$$

for $r > a$, $k = 1, 2$. By the convexity of the exponential function,

$$\begin{aligned}\tfrac{1}{2}Q_1(r)^2 + \tfrac{1}{2}Q_2(r)^2 &\geq c \int_a^r \exp\left(\int_a^u \tfrac{1}{2}v^{-1}[P_1(v)^{-1} + P_2(v)^{-1}]dv\right)u^{-1}du \\ &\geq c \int_a^r \exp\left(\int_a^u 2(v+1)^{-1}dv\right)u^{-1}du \\ &\geq c \int_a^r (a+1)^{-2}(u+1)^2u^{-1}du \\ &\geq \tfrac{1}{2}c(r+1)^2/(a+1)^2 - \tfrac{1}{2}c,\end{aligned}$$

which contradicts the growth hypotheses on $Q_1(r)$ and $Q_2(r)$. So $Q_1(r)$ or $Q_2(r)$ is bounded.

If $Q_k(r)$ is bounded, then $F_k(z)$ is a constant by Liouville's theorem. For if $|z| < \epsilon a$ where $\epsilon < 1$,

$$\begin{aligned}\log|F_k(z)| &\leq \frac{a^2 - |z|^2}{2\pi} \int_0^{2\pi} \frac{\log|F_k(ae^{i\theta})|d\theta}{|ae^{i\theta} - z|^2} \\ &\leq (1+\epsilon)/(1-\epsilon)\ 1/(2\pi) \int_0^{2\pi} \log^+|F(ae^{i\theta})|d\theta \\ &\leq \sqrt{Q(a)}(1+\epsilon)/(1-\epsilon)\end{aligned}$$

by the Schwarz inequality. By the arbitrariness of a and ϵ, $F_k(z)$ is bounded in the complex plane and hence a constant.

We now know that one function, say $F_1(z)$, is a constant. If $F_1(z)$ does not vanish identically, the hypotheses imply that $\lim F_2(iy) = 0$ as $|y| \to \infty$. By Problem 38, $F_2(z)$ is a constant and so vanishes identically.

Proof of Theorem 35. The theorem follows from Problem 109 if $\mathcal{H}(E(a))$ or $\mathcal{H}(E(b))$ fills $L^2(\mu)$. Otherwise consider first the case in which the mean type of $E(b, z)/E(a, z)$ is positive. Then

$$\limsup_{y\to+\infty} |E(a, iy)/E(b, iy)| = 0$$

by Theorem 10. By Theorem 26 and Problem 69, $[F(z)E(a, w) - E(a, z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. Since $G^*(z)$ belongs to $\mathcal{H}(E(b))$ whenever $G(z)$ belongs to $\mathcal{H}(E(b))$,

$$[F(z)E^*(a, w) - E^*(a, z)F(w)]/(z - w)$$

belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. On taking a linear combination of these two functions, we obtain $[E(a, z)E^*(a, w) - E^*(a, z)E(a, w)]/(z - w)$ in $\mathcal{H}(E(b))$. Therefore any finite linear combination of the functions

$$K(a, w, z) = [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)]/[\pi(z - \bar{w})]$$

belongs to $\mathcal{H}(E(b))$. Since such linear combinations are dense in $\mathcal{H}(E(a))$, it follows that $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$. A similar argument will show that $\mathcal{H}(E(b))$ is contained in $\mathcal{H}(E(a))$ if the mean type of $E(a, z)/E(b, z)$ is positive. In the remainder of the proof we assume that $E(b, z)/E(a, z)$ has zero mean type in the upper half-plane.

Let $P(x)$ be any element of $L^2(\mu)$ which has norm 1 and is orthogonal to $\mathcal{H}(E(a))$. Let $Q(x)$ be any element of $L^2(\mu)$ which has norm 1 and is orthogonal to $\mathcal{H}(E(b))$. If $F(z)$ is in $\mathcal{H}(E(a))$, then by the proof of Theorem 26 there exists an entire function $f(z)$ such that

$$f(w)E(b, w) = \int_{-\infty}^{+\infty} \frac{F(t)E(b, w) - E(b, t)F(w)}{t - w} \bar{Q}(t)d\mu(t),$$

$$f(w)E^*(b, w) = \int_{-\infty}^{+\infty} \frac{F(t)E^*(b, w) - E^*(b, t)F(w)}{t - w} \bar{Q}(t)d\mu(t)$$

for all complex w. By the proof of Theorem 26, $f(z)$ and $f^*(z)$ are of bounded type in the upper half-plane. Write

$$f(z) = \int_{-\infty}^{+\infty} \frac{F(t)\bar{Q}(t)d\mu(t)}{t - z} - \frac{F(z)}{E(a, z)} \frac{E(a, z)}{E(b, z)} \int_{-\infty}^{+\infty} \frac{E(b, t)\bar{Q}(t)d\mu(t)}{t - z}$$

for $y > 0$. By Problem 65, the integrals represent functions which have nonpositive mean type in the upper half-plane. Since $E(a, z)/E(b, z)$ has zero mean type by hypothesis and since $F(z)/E(a, z)$ has nonpositive mean type by the definition of $\mathcal{H}(E(a))$, $f(z)$ has nonpositive mean type in the upper half-plane. A similar argument will show that $f^*(z)$ has nonpositive mean type in the upper half-plane. By Problems 35 and 36,

$$\lim_{r\to\infty} r^{-2} \int_0^{2\pi} [\log^+ |f(re^{i\theta})|]^2 d\theta = 0.$$

In what follows we assume that $F(z)$ does not vanish identically, so that these constructions are nontrivial.

If $G(z)$ is in $\mathcal{H}(E(b))$, then for the same reasons there exists an entire function $g(z)$ such that

$$\lim_{r\to\infty} r^{-2} \int_0^{2\pi} [\log^+ |g(re^{i\theta})|]^2 d\theta = 0$$

and such that

$$g(w)E(a, w) = \int_{-\infty}^{+\infty} \frac{G(t)E(a, w) - E(a, t)G(w)}{t - w} \bar{P}(t)d\mu(t),$$

$$g(w)E^*(a, w) = \int_{-\infty}^{+\infty} \frac{G(t)E^*(a, w) - E^*(a, t)G(w)}{t - w} \bar{P}(t)d\mu(t)$$

for all complex numbers w. We assume that $G(z)$ does not vanish identically. As in the proof of Theorem 26,

$$f(w)G(w) = \int_{-\infty}^{+\infty} \frac{F(t)G(w) - G(t)F(w)}{t - w} \bar{Q}(t)\,d\mu(t),$$

$$g(w)F(w) = \int_{-\infty}^{+\infty} \frac{G(t)F(w) - F(t)G(w)}{t - w} \bar{P}(t)\,d\mu(t)$$

for all complex numbers w. By the Schwarz inequality in $L^2(\mu)$,

$$|f(z)G(z)| \leq |y|^{-1}[\|F(t)\|\ |G(z)| + \|G(t)\|\ |F(z)|],$$
$$|g(z)F(z)| \leq |y|^{-1}[\|G(t)\|\ |F(z)| + \|F(t)\|\ |G(z)|]$$

for nonreal numbers z. It follows that

$$|y| \leq \|F(t)\|/|f(z)| + \|G(t)\|/|g(z)|$$

at all points z where $\|F(t)\|\ |G(z)| + \|G(t)\|\ |F(z)|$ is nonzero. The inequality follows by continuity for all nonreal z. It implies that

$$\min\left(\tfrac{1}{2}|f(z)|/\|F(t)\|,\ \tfrac{1}{2}|g(z)|/\|G(t)\|\right) \leq |y|^{-1}.$$

By Lemma 8, either $f(z)$ or $g(z)$ vanishes identically.

If $f(z)$ does not vanish identically for some choice of $F(z)$ and $Q(x)$, then $g(z)$ must vanish identically for every choice of $G(z)$ and $P(x)$. Thus either $f(z)$ vanishes identically in all cases or $g(z)$ vanishes identically in all cases. In the remainder of the proof we assume for definiteness that $g(z)$ vanishes identically in all cases. Then

$$0 = \int_{-\infty}^{+\infty} \frac{G(t)E(a, w) - E(a, t)G(w)}{t - w} \bar{P}(t)\,d\mu(t)$$

whenever $G(z)$ is in $\mathcal{H}(E(b))$ and $P(x)$ is an element of norm 1 in $L^2(\mu)$ which is orthogonal to $\mathcal{H}(E(a))$. By the arbitrariness of $P(x)$, $[G(z)E(a, w) - E(a, z)G(w)]/(z - w)$ coincides with an element of $\mathcal{H}(E(a))$ almost everywhere with respect to μ. By the proof of Theorem 26, the function actually belongs to $\mathcal{H}(E(a))$. So $[G(z)E(a, w) - E(a, z)G(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $G(z)$ belongs to $\mathcal{H}(E(b))$.

If $\mathcal{H}(E(b))$ is not contained in $\mathcal{H}(E(a))$, let $L(z)$ be an element of $\mathcal{H}(E(b))$ which does not belong to $\mathcal{H}(E(a))$. Let $P(x)$ be an element of norm 1 in $L^2(\mu)$ which is orthogonal to $\mathcal{H}(E(a))$ but which is not orthogonal to $L(z)$. Since

$$\frac{zL(z)E(a, w) - E(a, z)wL(w)}{z - w} = L(z)E(a, w) + w\,\frac{L(z)E(a, w) - E(a, z)L(w)}{z - w}$$

where $[L(z)E(a, w) - E(a, z)L(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$, we obtain

$$E(a, w)\langle L(t), P(t)\rangle = \int_{-\infty}^{+\infty} \frac{tL(t)E(a, w) - E(a, t)wL(w)}{t - w} \bar{P}(t)\, d\mu(t).$$

So when $y > 0$,

$$\langle L(t), P(t)\rangle =$$

$$\int_{-\infty}^{+\infty} \frac{tL(t)\bar{P}(t)\, d\mu(t)}{t - iy} - \frac{L(iy)}{E(b, iy)} \frac{iyE(b, iy)}{E(a, iy)} \int_{-\infty}^{+\infty} \frac{E(a, t)\bar{P}(t)\, d\mu(t)}{t - iy}.$$

By the Schwarz inequality,

$$\left|\int_{-\infty}^{+\infty} \frac{tL(t)\bar{P}(t)\, d\mu(t)}{t - iy}\right|^2 \leq \int_{-\infty}^{+\infty} \frac{t^2\, |L(t)|^2 d\mu(t)}{t^2 + y^2},$$

$$\left|\int_{-\infty}^{+\infty} \frac{E(a, t)\bar{P}(t)\, d\mu(t)}{t - iy}\right|^2 \leq \int_{-\infty}^{+\infty} \frac{|E(a, t)|^2 d\mu(t)}{t^2 + y^2}.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{y\to+\infty} \int_{-\infty}^{+\infty} \frac{tL(t)\bar{P}(t)\, d\mu(t)}{t - iy} = 0, \qquad \lim_{y\to+\infty} \int_{-\infty}^{+\infty} \frac{E(a, t)\bar{P}(t)\, d\mu(t)}{t - iy} = 0.$$

Since $L(z)$ belongs to $\mathcal{H}(E(b))$,

$$\limsup_{y\to+\infty} \sqrt{y}\, |L(iy)/E(b, iy)| < \infty$$

by Theorem 20. Since $\langle L(t), P(t)\rangle \neq 0$, we can conclude that

$$\lim_{y\to+\infty} \sqrt{y}\, |E(b, iy)/E(a, iy)| = \infty.$$

But if $F(z)$ belongs to $\mathcal{H}(E(a))$,

$$\limsup_{y\to+\infty} \sqrt{y}\, |F(iy)/E(a, iy)| < \infty$$

by Theorem 20. It follows that

$$\lim_{y\to+\infty} |F(iy)/E(b, iy)| = 0.$$

The same conclusion holds when $F(z)$ is replaced by $F^*(z)$. By Theorem 26, $[F(z)G(w) - G(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(b))$ whenever $G(z)$ belongs to $\mathcal{H}(E(b))$ if $F(z)$ is in $\mathcal{H}(E(a))$.

But we already know that $[F(z)G(w) - G(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a)$ if $G(z)$ is in $\mathcal{H}(E(b))$. If $\mathcal{H}(E(b))$ is not contained in $\mathcal{H}(E(a))$, we can choose a nonzero element $Q(z)$ of $\mathcal{H}(E(b))$

which is orthogonal to $[F(z)G(w) - G(z)F(w)]/(z - w)$ whenever $F(z)$ is in $\mathcal{H}(E(a))$ and $G(z)$ is in $\mathcal{H}(E(b))$. Then $Q(z)$ is orthogonal to the domain of multiplication by z in $\mathcal{H}(E(b))$. By Theorem 29 the closed span of the functions $[F(z)G(w) - G(z)F(w)]/(z - w)$ with $F(z)$ in $\mathcal{H}(E(a))$ and $G(z)$ in $\mathcal{H}(E(b))$ fills the orthogonal complement of $Q(z)$ in $\mathcal{H}(E(b))$. If $\mathcal{H}(E(b))$ is not contained in $\mathcal{H}(E(a))$, then $Q(z)$ cannot belong to $\mathcal{H}(E(a))$. We show that $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$.

It is sufficient to show that the closed span of the functions $[F(z)G(w) - G(z)F(w)]/(z - w)$ with $F(z)$ in $\mathcal{H}(E(a))$ and $G(z)$ in $\mathcal{H}(E(b))$ is all of $\mathcal{H}(E(a))$. So we need only show that there is no nonzero element $P(z)$ of $\mathcal{H}(E(a))$ which is orthogonal to all such functions $[F(z)G(w) - G(z)F(w)]/(z - w)$. Note that $P(z)$ and $Q(z)$ are then orthogonal to $[P(z)Q(w) - Q(z)P(w)]/(z - w)$ for all complex w. Since

$$\begin{aligned}(w - \bar{w})\, &\| [P(t)Q(w) - Q(t)P(w)]/(t - w) \|^2 \\ &= \langle P(t)Q(w) - Q(t)P(w), [P(t)Q(w) - Q(t)P(w)]/(t - w) \rangle \\ &\quad - \langle [P(t)Q(w) - Q(t)P(w)]/(t - w), P(t)Q(w) - Q(t)P(w) \rangle = 0,\end{aligned}$$

$[P(z)Q(w) - Q(z)P(w)]/(z - w)$ vanishes identically when w is not real. It follows that $P(z)$ and $Q(z)$ are linearly dependent. Since $P(z)$ belongs to $\mathcal{H}(E(a))$ and $Q(z)$ does not, $P(z) = 0$. The theorem follows.

PROBLEM 110. If $\mathcal{H}(M)$ is a given space which has finite dimension r, show that

$$M(z) = \begin{pmatrix} 1 - \beta_1 z & \alpha_1 z \\ -\gamma_1 z & 1 + \beta_1 z \end{pmatrix} \cdots \begin{pmatrix} 1 - \beta_r z & \alpha_r z \\ -\gamma_r z & 1 + \beta_r z \end{pmatrix} M(0)$$

where (α_k), (β_k), (γ_k) are real numbers such that $\alpha_k \geq 0$, $\gamma_k \geq 0$, and $\alpha_k \gamma_k = \beta_k^2$ for $k = 1, \cdots, r$. *Hint*: The transformation $f(z) \to [f(z) - f(0)]/z$ in $\mathcal{H}(M)$ has an eigenvalue.

PROBLEM 111. Let $\mathcal{H}(M(a))$, $\mathcal{H}(M(b))$, $\mathcal{H}(M(c))$ be spaces such that

$$M(c, z) = M(a, z)M(a, c, z) \quad \text{and} \quad M(c, z) = M(b, z)M(b, c, z)$$

for some spaces $\mathcal{H}(M(a, c))$ and $\mathcal{H}(M(b, c))$. If $\mathcal{H}(M(c))$ has dimension 0 or 1, show that either

$$M(b, z) = M(a, z)M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$ or

$$M(a, z) = M(b, z)M(b, a, z)$$

for some space $\mathcal{H}(M(b, a))$.

PROBLEM 112. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(E(c))$ be given spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z),$$
$$(A(c, z), B(c, z)) = (A(b, z), B(b, z))M(b, c, z)$$

for some spaces $\mathcal{H}(M(a, c))$ and $\mathcal{H}(M(b, c))$. Show that either

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

and

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(a, b))$, or

$$(A(a, z), B(a, z)) = (A(b, z), B(b, z))M(b, a, z)$$

and

$$M(b, c, z) = M(b, a, z)M(a, c, z)$$

for some space $\mathcal{H}(M(b, a))$.

PROBLEM 113. Let $\mathcal{H}(M(a))$, $\mathcal{H}(M(b))$, and $\mathcal{H}(M(c))$ be spaces such that

$$M(c, z) = M(a, z)M(a, c, z) \quad \text{and} \quad M(c, z) = M(b, z)M(b, c, z)$$

for some spaces $\mathcal{H}(M(a, c))$ and $\mathcal{H}(M(b, c))$. Show that either

$$M(b, z) = M(a, z)M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$ or

$$M(a, z) = M(b, z)M(b, a, z)$$

for some space $\mathcal{H}(M(b, a))$.

PROBLEM 114. The Schmidt norm of a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is defined to be

$$\sigma(M)^2 = |A|^2 + |B|^2 + |C|^2 + |D|^2.$$

Show that $\sigma(PQ) \leq \sigma(P)\sigma(Q)$ for all 2×2-matrices P and Q. Show also that

$$1 + \sigma(PQ - 1) \leq [1 + \sigma(P - 1)][1 + \sigma(Q - 1)].$$

PROBLEM 115. Let $\mathcal{H}(M)$ be a finite dimensional space such that $M(0) = 1$. Show that

$$M'(0)I = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \geq 0$$

and that

$$M(z) = \sum_{n=0}^{\infty} M^{(n)}(0) z^n/n!$$

where $\sigma(M^{(n)}(0)) \leq (\alpha + \gamma)^n$ for every $n = 1, 2, 3, \cdots$.

PROBLEM 116. If $\mathcal{H}(E(b))$ is a finite dimensional space and if $h \geq 0$, show that there exists a space $\mathcal{H}(M(a, b))$ such that $M(a, b, 0) = 1$, $B'(a, b, 0) - C'(a, b, 0) = h$, and

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some entire functions $A(a, z)$ and $B(a, z)$, which are real for real z, such that

$$[B(a, z)\bar{A}(a, z) - A(a, z)\bar{B}(a, z)]/(z - \bar{z}) \geq 0$$

for all complex z.

36. EXISTENCE OF SUBSPACES OF $\mathcal{H}(E)$

Similar results hold for infinite dimensional spaces.

THEOREM 36. If $\mathcal{H}(E(b))$ is a given space and if $h \geq 0$, then there exists a space $\mathcal{H}(M(a, b))$ such that $M(a, b, 0) = 1$, $B'(a, b, 0) - C'(a, b, 0) = h$, and

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some entire functions $A(a, z)$ and $B(a, z)$, which are real for real z, such that

$$[B(a, z)\bar{A}(a, z) - A(a, z)\bar{B}(a, z)]/(z - \bar{z}) \geq 0$$

for all complex z.

Proof of Theorem 36. By Problem 116 we can restrict explicit proof to the case in which $\mathcal{H}(E(b))$ has infinite dimension. We use Theorem 22 to approximate by finite dimensional spaces. The argument requires the choice of a real number α such that $e^{i\alpha}E(b, z) - e^{-i\alpha}E^*(b, z)$ does not belong to $\mathcal{H}(E(b))$. We assume for definiteness that $\alpha = 0$, but a similar argument can be given for any other choice of α. Let $\varphi(b, x)$ be a phase function associated with $E(b, z)$ and let (t_n) be an enumeration of the real numbers t such that $\varphi(b, t) \equiv 0$ modulo π. The functions $\{B(b, z)/(z - t_n)\}$ are then an orthogonal basis for $\mathcal{H}(E(b))$ and

$$\|B(b, t)/(t - t_n)\|^2 = \pi B'(b, t_n)/A(b, t_n)$$

for every n. If the span $\mathcal{H}_n$ of the first n elements of the basis is considered as a Hilbert space in the metric of $\mathcal{H}(E(b))$, then

$$K_n(b, w, z) = \frac{1}{\pi} \sum_{k=1}^{n} \frac{A(b, t_k)}{B'(b, t_k)} \frac{B(b, z)}{z - t_k} \frac{\bar{B}(b, w)}{\bar{w} - t_k}$$

belongs to $\mathcal{H}_n$ as a function of z for every w and

$$F(w) = \langle F(t), K_n(b, w, t)\rangle$$

for every $F(z)$ in $\mathcal{H}_n$. If we write

$$A_n(b, z) = \sum_{k=1}^{n} \frac{A(b, t_k)}{B'(b, t_k)} \frac{B(b, z)}{z - t_k} + h_n B(b, z)$$

for any real number h_n, then $A_n(b, z)$ is an entire function which is real for real z and

$$K_n(b, w, z) = [B(b, z)\bar{A}_n(b, w) - A_n(b, z)\bar{B}(b, w)]/[\pi(z - \bar{w})]$$

for all complex z and w. Since $K_n(b, w, w) > 0$ when w is not real, a space $\mathcal{H}(E_n(b))$ exists by the proof of Theorem 23, $E_n(b, z) = A_n(b, z) - iB(b, z)$, and the space is equal isometrically to $\mathcal{H}_n$. Since the starting orthogonal set is complete, the union of the spaces $\mathcal{H}(E_n(b))$ is dense in $\mathcal{H}(E(b))$. Since $K_n(b, w, z)$ is the projection of $K(b, w, z)$ in $\mathcal{H}(E_n(b))$ for every n,

$$K(b, w, z) = \lim_{n\to\infty} K_n(b, w, z)$$

in the metric of $\mathcal{H}(E(b))$ for every w. It follows that

$$B(b, z)\bar{A}(b, w) - A(b, z)\bar{B}(b, w) = \lim_{n\to\infty} [B(b, z)\bar{A}_n(b, w) - A_n(b, z)\bar{B}(b, w)]$$

for all complex z and w. If the numbers (h_n) are chosen so that

$$\operatorname{Re} A(b, w)/B(b, w) = \lim_{n\to\infty} \operatorname{Re} A_n(b, w)/B(b, w)$$

for some nonreal number w, then $A(b, z) = \lim A_n(b, z)$ as $n \to \infty$ for all complex z. The convergence is uniform on bounded sets. By Problem 116,

$$(A_n(b, z), B_n(b, z)) = (A_n(a, z), B_n(a, z))M_n(a, b, z)$$

for some space $\mathcal{H}(M_n(a, b))$ such that $M_n(a, b, 0) = 1$ and $B_n'(a, b, 0) - C_n'(a, b, 0) = h$, and for some entire functions $A_n(a, z)$ and $B_n(a, z)$, which are real for real z, such that

$$[B_n(a, z)\bar{A}_n(a, z) - A_n(a, z)\bar{B}_n(a, z)]/(z - \bar{z}) \geq 0$$

for all complex z. Because of Problem 115 there exists an increasing sequence $s(1), s(2), s(3), \cdots$, of positive integers such that

$$M(a, b, z) = \lim_{n\to\infty} M_{s(n)}(a, b, z)$$

exists in the sense of formal power series. (See the analogous SSPS Lemma 6 and Theorem 22.) By the estimate of Problem 115, the limit series converges for all complex z and convergence of the last limit takes place also in the sense of function values, uniformly for z in any bounded set. Since the entries of $M_{s(n)}(a, b, z)$ are real for real z for every n, the entries of $M(a, b, z)$ are real for real z. Since

$$M_{s(n)}(a, b, z)I\bar{M}_{s(n)}(a, b, \bar{z}) = I$$

and

$$[M_{s(n)}(a, b, z)I\bar{M}_{s(n)}(a, b, z) - I]/(z - \bar{z}) \geq 0$$

for every n,

$$M(a, b, z)I\bar{M}(a, b, \bar{z}) = I$$

and

$$[M(a, b, z)I\bar{M}(a, b, z) - I]/(z - \bar{z}) \geq 0$$

for all complex z. These are the conditions for the existence of a space $\mathcal{H}(M(a, b))$. Since

$$B'_{s(n)}(a, b, 0) - C'_{s(n)}(a, b, 0) = h$$

for every n,

$$B'(a, b, 0) - C'(a, b, 0) = h.$$

Since we have

$$(A_{s(n)}(a, z), B_{s(n)}(a, z)) = (A_{s(n)}(b, z), B_{s(n)}(b, z))I\bar{M}_{s(n)}(a, b, \bar{z})\bar{I}$$

for every n, where

$$A(b, z) = \lim_{n\to\infty} A_{s(n)}(b, z), \qquad B(b, z) = \lim_{n\to\infty} B_{s(n)}(b, z),$$

$$M(a, b, z) = \lim_{n\to\infty} M_{s(n)}(a, b, z),$$

the limits

$$A(a, z) = \lim_{n\to\infty} A_{s(n)}(a, z) \quad \text{and} \quad B(a, z) = \lim_{n\to\infty} B_{s(n)}(a, z)$$

exist for all complex z. Convergence is uniform on bounded sets. The limit functions $A(a, z)$ and $B(a, z)$ are entire functions which are real for real z. Since

$$[B_{s(n)}(a, z)\bar{A}_{s(n)}(a, z) - A_{s(n)}(a, z)\bar{B}_{s(n)}(a, z)]/(z - \bar{z}) \geq 0$$

for every n,

$$[B(a, z)\bar{A}(a, z) - A(a, z)\bar{B}(a, z)]/(z - \bar{z}) \geq 0$$

for all complex z.

PROBLEM 117. In Theorem 36 show that a space $\mathcal{H}(E(a))$ exists, $E(a, z) = A(a, z) - iB(a, z)$, if $A(a, z)$ and $B(a, z)$ are linearly independent.

PROBLEM 118. If $A(a, z)$ and $B(a, z)$ are linearly dependent in Theorem 36, show that $E(a, z) = A(a, z) - iB(a, z)$ has only real zeros and that $E(b, z)/E(a, z)$ is an entire function. Show that

$$[F(z)E(a, w) - E(a, z)F(w)]/(z - w)$$

belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$.

PROBLEM 119. Let $\mathcal{H}(E(b))$, $\mathcal{H}(E(c))$, and $\mathcal{H}(M(b, c))$ be spaces such that

$$(A(c, z), B(c, z)) = (A(b, z), B(b, z))M(b, c, z).$$

Let $A(a, z)$ and $B(a, z)$ be linearly dependent entire functions, which are real for real z, such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z)$$

for some space $\mathcal{H}(M(a, c))$. Show that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$
$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(a, b))$.

PROBLEM 120. Let $\mathcal{H}(E(c))$, $\mathcal{H}(M(a, c))$, and $\mathcal{H}(M(b, c))$ be spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z),$$
$$(A(c, z), B(c, z)) = (A(b, z), B(b, z))M(b, c, z)$$

for linearly dependent entire functions $A(a, z)$ and $B(a, z)$ and for linearly dependent entire functions $A(b, z)$ and $B(b, z)$ which are real for real z. Show that $E(a, z) = A(a, z) - iB(a, z)$ and $E(b, z) = A(b, z) - iB(b, z)$ are linearly dependent. Show that either

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$
$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(a, b))$ or

$$(A(a, z), B(a, z)) = (A(b, z), B(b, z))M(b, a, z),$$
$$M(b, c, z) = M(b, a, z)M(a, c, z)$$

for some space $\mathcal{H}(M(b, a))$.

PROBLEM 121. Let $\mathcal{H}(E(c))$, $\mathcal{H}(M(a, c))$, and $\mathcal{H}(M(b, c))$ be spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z),$$
$$(A(c, z), B(c, z)) = (A(b, z), B(b, z))M(b, c, z)$$

for entire functions $A(a, z)$, $B(a, z)$, $A(b, z)$, and $B(b, z)$, which are real for real z, such that

$$[B(a, z)\bar{A}(a, z) - A(a, z)\bar{B}(a, z)]/(z - \bar{z}) \geq 0,$$
$$[B(b, z)\bar{A}(b, z) - A(b, z)\bar{B}(b, z)]/(z - \bar{z}) \geq 0$$

for all complex z. If $M(a, c, 0) = M(b, c, 0) = 1$, and if

$$B'(a, c, 0) - C'(a, c, 0) \geq B'(b, c, 0) - C'(b, c, 0),$$

show that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$
$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(a, b))$. If

$$B'(a, c, 0) - C'(a, c, 0) = B'(b, c, 0) - C'(b, c, 0),$$

show that $M(a, b, z) = 1$, $A(a, z) = A(b, z)$, $B(a, z) = B(b, z)$, and $M(a, c, z) = M(b, c, z)$.

PROBLEM 122. Show that $A(a, z) = \lim A_n(a, z)$, $B(a, z) = \lim B_n(a, z)$, and $M(a, b, z) = \lim M_n(a, b, z)$ as $n \to \infty$ in the proof of Theorem 36.

PROBLEM 123. If $\mathcal{H}(M)$ is a given space and if $M(0) = 1$, show that there exists a sequence $\{\mathcal{H}(M_n)\}$ of finite dimensional spaces such that $M_n(0) = 1$ and $B_n'(0) - C_n'(0) = B'(0) - C'(0)$ for every n, and such that $M(z) = \lim M_n(z)$ for all complex z.

PROBLEM 124. If $\mathcal{H}(M)$ is a given space and if $M(0) = 1$, show that

$$M'(0)I = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \geq 0$$

and that

$$M(z) = \sum_{n=0}^{\infty} M^{(n)}(0)z^n/n!$$

where $\sigma(M^{(n)}(0)) \leq (\alpha + \gamma)^n$ for every $n = 1, 2, 3, \cdots$. Show that

$$1 + \sigma[M(z) - 1] \leq \exp[(\alpha + \gamma)\,|z|]$$

for all complex z.

PROBLEM 125. If $\mathcal{H}(M(b))$ is a given space, if $M(b, 0) = 1$, and if h is a given number, $0 \leq h \leq B'(b, 0) - C'(b, 0)$, show that there exist unique spaces $\mathcal{H}(M(a))$ and $\mathcal{H}(M(a, b))$ such that

$$M(b, z) = M(a, z)M(a, b, z),$$

$M(a, b, 0) = 1$, and $h = B'(a, 0) - C'(a, 0)$.

37. INTEGRAL EQUATION FOR $M(z)$

The factors of $M(z)$ satisfy an integral equation.

THEOREM 37. Let c be a given positive number and let $\mathcal{H}(M(c))$ be a given space such that $M(c, 0) = 1$. For each number a, $0 \leq a \leq c$, let

$$M(c, z) = M(a, z)M(a, c, z)$$

be the unique factorization such that $\mathcal{H}(M(a))$ and $\mathcal{H}(M(a, c))$ exist, $M(a, 0) = 1$, and

$$B'(a, 0) - C'(a, 0) = [B'(c, 0) - C'(c, 0)]a/c.$$

Then $m(a) = M'(a, 0)I$ is a continuous, nondecreasing function of a, $0 \leq a \leq c$, and

$$M(a, z)I - I = z \int_0^a M(t, z)dm(t)$$

for $0 \leq a \leq c$.

We show in the proof that the entries of $M(t, z)$ are continuous functions of t for each fixed z and that the entries of $m(t)$ are functions of bounded variation. The matrix integral equation is equivalent to four scalar Stieltjes integral equations:

$$\begin{aligned} B(a, z) &= z \int_0^a A(t, z)d\alpha(t) + z \int_0^a B(t, z)d\beta(t), \\ 1 - A(a, z) &= z \int_0^a A(t, z)d\beta(t) + z \int_0^a B(t, z)d\gamma(t), \\ D(a, z) - 1 &= z \int_0^a C(t, z)d\alpha(t) + z \int_0^a D(t, z)d\beta(t), \\ -C(a, z) &= z \int_0^a C(t, z)d\beta(t) + z \int_0^a D(t, z)d\gamma(t). \end{aligned}$$

Proof of Theorem 37. For notational convenience we restrict the proof to the case in which $\alpha(c) + \gamma(c) = c$, but the general case can be obtained by an obvious change of variable. It follows from Problem 113 that when $a \leq b$ there is a space $\mathcal{H}(M(a, b))$ such that $M(b, z) = M(a, z)M(a, b, z)$. This condition implies that $M(a, b, 0) = 1$ and that $m(b) = m(a) + m(a, b)$ where the matrix $m(a, b)$ is nonnegative. So $m(t)$ is a nondecreasing function of t. Since $\alpha(t)$ and $\gamma(t)$ are nondecreasing functions of t and since $\alpha(t) + \gamma(t) = t$, they are continuous. Since

$$[\beta(b) - \beta(a)]^2 \leq [\alpha(b) - \alpha(a)][\gamma(b) - \gamma(a)] \leq (b - a)^2$$

when $a \leq b$, $\beta(t)$ is a continuous function of t of total variation at most c in $[0, c]$. Since

$$\sigma[M(a, b, z) - 1] \leq \exp[(b - a)|z|] - 1$$

and $\sigma[M(a, z) - 1] \leq \exp[a|z|] - 1$, we obtain

$$\sigma[M(b, z) - M(a, z)] \leq \exp(b|z|) - \exp(a|z|)$$

when $a \leq b$, which implies that the entries of $M(t, z)$ are continuous functions of t for each fixed z. These conditions are sufficient for the existence of the Stieltjes integral $\int_0^a M(t, z)dm(t)$ for all complex z where $0 \leq a \leq c$. It remains to show that the integral is equal to $M(a, z)I - I$. We do this by showing that

$$M(a, z)I - I - z\int_0^a M(t, z)dm(t)$$

vanishes identically. When $a \leq b$,

$$M(b, z)I - M(a, z)I - z\int_a^b M(t, z)\, dm(t) = M(a, z)[M(a, b, z)I - I - z\int_a^b M(a, t, z)dm(t)]$$

where

$$M(a, b, z)I - I - z\int_a^b M(a, t, z)dm(t) = M(a, b, z)I - I - z(m(b) - m(a)) - z\int_a^b [M(a, t, z) - 1]dm(t).$$

By Problem 124,

$$\sigma[M(a, b, z)I - I - z(m(b) - m(a))] \leq \exp[(b - a)|z|] - 1 - (b - a)|z| \leq (b - a)^2|z|^2 \exp[(b - a)|z|],$$

$$\sigma\left[z\int_a^b [M(a, t, z) - 1]dm(t)\right] \leq (b - a)|z| \exp[(b - a)|z|] - (b - a)|z|.$$

So we obtain

$$\sigma\left[M(b, z)I - M(a, z)I - z\int_a^b M(t, z)dm(t)\right] \leq 2(b - a)^2|z|^2 \exp[c|z|].$$

If $0 = t_0 < t_1 < \cdots < t_r = a$ is a partition of the interval $[0, a]$ of mesh at most δ, then

$$\sigma\left[M(t_k, z)I - M(t_{k-1}, z)I - z\int_{t_{k-1}}^{t_k} M(t, z)dm(t)\right] \leq 2\delta(t_k - t_{k-1})|z|^2 \exp[c|z|]$$

for every k. By the triangle inequality

$$\sigma\left[M(a, z)I - I - z \int_0^a M(t, z)dm(t)\right] \leq 2\delta a\, |z|^2 \exp\,[c\,|z|].$$

By the arbitrariness of δ,

$$M(a, z)I - I - z \int_0^a M(t, z)dm(t)$$

vanishes identically.

PROBLEM 126. If $\mathcal{H}(E(0))$ is a given space and if $t \leq 0$, let $\mathcal{H}(M(t, 0))$ be the unique space such that $M(t, 0, 0) = 1$,

$$M'(t, 0, 0)I = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix} = m(t)$$

where $\alpha(t) + \gamma(t) = t$, and

$$(A(0, z), B(0, z)) = (A(t, z), B(t, z))M(t, 0, z)$$

for entire functions $A(t, z)$ and $B(t, z)$, which are real for real z, such that

$$[B(t, z)\bar{A}(t, z) - A(t, z)\bar{B}(t, z)]/(z - \bar{z}) \geq 0$$

for all complex z. Show that $m(t)$ is a nondecreasing function of t and that its entries are continuous, real valued functions of t. Show that $A(t, w)$ and $B(t, w)$ are continuous functions of t for every w and that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $-\infty < a < b \leq 0$. Show that $A(a, z)$ and $B(a, z)$ are linearly dependent if $a < b$ and if $A(b, z)$ and $B(b, z)$ are linearly dependent. If there exists a value of t such that $A(t, z)$ and $B(t, z)$ are linearly dependent, show that there exists a largest value of t with this property, say $t = s_-$. Otherwise define $s_- = -\infty$. Show that a space $\mathcal{H}(E(t))$ exists when $> s_-$.

38. SOLUTION OF THE INTEGRAL EQUATION FOR $M(z)$

The integral equation of Theorem 37 has a unique solution for any given choice of $m(t)$, and a space $\mathcal{H}(M)$ exists.

THEOREM 38. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a continuous, nondecreasing function of t in a finite interval $[a, c]$. Then for each complex number w, there exists a unique continuous, matrix valued function

$$M(a, t, w) = \begin{pmatrix} A(a, t, w) & B(a, t, w) \\ C(a, t, w) & D(a, t, w) \end{pmatrix}$$

such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)\, dm(t)$$

for $a \le b \le c$. For each fixed b, the entries of $M(a, b, z)$ are entire functions which are real for real z, and a space $\mathcal{H}(M(a, b))$ exists. Also

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(b, c))$.

Proof of Theorem 38. Define a sequence $(M_n(a, t))$ of functions of t, $a \le t \le c$, inductively by $M_0(a, t) = 1$ and

$$M_{n+1}(a, b)I = \int_a^b M_n(a, t)\, dm(t)$$

for $n = 0, 1, 2, \cdots$. The integrals exist because the integrand is always a continuous, matrix valued function of t and because $m(t)$ is nondecreasing. Since

$$\sigma[M_{n+1}(a, b)] \le \int_a^b \sigma[M_n(a, t)]\, d[\alpha(t) + \gamma(t)],$$

we find inductively that

$$\sigma[M_n(a, b)] \le [\alpha(b) + \gamma(b) - \alpha(a) - \gamma(a)]^n/n!.$$

It follows that the series

$$M(a, b, w) = \sum_0^\infty M_n(a, b) w^n$$

converges for all complex w. The sum $M(a, b, w)$ is a continuous function of b for every w. The uniform convergence of the series allows us to integrate term-by-term for any fixed w to obtain

$$\begin{aligned} w \int_a^b M(a, t, w)\, dm(t) &= \sum_0^\infty \int_a^b M_n(a, t)\, dm(t) w^{n+1} \\ &= \sum_0^\infty M_{n+1}(a, b) I w^{n+1} \\ &= M(a, b, w)I - I. \end{aligned}$$

So $M(a, b, w)$ is the desired solution of the integral equation. To prove uniqueness, consider a possibly different solution $M_1(a, b, w)$. Then

$$
\begin{aligned}
M(a, b, z)I\bar{M}_1(a, b, w) - I
&= z\int_a^b M(a, t, z)dm(t) - \bar{w}\int_a^b dm(t)\bar{M}_1(a, t, w) \\
&\quad + z\int_a^b M(a, t, z)dm(t)I\bar{w}\int_a^b dm(s)\bar{M}_1(a, s, w) \\
&= z\int_a^b M(a, t, z)dm(t) - \bar{w}\int_a^b dm(t)\bar{M}_1(a, t, w) \\
&\quad + z\int_a^b M(a, t, z)dm(t)I\bar{w}\int_a^t dm(s)\bar{M}_1(a, s, w) \\
&\quad + z\int_a^b\int_a^s M(a, t, z)dm(t)I\, dm(s)\bar{M}_1(a, s, w) \\
&= z\int_a^b M(a, t, z)dm(t) - \bar{w}\int_a^b dm(t)\bar{M}_1(a, t, w) \\
&\quad + z\int_a^b M(a, t, z)dm(t)[\bar{M}_1(a, t, w) - 1] \\
&\quad - \bar{w}\int_a^b [M(a, t, z) - 1]dm(t)\bar{M}_1(a, t, w) \\
&= (z - \bar{w})\int_a^b M(a, t, z)dm(t)\bar{M}_1(a, t, w).
\end{aligned}
$$

When $z = \bar{w}$, we obtain

$$M(a, b, \bar{w})I\bar{M}_1(a, b, w) = I.$$

The equation

$$M(a, b, \bar{w})I\bar{M}(a, b, w) = I$$

holds for the same reason. It implies that

$$\bar{M}(a, b, w)IM(a, b, \bar{w}) = I.$$

It follows that $M_1(a, b, w) = M(a, b, w)$, which completes the proof of uniqueness.

The entries of $M(a, b, z)$ are represented by power series with real coefficients and so are entire functions which are real for real z. The identity

$$M(a, b, z)I\bar{M}(a, b, \bar{z}) = I$$

implies that $M(a, b, z)$ has determinant 1. We have seen that

$$[M(a, b, z)I\bar{M}(a, b, w) - I]/(z - \bar{w}) = \int_a^b M(a, t, z)dm(t)\bar{M}(a, t, w)$$

for all complex z and w. Since the Stieltjes integral on the right is a limit of nonnegative matrix sums when $z = w$, it is a nonnegative matrix in this case. The conditions for the existence of a space $\mathcal{H}(M(a, b))$ have now been verified.

Since $M(a, b, z)$ has determinant 1, we can define a matrix $M(b, t, z)$ of entire functions by

$$M(b, t, z) = M(a, b, z)^{-1}M(a, t, z)$$

when $b \le t \le c$. It is easily verified that the integral equation

$$M(b, t, z)I - I = z\int_b^t M(b, s, z)dm(s)$$

holds. As we have seen, these conditions imply the existence of a space $\mathcal{H}(M(b, t))$ when $b \le t \le c$. The theorem follows.

PROBLEM 127. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a continuous, nondecreasing, matrix valued function defined in a finite interval $[a, b]$. Show that there exists a real valued function $\tau(t)$ in $[a, b]$ with the following property: If $h(t)$ is a real valued function of t in $[a, b]$, then the matrix valued function

$$\begin{pmatrix} \alpha(t) & \beta(t) + ih(t) \\ \beta(t) - ih(t) & \gamma(t) \end{pmatrix}$$

is nondecreasing if, and only if, the numerical functions $\tau(t) + h(t)$ and $\tau(t) - h(t)$ are nondecreasing. Show that $\tau(t)$ is a continuous, nondecreasing function of t which is absolutely continuous if $\alpha(t) + \gamma(t)$ is an absolutely continuous function of t. Show that $\tau'(t)$ exists at all points where $\alpha'(t)$, $\beta'(t)$, and $\gamma'(t)$ exist, and that

$$\tau'(t)^2 = \alpha'(t)\gamma'(t) - \beta'(t)^2$$

at such points. The function $\tau(t)$ is called the largest nondecreasing function such that

$$\begin{pmatrix} \alpha(t) & \beta(t) + i\tau(t) \\ \beta(t) - i\tau(t) & \gamma(t) \end{pmatrix}$$

is nondecreasing. It is unique within an added constant.

PROBLEM 128. If $\mathcal{H}(M)$ is a given space, show that the functions $A(z) - iB(z)$ and $D(z) + iC(z)$ are of bounded type in the upper half-plane and have equal mean types in the half-plane. Show that each of the functions $A(z)$, $B(z)$, $C(z)$, $D(z)$ is of bounded type in the upper half-plane and that it has the same mean type in the half-plane as $A(z) - iB(z)$ and $D(z) + iC(z)$ unless it vanishes identically. The common mean type of these functions is taken as the definition of the mean type of $M(z)$.

PROBLEM 129. If $\mathcal{H}(M)$ is a given space, show that the mean type of $M(z)$ is nonnegative and that it is zero if $A(z)$ and $B(z)$ are linearly dependent.

39. MEAN TYPE OF $M(z)$

We now determine mean type from a knowledge of $m(t)$.

THEOREM 39. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a continuous, nondecreasing, matrix valued function of t in a finite interval $[a, c]$. Let $(M(a, t, z))$ be the corresponding family of matrix valued functions such that

$$M(a, b, z)I - I = z \int_a^b M(a, t, z)\, dm(t)$$

for $a \leq b \leq c$. Let $\tau(t)$ be the largest nondecreasing function of t such that

$$\begin{pmatrix} \alpha(t) & \beta(t) + i\tau(t) \\ \beta(t) - i\tau(t) & \gamma(t) \end{pmatrix}$$

is nondecreasing. Then the mean type of $M(a, b, z)$ is $\tau(b) - \tau(a)$.

LEMMA 9. If $\mathcal{H}(M)$ is a given space, if τ is the mean type of $M(z)$, and if $-\tau \leq h \leq \tau$, then there exists a space $\mathcal{H}_S(M)$ corresponding to $S(z) = e^{ihz}$. If $M(0) = 1$ and if $M'(0)I = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, then $\tau^2 \leq \alpha\gamma - \beta^2$.

Proof of Lemma 9. If $A(z)$ and $B(z)$ are linearly dependent, $\tau = 0$ by Problem 129 and $\alpha\gamma - \beta^2 \geq 0$ because $M'(0)I \geq 0$. If $A(z)$ and $B(z)$ are linearly independent and if $E(z) = A(z) - iB(z)$, a space $\mathcal{H}(E)$ exists by Problem 14. Since we assume that $-\tau \leq h \leq \tau$, $S(z)/E(z)$ and $S^*(z)/E(z)$ are of bounded type and of nonpositive mean type in the upper half-plane. By the proof of Theorem 27, there exists a number $p \geq 0$ such that

$$\operatorname{Re} \frac{D(z) + iC(z)}{A(z) - iB(z)} = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(t)|^{-2} dt}{(t - x)^2 + y^2}$$

for $y > 0$. Since $S(z)$ has absolute value 1 on the real axis,

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \, |S(t)/E(t)|^2 dt < \infty.$$

By Theorem 25, $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. By Theorem 27, there exist entire functions $C_1(z)$ and

$D_1(z)$, which are real for real z, such that

$$A(z)D_1(z) - B(z)C_1(z) = S(z)S^*(z),$$
$$\text{Re}\,[A(z)\bar{D}_1(z) - B(z)\bar{C}_1(z)] \geq \tfrac{1}{2}\,|S(z)|^2 + \tfrac{1}{2}\,|S^*(z)|^2$$

for all complex z. By the proof of Theorem 27,

$$\text{Re}\,\frac{D_1(z) + iC_1(z)}{A(z) - iB(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E(t)|^{-2}dt}{(t-x)^2 + y^2}$$

for $y > 0$. It follows that

$$\text{Re}\,\frac{D(z) + iC(z)}{A(z) - iB(z)} = py + \text{Re}\,\frac{D_1(z) + iC_1(z)}{A(z) - iB(z)}$$

for $y > 0$. Since we can add an imaginary multiple of $E(z)$ to $D_1(z) + iC_1(z)$ without changing the defining property of the functions, we can choose them so that

$$\frac{D(z) + iC(z)}{A(z) - iB(z)} = -ipz + \frac{D_1(z) + iC_1(z)}{A(z) - iB(z)}\,.$$

It follows that $D(z) = D_1(z) - pzB(z)$, $C(z) = C_1(z) - pzA(z)$,

$$A(z)D(z) - B(z)C(z) = S(z)S^*(z),$$
$$\begin{aligned}\text{Re}\,[A(z)\bar{D}(z) - B(z)\bar{C}(z)] &\geq \text{Re}\,[A(z)\bar{D}_1(z) - B(z)\bar{C}_1(z)]\\ &\geq \tfrac{1}{2}\,|S(z)|^2 + \tfrac{1}{2}\,|S^*(z)|^2\end{aligned}$$

for all complex z. This verifies the conditions for the existence of a space $\mathcal{H}_S(M)$. Since

$$[M(w)I\bar{M}(w) - S(w)I\bar{S}(w)]/(w - \bar{w}) \geq 0$$

for all complex w, we obtain

$$M'(0)I - S'(0)I \geq 0$$

when $w = 0$. It follows that $h^2 \leq \alpha\gamma - \beta^2$. Since h is an arbitrary number in the interval $[-\tau, \tau]$, $\tau^2 \leq \alpha\gamma - \beta^2$.

Proof of Theorem 39. Let $\tau(s, t)$ be the mean type of $M(s, t, z)$ when $a \leq s \leq t \leq c$. Since $M(a, t, z) = M(a, s, z)M(s, t, z)$ when $a \leq s \leq t$, it is clear that

$$\tau(a, t) \leq \tau(a, s) + \tau(s, t).$$

By Lemma 9,

$$\begin{aligned}[\tau(a, t) - \tau(a, s)]^2 &\leq \tau(s, t)^2\\ &\leq [\alpha(t) - \alpha(s)][\gamma(t) - \gamma(s)] - [\beta(t) - \beta(s)]^2.\end{aligned}$$

It follows that

$$\begin{pmatrix} \alpha(t) & \beta(t) + i\tau(a, t) \\ \beta(t) - i\tau(a, t) & \gamma(t) \end{pmatrix}$$

is a nondecreasing function of t, $a \leq t \leq c$. By the definition of $\tau(t)$ we obtain the inequality

$$\tau(a, t) - \tau(a, s) \leq \tau(t) - \tau(s)$$

whenever $a \leq s \leq t \leq c$.

To obtain the reverse inequality, consider any function $h(t)$, $a \leq t \leq c$, such that

$$\tau(s) - \tau(t) \leq h(t) - h(s) \leq \tau(t) - \tau(s)$$

whenever $a \leq s \leq t \leq c$. Then $h(t)$ is a continuous function of t which is of bounded variation in $[a, c]$. Integrating by parts in the integral equation

$$M(a, b, z)I - I = z \int_a^b M(a, t, z)\, dm(t),$$

we obtain the identity

$$z \int_a^b e^{ih(t)z} M(a, t, z)\, dm(t) + iz \int_a^b e^{ih(t)z} M(a, t, z) I\, dh(t) \\ = e^{ih(b)z} M(a, b, z)I - e^{ih(a)z} I.$$

By the proof of Theorem 38, this implies the more general identity

$$(z - \bar{w}) \int_a^b e^{ih(t)z} M(a, t, z)\, d[m(t) + iIh(t)] e^{-ih(t)\bar{w}} \bar{M}(a, t, w) \\ = e^{ih(b)z} M(a, b, z) I e^{-ih(b)\bar{w}} \bar{M}(a, b, w) - e^{ih(a)z} I e^{-ih(a)\bar{w}}.$$

When $z = \bar{w}$ we obtain the inequality

$$[M(a, b, z)I\bar{M}(a, b, z) - S(a, b, z)I\bar{S}(a, b, z)]/(z - \bar{z}) \geq 0$$

with

$$S(a, b, z) = e^{ih(b)z} e^{-ih(a)z},$$

since the integral then represents a nonnegative matrix. (It is defined as a limit of Stieltjes sums, each of which is a nonnegative matrix.) A space $\mathcal{H}_{S(a,b)}(M(a, b))$ therefore exists. This implies that the mean type $h(b) - h(a)$ of $S^*(a, b, z)$ in the upper half-plane does not exceed the mean type $\tau(a, b)$ of $M(a, b, z)$. By the arbitrariness of $h(t)$, $\tau(a, b) \geq \tau(b) - \tau(a)$. Equality holds since the reverse inequality was obtained earlier in the proof.

PROBLEM 130. Let $\mu(x)$ be a nondecreasing function of real x which has $r + 1$ points of increase, $r = 0, 1, 2, \cdots$. Show that the polynomials of degree at most r are a Hilbert space which satisfies the axioms (H1), (H2), and (H3) in the metric of $L^2(\mu)$. Show that the space is a space $\mathcal{H}(E)$ for some polynomial $E(z)$ of degree $r + 1$ which has no real zeros. Show that there exist entire functions $C(z)$ and $D(z)$, which are real for real z, such that

$$A(z)D(z) - B(z)C(z) = 1,$$
$$\operatorname{Re}[A(z)\bar{D}(z) - B(z)\bar{C}(z)] \geq 1$$

for all complex z, $[D(z) + iC(z)]/E(z)$ has no real singularities and

$$\lim_{y \to +\infty} \operatorname{Re} y^{-1}[D(iy) + iC(iy)]/E(iy) = 0.$$

Show that the corresponding space $\mathcal{H}(M)$ has dimension $r + 1$ and that $D(z) + iC(z)$ is a polynomial of degree $r + 1$. Show that there exists a number W of absolute value 1 such that

$$\frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2} = \operatorname{Re} \frac{[D(z) + iC(z)] + [D(z) - iC(z)]W}{[A(z) - iB(z)] - [A(z) + iB(z)]W}$$

for $y > 0$.

PROBLEM 131. Show that a space $\mathcal{L}(\varphi)$ is finite dimensional if, and only if, $\varphi(z)$ can be written as a ratio of polynomials.

PROBLEM 132. If $\mathcal{L}(\varphi)$ is a finite dimensional space and if $a \geq 0$, show that there exists a space $\mathcal{H}(M(a))$ and a function $W(a, z)$, analytic and bounded by 1 for $y > 0$ such that

$$\varphi(z) = \frac{[D(a, z) + iC(a, z)] + [D(a, z) - iC(a, z)]W(a, z)}{[A(a, z) - iB(a, z)] - [A(a, z) + iB(a, z)]W(a, z)}$$

for $y > 0$, $M(a, 0) = 1$, and $B'(a, 0) - C'(a, 0) = a$.

PROBLEM 133. If $\mathcal{L}(\varphi)$ is a given space, show that there exists a sequence $\{\mathcal{L}(\varphi_n)\}$ of finite dimensional spaces such that $\varphi(z) = \lim \varphi_n(z)$ when z is not real. *Hint*: See SSPS Theorem 21.

PROBLEM 134. If $\mathcal{L}(\varphi)$ is a given space and if $a \geq 0$, show that there exists a space $\mathcal{H}(M(a))$ and a function $W(a, z)$, analytic and bounded by 1 for $y > 0$, such that

$$\varphi(z) = \frac{[D(a, z) + iC(a, z)] + [D(a, z) - iC(a, z)]W(a, z)}{[A(a, z) - iB(a, z)] - [A(a, z) + iB(a, z)]W(a, z)}$$

for $y > 0$, $M(a, 0) = 1$, and $B'(a, 0) - C'(a, 0) = a$.

PROBLEM 135. Let $\mathcal{H}(E(a))$ be a given space and let $W(a, z)$ be a function which is analytic and bounded by 1 for $y > 0$. Assume that $W(a, z)$ is not identically 1 and that

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \frac{[D(a, b, z) + iC(a, b, z)] + [D(a, b, z) - iC(a, b, z)]W(b, z)}{[A(a, b, z) - iB(a, b, z)] - [A(a, b, z) + iB(a, b, z)]W(b, z)}$$

where $\mathcal{H}(M(a, b))$ exists and $W(b, z)$ is analytic and bounded by 1 for $y > 0$. If $C(a, z) = -B(a, z)$, $D(a, z) = A(a, z)$, and

$$M(b, z) = M(a, z)M(a, b, z),$$

show that

$$\frac{E(a, z) + E^*(a, z)W(a, z)}{E(a, z) - E^*(a, z)W(a, z)} = \frac{[D(b, z) + iC(b, z)] + [D(b, z) - iC(b, z)]W(b, z)}{[A(b, z) - iB(b, z)] - [A(b, z) + iB(b, z)]W(b, z)}$$

for $y > 0$.

PROBLEM 136. Let $\varphi(z)$ be a function which is analytic and has a non-negative real part in the upper half-plane. Assume that

$$\varphi(z) = \frac{[D(a, b, z) + iC(a, b, z)] + [D(a, b, z) - iC(a, b, z)]W(b, z)}{[A(a, b, z) - iB(a, b, z)] - [A(a, b, z) + iB(a, b, z)]W(b, z)}$$

and that

$$\varphi(z) = \frac{[D(a, c, z) + iC(a, c, z)] + [D(a, c, z) - iC(a, c, z)]W(c, z)}{[A(a, c, z) - iB(a, c, z)] - [A(a, c, z) + iB(a, c, z)]W(c, z)}$$

for $y > 0$ where $\mathcal{H}(M(a, b))$ and $\mathcal{H}(M(a, c))$ exist and where $W(b, z)$ and $W(c, z)$ are analytic and bounded by 1 in the upper half-plane. Show that either

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

for some space $\mathcal{H}(M(b, c))$ or that

$$M(a, b, z) = M(a, c, z)M(c, b, z)$$

for some space $\mathcal{H}(M(c, b))$.

PROBLEM 137. Let $\mu(x)$ be a nondecreasing function of real x, which is not a constant, such that $\int_{-\infty}^{+\infty} (1 + t^2)^{-1} d\mu(t) < \infty$. Show that there exists a space $\mathcal{H}(E)$ contained isometrically in $L^2(\mu)$ such that $E(z)$ is of bounded type in the upper half-plane and has no real zeros.

PROBLEM 138. Let $\mathcal{H}(E(0))$ be a given space such that $E(0, z)$ has no real zeros and let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{H}(E(0))$ is contained isometrically in $L^2(\mu)$. For each number $b \geq 0$ show that there exists a unique space $\mathcal{H}(E(b))$ such that

$$(A(b, z), B(b, z)) = (A(0, z), B(0, z))M(0, b, z)$$

for a space $\mathcal{H}(M(0, b))$ with $M(0, b, 0) = 1$,

$$M'(0, b, 0)I = \begin{pmatrix} \alpha(b) & \beta(b) \\ \beta(b) & \gamma(b) \end{pmatrix} = m(b),$$

and $\alpha(b) + \gamma(b) = b$, and such that there exists a function $W(b, z)$, analytic and bounded by 1 for $y > 0$, and a number $p(b) \geq 0$ such that

$$\operatorname{Re} \frac{E(b, z) + E^*(b, z)W(b, z)}{E(b, z) - E^*(b, z)W(b, z)} = p(b)y + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(b, t)|^2 d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. Show that $m(t)$ is a nondecreasing function of t and that its entries are real valued, continuous functions of t. Show that $E(t, w)$ is a continuous function of $t \geq 0$ for every w and that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

for $0 \leq a < b < \infty$.

PROBLEM 139. Let $\{\mathcal{H}(E(t))\}$ be a family of spaces and let

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

be a nondecreasing, matrix valued function of t, both defined in an interval $s_- < t < s_+$. Assume that the entries of $m(t)$ are continuous, real valued functions of t, that $E(t, w)$ is a continuous function of t for every w, and that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $s_- < a < b < s_+$. Show that

$$[B(b, z)\bar{A}(b, w) - A(b, z)\bar{B}(b, w)] - [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)]$$
$$= (z - \bar{w}) \int_a^b (A(t, z), B(t, z))dm(t)(A(t, w), B(t, w))^-$$

for all complex z and w. *Hint*: See the proof of Theorem 38 for an analogous matrix identity.

PROBLEM 140. Let $\{\mathcal{H}(E_+(t))\}$ and $\{\mathcal{H}(E_-(t))\}$ be families of spaces and let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of t, both defined in an interval $s_- < t < s_+$. Assume that the entries of $m(t)$ are continuous, real valued functions of t, that $E_+(t, w)$ and $E_-(t, w)$ are continuous functions of t for every w, and that

$$(A_+(b, w), B_+(b, w))I - (A_+(a, w), B_+(a, w))I = w \int_a^b (A_+(t, w), B_+(t, w))dm(t),$$

$$(A_-(b, w), B_-(b, w))I - (A_-(a, w), B_-(a, w))I = w \int_a^b (A_-(t, w), B_-(t, w))dm(t)$$

whenever $s_- < a < b < s_+$. Show that

$$[B_+(b, z)\bar{A}_-(b, w) - A_+(b, z)\bar{B}_-(b, w)] - [B_+(a, z)\bar{A}_-(a, w) - A_+(a, z)\bar{B}_-(a, w)] = (z - \bar{w}) \int_a^b (A_+(t, z), B_+(t, z))dm(t)(A_-(t, w), B_-(t, w))^-$$

for all complex z and w. If there is some choice of a such that $E_+(a, z) = E_-(a, z)$ for all complex z, show that $E_+(t, z) = E_-(t, z)$ for all $t, s_- < t < s_+$, and for all complex z. *Hint*: See the proof of Theorem 38 for analogous matrix results.

PROBLEM 141. In Problem 139 let $M(a, t, w)$ be the unique, continuous, matrix valued function of t, $s_- < a \le t < s_+$, such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)dm(t)$$

for $a \le b < s_+$. By Theorem 38 the entries of $M(a, b, z)$ are entire functions of z for any fixed a and b, and a space $\mathcal{H}(M(a, b))$ exists. Show that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

whenever $s_- < a \le b < s_+$. *Hint*: Use the results of Problem 140.

PROBLEM 142. In Problem 139 let $h(t)$ be a real valued function of t such that $m(t) + iIh(t)$ is nondecreasing, $s_- < t < s_+$. Show that $h(t)$ is a continuous function of t which has finite total variation in any interval (a, b), $s_- < a < b < s_+$. Show that

$$\exp[ih(b)z][B(b, z)\bar{A}(b, w) - A(b, z)\bar{B}(b, w)] \exp[-ih(b)\bar{w}] - \exp[ih(a)z][B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] \exp[-ih(a)\bar{w}]$$
$$= (z - \bar{w}) \int_a^b \exp[ih(t)z](A(t, z), B(t, z)) \times d[m(t) + iIh(t)](A(t, w), B(t, w))^- \exp[-ih(t)\bar{w}]$$

for all complex z and w when $s_- < a < b < s_+$.

PROBLEM 143. In Problem 139 let $\tau(t)$ be the largest nondecreasing function of t such that

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) + i\tau(t) \\ \beta(t) - i\tau(t) & \gamma(t) \end{pmatrix}$$

is nondecreasing, $s_- < t < s_+$. Show that the mean type of $E(b, z)/E(a, z)$ in the upper half-plane is $\tau(b) - \tau(a)$ whenever $s_- < a < b < s_+$. *Hint*: See the proof of Theorem 39.

PROBLEM 144. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(M(a, b))$ be spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

Show that the mean type of $E(b, z)/E(a, z)$ in the upper half-plane is equal to the mean type of $M(a, b, z)$.

PROBLEM 145. Let $\mathcal{H}(M(a, b))$, $\mathcal{H}(M(b, c))$, and $\mathcal{H}(M(a, c))$ be spaces such that $M(a, c, z) = M(a, b, z)M(b, c, z)$. Show that the mean type of $M(a, c, z)$ is the sum of the mean types of $M(a, b, z)$ and $M(b, c, z)$.

PROBLEM 146. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(M(a, b))$ be spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

and such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$. Let τ be the mean type of $E(b, z)/E(a, z)$ in the upper half-plane and let h be a given number, $-\tau \leq h \leq \tau$. Let $\mathcal{M}$ be the set of elements $F(z)$ of $\mathcal{H}(E(b))$ such that $e^{ihz}F(z)$ belongs to $\mathcal{H}(E(a))$. Show that $\mathcal{M}$ is a closed subspace of $\mathcal{H}(E(b))$ which satisfies the axioms (H1) and (H2) in the metric of $\mathcal{H}(E(b))$. Show that the transformation $F(z) \to e^{ihz}F(z)$ takes $\mathcal{M}$ isometrically onto $\mathcal{H}(E(a))$. Show that $F(z)/(z - w)$ belongs to $\mathcal{M}$ whenever $F(z)$ belongs to $\mathcal{M}$ and $F(z)/(z - w)$ belongs to $\mathcal{H}(E(b))$.

PROBLEM 147. Let $\mathcal{H}(E(b))$ be a given space and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}(E(b))$ which contains a nonzero element. Assume that $F(z)/(z - w)$ belongs to $\mathcal{M}$ whenever $F(z)$ belongs to $\mathcal{M}$ and $F(z)/(z - w)$ belongs to $\mathcal{H}(E(b))$. Show that $\mathcal{M}$ satisfies the axioms (H1) and (H2) in the metric of $\mathcal{H}(E(b))$. Show that there exists a unique element $L(w, z)$ of $\mathcal{M}$ for every complex number w such that $F(w) = \langle F(t), L(w, t)\rangle$ for every $F(z)$ in $\mathcal{M}$. Show that $L(w, z)$ satisfies the identity of Problem 51 for any nonreal number α. Show that

$$L(w, z) = [B_1(z)\bar{A}_1(w) - A_1(z)\bar{B}_1(w)]/[\pi(z - \bar{w})]$$

for some entire functions $A_1(z)$ and $B_1(z)$ such that $B_1(z)A_1^*(z) = A_1(z)B_1^*(z)$. Show that $[A_1^*(z) - iB_1^*(z)]/[A_1(z) - iB_1(z)]$ is an entire function which has no zeros and which is of bounded type in the upper half-plane. Show that there exists a real number h such that

$$A(a, z) = e^{ihz}A_1(z) \quad \text{and} \quad B(a, z) = e^{ihz}B_1(z)$$

are entire functions which are real for real z. Show that a space $\mathcal{H}(E(a))$ exists, that it is contained isometrically in $\mathcal{H}(E(b))$, and that the transformation $F(z) \to e^{ihz}F(z)$ takes $\mathcal{M}$ isometrically onto $\mathcal{H}(E(a))$. Show that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$. If τ is the mean type of $E(b, z)/E(a, z)$ in the upper half-plane, show that $-\tau \le h \le \tau$.

40. INTEGRAL EQUATION FOR $E(z)$

A fundamental problem is to determine all spaces $\mathcal{H}(E(a))$ contained isometrically in a given space $\mathcal{H}(E(c))$, $E(a, z)$ and $E(c, z)$ having no real zeros. The solution is determined by a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

whose entries are continuous, real valued functions of t defined in a finite or infinite interval (s_-, s_+). A number b is said to be singular with respect to $m(t)$ if it belongs to an interval (a, c) such that $m(a) \ne m(b)$, $m(b) \ne m(c)$, and

$$[\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)] = [\beta(c) - \beta(a)]^2.$$

Otherwise a number b in the interval (s_-, s_+) is said to be regular with respect to $m(t)$.

THEOREM 40. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Then there exists a family $\{\mathcal{H}(E(t))\}$ of spaces, $s_- < t < s_+$, and a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix},$$

whose entries are continuous, real valued functions defined in (s_-, s_+), with these properties:

(1) $E(z) = E(c, z)$ for some number c which is regular with respect to $m(t)$.

(2) $E(t, w)$ is a continuous function of t for every complex number w, and

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $s_- < a < b < s_+$.

(3) $$\lim_{t \nearrow s_+} [\alpha(t) + \gamma(t)] = \infty.$$

(4) $E(a, z)$ has no real zeros and $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$ when a is regular with respect to $m(t)$.

(5) $\lim_{t \searrow s_-} K(t, w, w) = 0$ for all complex w.

Proof of Theorem 40. Let $E(0, z) = E(z)$. Define $E(a, z)$ by Problem 126 in a finite or infinite interval $(s_-, 0)$. Define $E(a, z)$ by Problem 138 for $a > 0$ so that $s_+ = +\infty$. Conditions (1), (2), and (3) are satisfied by construction and by the results of these problems. When $s_- < a < b < s_+$, let

$$M(a, t, w) = \begin{pmatrix} A(a, t, w) & B(a, t, w) \\ C(a, t, w) & D(a, t, w) \end{pmatrix}$$

be the unique, continuous, matrix valued function of $t \geq a$ such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)dm(t)$$

whenever $b \geq a$. By Theorem 38 the entries of $M(a, b, z)$ are entire functions of z for any a and b, and a space $\mathcal{H}(M(a, b))$ exists. By Problem 141,

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for all complex z when $s_- < a \leq b$.

If $b > s_-$ and if $\mathcal{H}(E(b))$ is not contained isometrically in $L^2(\mu)$, then multiplication by z is not densely defined in $\mathcal{H}(E(b))$, by Theorem 32. By the theorem the domain of multiplication by z in $\mathcal{H}(E(b))$ is contained isometrically in $L^2(\mu)$. If $\mathcal{H}(E(b))$ is a one-dimensional space and if $s_- < a < b$, then $\mathcal{H}(E(a))$ is not contained isometrically in $\mathcal{H}(E(b))$ since then it would fill $\mathcal{H}(E(b))$, $\mathcal{H}(M(a, b))$ would contain no nonzero element, and $M(a, b, z)$ would be a constant. This contradicts the construction of $\mathcal{H}(M(a, b))$ such that $B'(a, b, 0) - C'(a, b, 0) = b - a > 0$. Since the inclusion of $\mathcal{H}(E(b))$ in $L^2(\mu)$ does not increase norms, the inclusion of $\mathcal{H}(E(a))$ in $L^2(\mu)$ is not isometric when $s_- < a < b$. If $\mathcal{H}(E(b))$ is not a one-dimensional space, then by Problems 87 and 126 there exists an index b_-, $s_- < b_- < b$, such that $\mathcal{H}(E(b_-))$ is contained isometrically in $\mathcal{H}(E(b))$ and is the closure of the domain of multiplication by z in $\mathcal{H}(E(b))$. The space $\mathcal{H}(E(b_-))$ is also contained isometrically in $L^2(\mu)$, and the space $\mathcal{H}(M(b_-, b))$

is one-dimensional. If $b_- < a < b$, then $\mathcal{H}(E(b_-))$ is contained in $\mathcal{H}(E(a))$, $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$, and these inclusions do not increase norms. Since $\mathcal{H}(E(b_-))$ is contained isometrically in $\mathcal{H}(E(b))$, it is contained isometrically in $\mathcal{H}(E(a))$. An argument earlier in the proof will show that $\mathcal{H}(E(b_-))$ is not equal isometrically to $\mathcal{H}(E(a))$ and that $\mathcal{H}(E(a))$ is not equal isometrically to $\mathcal{H}(E(b))$. It follows that $\mathcal{H}(E(b_-))$ is contained properly in $\mathcal{H}(E(a))$ and that $\mathcal{H}(E(a))$ fills $\mathcal{H}(E(b))$ when $b_- < a < b$. It may be that there is no index $c > b$ such that $\mathcal{H}(E(c))$ is contained isometrically in $L^2(\mu)$. If, on the other hand, such an index exists, then $\mathcal{H}(E(b))$ is not contained isometrically in $\mathcal{H}(E(c))$. By Theorem 34 there exists a nonzero constant $\begin{pmatrix} u \\ v \end{pmatrix}$ in $\mathcal{H}(M(b, c))$ such that $A(b, z)u + B(b, z)v$ belongs to $\mathcal{H}(E(b))$. By Problems 102 and 138, there exists an index b_+, $b < b_+ \leq c$, such that $\mathcal{H}(E(b_+))$ is contained isometrically in $\mathcal{H}(E(c))$ and $\mathcal{H}(M(b, b_+))$ is a one-dimensional space spanned by $\begin{pmatrix} u \\ v \end{pmatrix}$. The form of $M(b, b_+, z)$ is given in Problem 101. If $b < a < b_+$, then it is easily verified that $\begin{pmatrix} u \\ v \end{pmatrix}$ belongs to $\mathcal{H}(M(a, b_+))$ and that

$$\begin{pmatrix} u \\ v \end{pmatrix} = M(b, a, z) \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to $\mathcal{H}(M(b, a))$. Since

$$(A(b, z), B(b, z))M(b, a, z) \begin{pmatrix} u \\ v \end{pmatrix} = A(a, z)u + B(a, z)v$$

belongs to $\mathcal{H}(E(a))$, $\mathcal{H}(E(a))$ is not contained isometrically in $\mathcal{H}(E(b_+))$ when $b < a < b_+$. It follows that $\mathcal{H}(E(a))$ is not contained isometrically in $L^2(\mu)$ when $b < a < b_+$. Thus if $\mathcal{H}(E(b))$ is not contained isometrically in $L^2(\mu)$, we can always find an interval (a, c), $s_- < a < b < c$, such that $\mathcal{H}(E(t))$ is not contained isometrically in $L^2(\mu)$ when $a < t < c$. For any such interval (a, c), $\mathcal{H}(M(a, c))$ is a one-dimensional space. By the form of one-dimensional spaces given in Problem 101,

$$[\alpha(c) - \alpha(a)][\gamma(c) - \gamma(a)] = [\beta(c) - \beta(a)]^2$$

and $m(c) - m(a) \neq 0$. This completes the proof that b is a singular point with respect to $m(t)$ if $\mathcal{H}(E(b))$ is not contained isometrically in $L^2(\mu)$.

To obtain (5) consider first the case in which there exists a smallest regular point b, $s_- < b < s_+$. Then the space $\mathcal{H}(E(b))$ is one-dimensional and the desired limit is obtained by Problem 86. If, on the other hand, there is no smallest regular point, consider the intersection $\mathcal{M}$ of the spaces $\mathcal{H}(E(a))$, a regular. Then $\mathcal{M}$ is a Hilbert space of entire functions which satisfies (H1), (H2), and (H3). If $F(z)$ belongs to $\mathcal{M}$ and has a zero w, then $F(z)/(z - w)$ belongs to $\mathcal{H}(E(a))$ for every regular number a and so belongs to $\mathcal{M}$. From this we can see that $\mathcal{M}$ contains no nonzero element. Otherwise it would be equal to a space $\mathcal{H}(E)$ such that $E(z)$ has no real zeros, and by Theorem 33 this contradicts the construction of Problem 126. If w is any complex number and if $a < b$ are regular points with respect to $m(t)$, $K(a, w, z)$ is the projection of $K(b, w, z)$ in $\mathcal{H}(E(a))$. Since $\mathcal{M}$ contains no nonzero element,

$$\lim_{a \searrow s_-} K(a, w, w) = \lim_{a \searrow s_-} \|K(a, w, t)\|^2 = 0.$$

This obtains (5) in all cases, and the theorem follows.

PROBLEM 148. In Theorem 40 show that $\mathcal{H}(E(a))$ is not contained isometrically in $L^2(\mu)$ when the index a is singular with respect to $m(t)$.

PROBLEM 149. In Theorem 40 let b be a regular point which is not the left end point of an interval of singular points. Show that $\mathcal{H}(E(b))$ is the intersection of the spaces $\mathcal{H}(E(c))$ such that c is regular and $b < c$.

PROBLEM 150. In Theorem 40 let b be a regular point which is not the right end point of an interval of singular points. Show that $\mathcal{H}(E(b))$ is the closed span of the spaces $\mathcal{H}(E(a))$ such that a is regular and $a < b$.

PROBLEM 151. If the regular points have an upper bound in Theorem 40, show that there is a largest regular point b and that $\mathcal{H}(E(b))$ fills $L^2(\mu)$.

PROBLEM 152. If $E(0) = 1$ in Theorem 40, show that $E(t, 0) = 1$ for all indices t. Show that

$$\alpha(b) - \alpha(a) = B'(b, 0) - B'(a, 0)$$

when $a < b$. Show that

$$\alpha(s_-) = \lim_{t \searrow s_-} \alpha(t)$$

exists and is finite.

PROBLEM 153. In Theorem 40 let $\mathcal{H}(E_1)$ be a given space which is contained isometrically in $L^2(\mu)$ such that $E_1(z)/E(z)$ is of bounded type in the upper half-plane and has no real zeros or singularities. Show that $\mathcal{H}(E_1)$ is equal isometrically to $\mathcal{H}(E(a))$ for some regular number a.

PROBLEM 154. Assume that $E(0) = 1$ in Theorem 40. For each index t, let $\varphi(t, x)$ be the phase function associated with $E(t, z)$ which has value 0 at the origin. Show that $\varphi(t, x)$ is a continuous function of t for each fixed x and that

$$\varphi(b, x)/x - \varphi(a, x)/x = \int_a^b \cos^2 \varphi(t, x) d\alpha(t) + 2 \int_a^b \cos \varphi(t, x) \sin \varphi(t, x) d\beta(t) + \int_a^b \sin^2 \varphi(t, x) d\gamma(t)$$

when $s_- < a < b < s_+$. Show that $\varphi(t, x)/x$ is a nonnegative, nondecreasing function of t for each fixed x.

PROBLEM 155. Let $\mathcal{H}(M(a))$, $\mathcal{H}(M(a, b))$, and $\mathcal{H}(M(b))$ be spaces such that $M(b, z) = M(a, z)M(a, b, z)$ and such that $A(a, z)$ and $B(a, z)$ are linearly independent. Show that when z is in the upper half-plane,

$$w \to \frac{[D(c, z) + iC(c, z)] + [D(c, z) - iC(c, z)]w}{[A(c, z) - iB(c, z)] - [A(c, z) + iB(c, z)]w}$$

is a mapping of the unit disk $|w| < 1$ onto the disk $\mathfrak{D}(c, z)$ of center

$$[D(c, z)\bar{A}(c, z) - C(c, z)\bar{B}(c, z)]/[iA(c, z)\bar{B}(c, z) - iB(c, z)\bar{A}(c, z)]$$

and radius

$$1/[iA(c, z)\bar{B}(c, z) - iB(c, z)\bar{A}(c, z)]$$

for $c = a$ and $c = b$, and show that $\mathfrak{D}(a, z)$ contains $\mathfrak{D}(b, z)$.

PROBLEM 156. Let $(E_n(z))$ be a sequence of entire functions of Pólya class such that $E_n(0) = 1$ and $\operatorname{Re} E_n'(0) = 0$ for every n, and such that $\lim \operatorname{Re} iE_n'(0) = 0$ and $\lim \operatorname{Re} E_n''(0) = 0$ as $n \to \infty$. Show that $\lim E_n(z) = 1$ uniformly on bounded sets as $n \to \infty$.

41. SOLUTION OF THE INTEGRAL EQUATION FOR $E(z)$

A fundamental problem is to determine the functions $m(t)$ which appear in the solution of the structure problem for some space $\mathcal{H}(E)$. The condition for $m(t)$ to correspond to a function $E(z)$ of Pólya class is known.

THEOREM 41. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t > 0$ whose entries are continuous, real valued functions of t. Assume that $\alpha(t) > 0$ for $t > 0$, that $\lim \alpha(t) = 0$ as $t \searrow 0$, and that $\int_0^1 \alpha(t) d\gamma(t) < \infty$. Then there exists a unique family $(E(t, z))$ of entire functions of Pólya class, $t > 0$, such that $E(t, w)$ is a continuous function of $t > 0$ for every w,

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $0 < a < b < \infty$, and

$$\lim_{t \searrow 0} E(t, w) \exp [\beta(t)w] = 1.$$

A space $\mathcal{H}(E(a))$ exists for every $a > 0$, $E(a, z)$ has no real zeros, and $E(a, 0) = 1$.

Proof of Theorem 41. For each $a > 0$ let $M(a, t, w)$ be the unique continuous, matrix valued function of $t \geq a$ such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)dm(t)$$

whenever $a < b$. By Theorem 38 the entries of $M(a, b, z)$ are entire functions of z for each fixed a and b and a space $\mathcal{H}(M(a, b))$ exists. When $a < b$ the function $E(a, b, z) = A(a, b, z) - iB(a, b, z)$ is of bounded type in the upper half-plane, it has no zeros on or above the real axis, and $|E(a, b, x - iy| \leq |E(a, b, x + iy)|$ for $y > 0$. By Problem 34 the function is of Pólya class. It has value 1 at the origin,

$$B'(a, b, 0) = \alpha(b) - \alpha(a), \quad A'(a, b, 0) = \beta(a) - \beta(b),$$

and

$$A'(a, b, 0)^2 - A''(a, b, 0) = 2 \int_a^b [\alpha(t) - \alpha(a)]d\gamma(t).$$

If we define

$$S(a, b, z) = E(a, b, z) \exp [\beta(b)z - \beta(a)z],$$

then $S(a, b, z)$ is of Pólya class, $S(a, b, 0) = 1$, $\operatorname{Re} S'(a, b, 0) = 0$,

$$\operatorname{Re} iS'(a, b, 0) = \alpha(b) - \alpha(a),$$

and

$$\operatorname{Re} S''(a, b, 0) = -2 \int_a^b [\alpha(t) - \alpha(a)]d\gamma(t).$$

By hypothesis

$$\int_a^b [\alpha(t) - \alpha(a)]d\gamma(t) \leq \int_0^b \alpha(t)d\gamma(t) < \infty.$$

By Problem 13, the estimate of Problem 10 holds for all functions of Pólya class. It follows that there exists a decreasing sequence (a_n) of positive numbers such that $\lim a_n = 0$ and

$$S(1, z) = \lim_{n \to \infty} S(a_n, 1, z)$$

exists for all complex z. The limit $S(1, z)$ is an entire function of Pólya class which has value 1 at the origin, $S'(1, 0) = \lim S'(a_n, 1, 0)$ and $S''(1, 0) = \lim S''(a_n, 1, 0)$ as $n \to \infty$. Since

$$M(a, c, z) = M(a, b, z)M(b, c, z)$$

when $a < b < c$,

$$S(b, z) = \lim_{n\to\infty} S(a_n, b, z)$$

exists for every $b > 0$. The limit $S(b, z)$ is an entire function of Pólya class, it has value 1 at the origin, $\operatorname{Re} S'(b, 0) = 0$, $\operatorname{Re} iS'(b, 0) = \alpha(b)$, and

$$\operatorname{Re} S''(b, 0) = -2 \int_0^b \alpha(t)d\gamma(t).$$

By Problem 156, $\lim S(b, z) = 1$ uniformly on bounded sets as $b \searrow 0$. Define

$$E(b, z) = S(b, z) \exp [-\beta(b)z].$$

Then $E(b, z)$ is of Pólya class, it has value 1 at the origin, and

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

when $a < b$. It follows that $E(t, w)$ is a continuous function of $t > 0$ for every w and that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $0 < a < b$. This condition implies that if w is a real zero of $E(t, z)$ for any fixed t, then it is a zero of $E(t, z)$ for all t. Since

$$\lim_{t\searrow 0} E(t, w) \exp [\beta(t)w] = 1$$

by construction, $E(t, z)$ has no real zeros for any $t > 0$. Since $B'(a, 0) = \alpha(a) > 0$ when $a > 0$, $B(a, z)$ does not vanish identically. Since $A(a, 0) = 1$ and $B(a, 0) = 0$, $A(a, z)$ and $B(a, z)$ are linearly independent. By Problem 14, a space $\mathcal{H}(E(a))$ exists for every index a.

We prove uniqueness of the family $(E(t, z))$ with these properties. Let $(E_+(t, z))$ and $(E_-(t, z))$ be families of entire functions of Pólya class, $t > 0$, such that $E_+(t, w)$ and $E_-(t, w)$ are continuous functions of t for every w and

$$(A_+(b, w), B_+(b, w))I - (A_+(a, w), B_+(a, w))I = w \int_a^b (A_+(t, w), B_+(t, w))dm(t),$$

$$(A_-(b, w), B_-(b, w))I - (A_-(a, w), B_-(a, w))I = w \int_a^b (A_-(t, w), B_-(t, w))dm(t)$$

whenever $0 < a < b < \infty$. As in Problem 140,

$$[B_+(b, z)\bar{A}_-(b, w) - A_+(b, z)\bar{B}_-(b, w)]/(z - \bar{w})$$
$$- [B_+(a, z)\bar{A}_-(a, w) - A_+(a, z)\bar{B}_-(a, w)]/(z - \bar{w})$$
$$= \int_a^b (A_+(t, z), B_+(t, z))dm(t)(A_-(t, w), B_-(t, w))^-$$

for all complex z and w when $0 < a < b < \infty$. When $z = \bar{w}$, we obtain the identity

$$B_+(b, z)A_-(b, z) - A_+(b, z)B_-(b, z) = B_+(a, z)A_-(a, z) - A_+(a, z)B_-(a, z)$$

for all complex z. If

$$\lim_{t \searrow 0} E_+(t, z) \exp [\beta(t)z] = 1 \quad \text{and} \quad \lim_{t \searrow 0} E_-(t, z) \exp [\beta(t)z] = 1,$$

then

$$\lim_{a \searrow 0} [B_+(a, z)A_-(a, z) - A_+(a, z)B_-(a, z)] = 0$$

when z is on the imaginary axis. Since

$$B_+(b, z)A_-(b, z) - A_+(b, z)B_-(b, z)$$

vanishes on the imaginary axis, it vanishes identically. Since $E_+(b, z)$ and $E_-(b, z)$ can have no real zeros under these conditions,

$$E_-(b, z) = S(b, z)E_+(b, z)$$

for some entire function $S(b, z)$ which is real for real z and which has no zeros. Since

$$(A_+(b, z), B_+(b, z)) = (A_+(a, z), B_+(a, z))M(a, b, z),$$
$$(A_-(b, z), B_-(b, z)) = (A_-(a, z), B_-(a, z))M(a, b, z)$$

when $a < b$, $S(a, z) = S(b, z)$. Since

$$\lim_{t \searrow 0} E_-(t, z)/E_+(t, z) = 1,$$

$S(t, z) = 1$ identically and $E_-(t, z) = E_+(t, z)$ for all t.

PROBLEM 157. Assume that $E(z)$ is of Pólya class and that $E(0) = 1$ in Theorem 40. Show that

$$2 \int_{s-}^{c} [\alpha(t) - \alpha(s_-)]d\gamma(t) \leq A'(0)^2 - A''(0).$$

Hint: Obtain the inequality first for approximating finite dimensional spaces and pass to a limit using Fatou's theorem.

PROBLEM 158. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function whose entries are continuous, real valued functions of t in some interval (s_-, s_+). Assume that there exists a family $(E(t, z))$ of entire functions, which have no real zeros, such that $E(t, w)$ is a continuous function of t for every w and

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $s_- < a < b < s_+$. If a space $\mathcal{H}(E(a))$ exists for every a, $s_- < a < s_+$, show that there exists a family $(W(a, z))$ of functions, analytic and bounded by 1 for $y > 0$, such that

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \frac{[D(a, b, z) + iC(a, b, z)] + [D(a, b, z) - iC(a, b, z)]W(b, z)}{[A(a, b, z) - iB(a, b, z)] - [A(a, b, z) + iB(a, b, z)]W(b, z)}$$

when $s_- < a < b < s_+$. (If $W(a, z)$ is identically 1, the formula is meaningless as written but has an obvious interpretation on solving for $W(a, z)$.) Show that there exists a nondecreasing function $\mu(x)$ of real x such that

$$\operatorname{Re} \frac{E(a, z) + E^*(a, z)W(a, z)}{E(a, z) - E^*(a, z)W(a, z)} = p(a)y + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(a, t)|^2 d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$ and all indices a, where $p(a)$ is a nonnegative constant which depends only on a.

PROBLEM 159. If $E(z)$ is an entire function of Pólya class such that $\mathcal{H}(E)$ exists, show that $K(w, z)$ is of Pólya class when w is in the upper half-plane.

PROBLEM 160. If $E(z)$ is an entire function of Pólya class such that $\mathcal{H}(E)$ exists, show that $K(x + iy, x + iy)$ is a nondecreasing function of $y > 0$ for each fixed x.

PROBLEM 161. If $\lim [\alpha(t) + \gamma(t)] = \infty$ as $t \nearrow s_+$ in Problem 158 and if there exists an index c such that the interval (c, s_+) contains only singular points, show that

$$\lim_{t \nearrow s_+} [B(t, z)\bar{A}(t, z) - A(t, z)\bar{B}(t, z)]/(z - \bar{z}) = \infty$$

for all nonreal z. Show that the functions $W(a, z)$ are unique and are given by

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \lim_{b \nearrow s_+} \frac{D(a, b, z) + iC(a, b, z)}{A(a, b, z) - iB(a, b, z)}$$

for $y > 0$, $s_- < a < s_+$. Show that the union of the spaces $\mathcal{H}(E(a))$, a regular, is dense in $L^2(\mu)$.

42. MEASURES DETERMINED BY INTEGRAL EQUATIONS

The same conclusion holds also when there is no such interval of singular points.

THEOREM 42. If $\lim [\alpha(t) + \gamma(t)] = \infty$ as $t \nearrow \infty$ in Theorem 41 and if the set of regular points does not have an upper bound, then

$$\lim_{t \nearrow \infty} [B(t, z)\bar{A}(t, z) - A(t, z)\bar{B}(t, z)]/(z - \bar{z}) = \infty$$

for all nonreal z. There exists an essentially unique nondecreasing function $\mu(x)$ such that $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$ whenever a is regular with respect to $m(t)$.

Essential uniqueness means that any other nondecreasing function $\nu(x)$ with the same property is related in such a way that $\nu(b) - \nu(a) = \mu(b) - \mu(a)$ whenever a and b are points of continuity of $\mu(x)$.

Proof of Theorem 42. The existence of at least one nondecreasing function $\mu(x)$ with these properties follows from Theorem 40 and Problem 158. We use the choice of such a function $\mu(x)$ in showing that

$$\pi T(w) = \lim_{t \to \infty} [B(t, w)\bar{A}(t, w) - A(t, w)\bar{B}(t, w)]/(w - \bar{w})$$

cannot be finite for any nonreal number w. Let $\mathcal{M}$ be the union of the spaces $\mathcal{H}(E(a))$, a regular, and note that

$$T(w) = \sup |F(w)|$$

where the supremum is taken over all elements $F(z)$ of $\mathcal{M}$ such that $\|F(t)\| \leq 1$ in $L^2(\mu)$. If $F(z)$ and $G(z)$ belong to $\mathcal{M}$, then

$$[F(z)G(\beta) - G(z)F(\beta)]/(z - \beta)$$

belongs to $\mathcal{M}$ for every complex number β. If $\|F(t)\| \leq 1$ and $\|G(t)\| \leq 1$, then

$$\begin{aligned} |[F(\alpha)G(\beta) - G(\alpha)F(\beta)]/(\alpha - \beta)| &\leq T(\alpha) \, \|[F(t)G(\beta) - G(t)F(\beta)]/(t - \beta)\| \\ &\leq 2T(\alpha)[|F(\beta)| + |G(\beta)|]/|\beta - \bar{\beta}|. \end{aligned}$$

If $T(\alpha)$ is finite and if α is not real, consider any number β such that $|\beta - \alpha| < \frac{1}{2} |\beta - \bar{\beta}|$ and choose $G(z)$ so that

$$|G(\alpha)| > T(\alpha)2 \, |\beta - \alpha|/|\beta - \bar{\beta}|.$$

It follows that

$$|F(\beta)|\,[|G(\alpha)| - T(\alpha)2|\beta - \alpha|/|\beta - \bar{\beta}|] \le T(\alpha)|G(\beta)| + T(\alpha)|G(\beta)|2|\beta - \alpha|/|\beta - \bar{\beta}|.$$

By the arbitrariness of $F(z)$,

$$T(\beta)[\,|G(\alpha)| - T(\alpha)2|\beta - \alpha|/|\beta - \bar{\beta}|] \le T(\alpha)|G(\beta)| + T(\alpha)|G(\beta)|2|\beta - \alpha|/|\beta - \bar{\beta}|.$$

It follows that $T(\beta) < \infty$. On iterating this procedure we obtain the finiteness of $T(w)$ for all numbers w on the same side of the real axis as α. Since $T(\bar{w}) = T(w)$, $T(w)$ is finite for all nonreal numbers w. The above estimate shows that $T(w)$ remains bounded in a neighborhood of any nonreal point. By Problem 160, $K(a, x + iy, x + iy)$ is a nondecreasing function of $y > 0$ for each real x and each index a. By the arbitrariness of a, $T(x + iy)$ is a nondecreasing function of $y > 0$ for each real x. It now follows that $T(w)$ is finite for all complex w and that it remains bounded on any bounded set.

If $(F_n(z))$ is a Cauchy sequence in the metric of $\mathcal{M}$, the inequality

$$|F_n(z) - F_r(z)| \le T(z)\ \|F_n(t) - F_r(t)\|$$

shows that $(F_r(w))$ is a Cauchy sequence of numbers for every complex number w. A limit function $F(z)$ therefore exists. It is entire since $T(z)$ is bounded on bounded sets. Let $\mathcal{H}$ be the set of entire functions $F(z)$ which are obtained as limits $F(z) = \lim F_n(z)$ for some Cauchy sequence of elements of $\mathcal{M}$. Since

$$|F_n(z)| \le T(z)\ \|F_n(t)\|$$

for every n, the inequality

$$|F(z)| \le T(z)\ \|F(t)\|$$

holds for all complex z if $F(z)$ is in $\mathcal{H}$. It is clear that $\mathcal{H}$ is a vector space over the complex numbers and that it has a well-defined inner product inherited from $L^2(\mu)$. If $(F_n(z))$ is a Cauchy sequence of elements of $\mathcal{H}$, then for every n there exists an element $G_n(z)$ of $\mathcal{M}$ such that $\|F_n(t) - G_n(t)\| \le 1/n$. The sequence $(G_n(z))$ is then a Cauchy sequence of elements of $\mathcal{M}$. By the construction of $\mathcal{H}$ there exists an element $F(z)$ of $\mathcal{H}$ such that $F(z) = \lim G_n(z)$ for all complex z. Since $\lim \|F(t) - G_n(t)\| = 0$ and $\lim \|F_n(t) - G_n(t)\| = 0$ as $n \to \infty$, $\lim \|F(t) - F_n(t)\| = 0$ as $n \to \infty$. Thus every Cauchy sequence of elements of $\mathcal{H}$ converges to an element of $\mathcal{H}$, and $\mathcal{H}$ is a Hilbert space.

We show that the space $\mathcal{H}$ satisfies the axiom (H1). If $F(z)$ is in $\mathcal{H}$ and has a nonreal zero w, there exists a sequence $(F_n(z))$ of elements of $\mathcal{M}$ such that $F(z) = \lim F_n(z)$ in the metric of $\mathcal{H}$. If $L(z)$ is the choice of an element of $\mathcal{M}$ which has value 1 at w, then

$$G_n(z) = F_n(z) - F_n(w)L(z)$$

belongs to $\mathcal{M}$ for every n, $G_n(w) = 0$, and $F(z) = \lim G_n(z)$ in the metric of $\mathcal{H}$. It follows that $(G_n(z))$ is a Cauchy sequence in the metric of $\mathcal{H}$. This implies that $(G_n(z)(z - \bar{w})/(z - w))$ is a Cauchy sequence in the metric of $\mathcal{H}$. Since

$$F(z)(z - \bar{w})/(z - w) = \lim G_n(z)(z - \bar{w})/(z - w),$$

it belongs to $\mathcal{H}$. It clearly has the same norm as $F(z)$ and (H1) follows.

The axiom (H2) follows from the inequality

$$|F(w)| \leq T(w) \, \|F(t)\|,$$

which holds for every $F(z)$ in $\mathcal{H}$. The axiom (H3) has an obvious proof. By Theorem 23 the space $\mathcal{H}$ is equal isometrically to a space $\mathcal{H}(E(\infty))$. By construction $\mathcal{H}(E(\infty))$ contains $\mathcal{H}(E(a))$ isometrically for every regular number a. By Theorem 33 there exists a space $\mathcal{H}(M(a, \infty))$ such that

$$(A(\infty, z), B(\infty, z)) = (A(a, z), B(a, z))M(a, \infty, z).$$

When a and b are regular and $a < b$, then there exist spaces $\mathcal{H}(M(a, b))$ and $\mathcal{H}(M(b, \infty))$ such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$
$$(A(\infty, z), B(\infty, z)) = (A(b, z), B(b, z))M(b, \infty, z).$$

By Problem 100 we can conclude that

$$M(a, \infty, z) = M(a, b, z)M(b, \infty, z).$$

It follows that

$$[M(a, b, w)I\bar{M}(a, b, w) - I]/(w - \bar{w}) \leq [M(a, \infty, w)I\bar{M}(a, \infty, w) - I]/(w - \bar{w})$$

for all complex w. Thus the matrices on the left have an upper bound as $b \nearrow \infty$ through regular points, for each fixed a and w. When $w = 0$ this implies that the matrices

$$m(b) - m(a) = \begin{pmatrix} \alpha(b) - \alpha(a) & \beta(b) - \beta(a) \\ \beta(b) - \beta(a) & \gamma(b) - \gamma(a) \end{pmatrix}$$

have an upper bound as $b \nearrow \infty$ through regular points. This is contrary to the hypothesis that $\lim [\alpha(t) + \gamma(t)] = \infty$ as $t \nearrow \infty$ through regular points. We must therefore grant that $T(w) = \infty$ for every nonreal number w.

To prove uniqueness of $\mu(x)$, consider any nondecreasing function $\nu(x)$ of real x such that $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\nu)$ whenever a is regular with respect to $m(t)$. If a is regular, then by Theorems 30 and 31 there exists a function $W(a, z)$, analytic and bounded by 1 in the upper half-plane, such that

$$\operatorname{Re} \frac{E(a, z) + E^*(a, z) W(a, z)}{E(a, z) - E^*(a, z) W(a, z)} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(a, t)|^2 d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. By Theorem 32 and Problem 135,

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \frac{[D(a, b, z) + iC(a, b, z)] + [D(a, b, z) - iC(a, b, z)] W(b, z)}{[A(a, b, z) - iB(a, b, z)] - [A(a, b, z) + iB(a, b, z)] W(b, z)}$$

for $y > 0$ when a and b are regular points, $a < b$. By the first part of the proof,

$$\lim_{b \nearrow \infty} [B(a, b, z)\bar{A}(a, b, z) - A(a, b, z)\bar{B}(a, b, z)]/(z - \bar{z}) = \infty$$

when z is in the upper half-plane. By Problem 155,

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \lim_{b \nearrow \infty} \frac{D(a, b, z) + iC(a, b, z)}{A(a, b, z) - iB(a, b, z)}.$$

Since a similar analysis applies with $\mu(x)$ replaced by $\nu(x)$,

$$\frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(a, t)|^2 d\mu(t)}{(t - x)^2 + y^2} = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{|E(a, t)|^2 d\nu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. By the Stieltjes inversion formula,

$$\int_u^v |E(a, t)|^2 d\mu(t) = \int_u^v |E(a, t)|^2 d\nu(t)$$

whenever u and v are points of continuity of $\mu(x)$. It follows that $\nu(b) - \nu(a) = \mu(b) - \mu(a)$ whenever a and b are points of continuity of $\mu(x)$.

PROBLEM 162. If $\mathcal{H}(E)$ is a given space and if w is a given point above the real axis, show that there exists a linear function $E_1(z)$ such that $\mathcal{H}(E_1)$ exists, $A_1(w) = A(w)$, and $B_1(w) = B(w)$.

PROBLEM 163. If $\lim [\alpha(t) + \gamma(t)] = \infty$ as $t \nearrow s_+$ in Problem 158, show that

$$\lim_{t \nearrow s_+} [B(t, z)\bar{A}(t, z) - A(t, z)\bar{B}(t, z)]/(z - \bar{z}) = \infty$$

for all nonreal z. If there is no index c such that the interval (c, s_+) contains only singular points, show that there exists an essentially unique nondecreasing function $\mu(x)$ of real x such that $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$ when a is regular with respect to $m(t)$. Show that the union of the spaces $\mathcal{H}(E(a))$, a regular, is dense in $L^2(\mu)$.

PROBLEM 164. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t > 0$ whose entries are continuous, real valued functions of t. Assume that $\alpha(t) > 0$ for $t > 0$ and that $\lim \alpha(t) = 0$ as $t \searrow 0$. Assume that $(E_+(t, z))$ and $(E_-(t, z))$ are given families of entire functions, which have no real zeros and which have value 1 at the origin, such that spaces $\mathcal{H}(E_+(t))$ and $\mathcal{H}(E_-(t))$ exist for every $t > 0$, $E_+(t, w)$ and $E_-(t, w)$ are continuous functions of t for every w, and

$$(A_+(b, w), B_+(b, w))I - (A_+(a, w), B_+(a, w))I = w \int_a^b (A_+(t, w), B_+(t, w))dm(t),$$

$$(A_-(b, w), B_-(b, w))I - (A_-(a, w), B_-(a, w))I = w \int_a^b (A_-(t, w), B_-(t, w))dm(t)$$

for $0 < a < b < \infty$, and such that $\pi K_+(a, 0, 0) = \alpha(a) = \pi K_-(a, 0, 0)$ for $a > 0$. If

$$P(t, z) = \begin{pmatrix} A_+(t, z) & B_+(t, z) \\ A_-(t, z) & B_-(t, z) \end{pmatrix},$$

show that

$$P(b, z)I\bar{P}(b, w) - P(a, z)I\bar{P}(a, w) = (z - \bar{w}) \int_a^b P(t, z)dm(t)\bar{P}(t, w)$$

whenever $0 < a < b < \infty$ and that

$$\lim_{t \searrow 0} P(t, z)I\bar{P}(t, w) = T_+(z)I\bar{T}_-(w)$$

for some entire functions $T_+(z)$ and $T_-(z)$ which are real for real z. Show that $T_+(z)T_-(z)$ vanishes at the origin, and use this fact to show that it vanishes identically. Show that there exists an entire function $S(z)$, which is real for real z and which has no zeros, such that $E_-(t, z) = S(z)E_+(t, z)$ for all $t > 0$.

PROBLEM 165. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t > 0$ whose entries are absolutely continuous, real valued functions of t. Let $\sigma(t)$ be a nondecreasing, continuous function with respect to which the entries of $m(t)$ are absolutely continuous. Let $\tau(t)$ be a largest nondecreasing function such that $m(t) - iI\tau(t)$ is nondecreasing. Construct Borel measurable functions $u(t)$ and $v(t)$ of $t \geq 0$ such that

$$\alpha(b) - \alpha(a) = \int_a^b u(t)\bar{u}(t)d\sigma(t),$$

$$\beta(b) - \beta(a) = \operatorname{Re} \int_a^b u(t)\bar{v}(t)d\sigma(t),$$

$$\gamma(b) - \gamma(a) = \int_a^b v(t)\bar{v}(t)d\sigma(t),$$

$$\tau(b) - \tau(a) = \operatorname{Re} \int_a^b iu(t)\bar{v}(t)d\sigma(t)$$

when $0 < a < b < \infty$. Show that these functions can be chosen so that $u(t)$ has real values and so that the values of $v(t)$ are real in the set where $u(t) = 0$. Show that an interval (a, b) contains only singular points with respect to $m(t)$ if, and only if, $u(t)$ and $v(t)$ are equivalent to linearly dependent, real valued functions in (a, b).

43. COMPLETENESS OF $L^2(m)$

The solution of the structure problem for a given space $\mathcal{H}(E)$ is determined by knowledge of a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

whose entries are continuous, real valued functions of t in an interval. The structure space $L^2(m)$ associated with $m(t)$ is constructed from the set of all pairs $(f(t), g(t))$ of Borel measurable functions of t, defined in the interval, which are constant in each interval of singular points, such that

$$\|(f, g)\|_m^2 = \int (f(t), g(t))dm(t)(f(t), g(t))^- < \infty.$$

In giving the meaning of this last integral, we assume for definiteness that the interval of parametrization is $(0, \infty)$. The definition of the integral uses the choice of a nondecreasing, continuous function $\sigma(t)$ with respect to which $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are absolutely continuous. For example, $\sigma(t) = \alpha(t) + \gamma(t)$ has this property. If $u(t)$ and $v(t)$ are constructed for $m(t)$ as in Problem 165, the definition of the m-integral is

$$\|(f, g)\|_m^2 = \tfrac{1}{2} \int_0^\infty |f(t)u(t) + g(t)v(t)|^2 d\sigma(t) + \tfrac{1}{2} \int_0^\infty |f(t)u(t) + g(t)\bar{v}(t)|^2 d\sigma(t).$$

It is easily verified that this definition does not depend on the choice of $\sigma(t)$, $u(t)$, and $v(t)$. In the same way, we define the inner product

$$\langle (f, g), (p, q)\rangle_m = \tfrac{1}{2}\int_0^\infty [f(t)u(t) + g(t)v(t)][\bar{p}(t)u(t) + \bar{q}(t)\bar{v}(t)]d\sigma(t) + \tfrac{1}{2}\int_0^\infty [f(t)u(t) + g(t)\bar{v}(t)][\bar{p}(t)u(t) + \bar{q}(t)v(t)]d\sigma(t)$$

when $\|(f, g)\|_m < \infty$ and $\|(p, q)\|_m < \infty$. Two pairs of functions (f, g) and (p, q) are identified with respect to $m(t)$ if $\|(f, g) - (p, q)\|_m = 0$. The space of equivalence classes so obtained is a vector space with a well-defined inner product. The definition of $L^2(m)$ uses only those pairs $(f(t), g(t))$, $\|(f, g)\|_m < \infty$, which are equivalent to a constant in any interval (a, b) of singular points. By this we mean that for any interval (a, b) of singular points, there exists a pair (x, y) of numbers such that

$$\int_a^b (x - f(t), y - g(t))dm(t)(x - f(t), y - g(t))^- = 0.$$

Thus the space $L^2(m)$ is finite dimensional if there exist only a finite number of regular points. We show that the space is complete.

THEOREM 43. If $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ is a nondecreasing, matrix valued function whose entries are continuous, real valued functions of t defined in an interval, then the corresponding space $L^2(m)$ is a Hilbert space.

Proof of Theorem 43. Explicit proof is restricted to the special case in which $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are absolutely continuous functions of t. But the argument has an obvious generalization to the case in which $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are absolutely continuous with respect to a nondecreasing, continuous function $\sigma(t)$. For convenience we assume that the interval of parametrization is the half-line $(0, \infty)$. The general case is reducible to this one by a change of variable. Let $u(t)$ and $v(t)$ be Borel measurable functions of $t > 0$ such that

$$\alpha(b) - \alpha(a) = \int_a^b u(t)\bar{u}(t)dt,$$

$$\beta(b) - \beta(a) = \operatorname{Re}\int_a^b u(t)\bar{v}(t)dt,$$

$$\gamma(b) - \gamma(a) = \int_a^b v(t)\bar{v}(t)dt,$$

$$\tau(b) - \tau(a) = \operatorname{Re} i\int_a^b u(t)\bar{v}(t)dt$$

when $0 < a < b < \infty$, where $\tau(t)$ is a largest nondecreasing function of t such that $m(t) - iI\tau(t)$ is nondecreasing. Choose them so that $u(t)$ is real valued and so that $v(t)$ has real values in the set where $u(t) = 0$. If $(f(t), g(t))$

belongs to $L^2(m)$ and if $h(t)$ is the function of real t defined by

$$h(t) = f(t)u(t) + g(t)v(t),$$
$$h(-t) = f(t)u(t) + g(t)\bar{v}(t)$$

for $t > 0$, then $h(t)$ belongs to $L^2(-\infty, +\infty)$ and

$$\|(f, g)\|_m^2 = \tfrac{1}{2}\int_{-\infty}^{+\infty} |h(t)|^2 dt.$$

Such a function $h(t)$ clearly has these properties:

(1) $h(-t) = h(t)$ almost everywhere in the subset of $(0, \infty)$ where $v(t) = \bar{v}(t)$.

(2) $h(t) = 0$ almost everywhere in the subset of $(0, \infty)$ where $u(t)$ and $v(t)$ both vanish.

(3) $h(t)$ depends linearly on $u(t)$ and $v(t)$ in any interval (a, b), $0 < a < b < \infty$, which contains only singular points with respect to $m(t)$.

Conversely, if $h(t)$ is an element of $L^2(-\infty, +\infty)$ which satisfies these three conditions, define $f(t)$ and $g(t)$ by

$$f(t) = [h(-t)v(t) - h(t)\bar{v}(t)]/[u(t)v(t) - u(t)\bar{v}(t)],$$
$$g(t) = [h(t) - h(-t)]/[v(t) - \bar{v}(t)]$$

in the set where $v(t) \neq \bar{v}(t)$, by

$$f(t) = h(t)u(t)/[u(t)^2 + v(t)^2],$$
$$g(t) = h(t)v(t)/[u(t)^2 + v(t)^2]$$

in the set where $v(t) = \bar{v}(t)$ and $u(t)^2 + v(t)^2 \neq 0$, and by $f(t) = g(t) = 0$ in the set where $u(t) = v(t) = 0$. It is easily verified that $(f(t), g(t))$ is an element of $L^2(m)$ such that the corresponding element of $L^2(-\infty, +\infty)$ is $h(t)$. Since the set of elements of $L^2(-\infty, +\infty)$ which satisfy conditions (1), (2), and (3) is closed in $L^2(-\infty, +\infty)$, it is a Hilbert space in the metric of $L^2(-\infty, +\infty)$. Since there exists a linear isometric transformation of $L^2(m)$ onto this closed subspace of $L^2(-\infty, +\infty)$, $L^2(m)$ is a Hilbert space.

44. EXPANSION THEOREM FOR SPACES $\mathcal{H}(E)$

A space $\mathcal{H}(E)$ can be recovered from its structure space $L^2(m)$ by an isometric expansion. In describing the transformation, we use the function $\chi(a, t)$ which is equal to 1 when $t < a$ and which is equal to 0 when $t > a$.

THEOREM 44. Let $\{\mathcal{H}(E(t))\}$ be a family of spaces, $s_- < t < s_+$, and let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing function whose entries are continuous, real valued functions defined in (s_-, s_+). Assume that $\alpha(t) > 0$ for

$t > s_-$, that $\lim \alpha(t) = 0$ as $t \searrow s_-$, that each function $E(a, z)$ has no real zeros and has value 1 at the origin, that $E(t, w)$ is a continuous function of t for each complex number w, and that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $s_- < a < b < s_+$. Then $\chi(c, t)(A(t, w), B(t, w))$ belongs to $L^2(m)$ as a function of t for every regular number c and every complex number w. For each element $(f(t), g(t))$ of $L^2(m)$ which vanishes outside of (s_-, c), define a corresponding function $F(z)$ by

$$\pi F(w) = \int_{s_-}^{s_+} (f(t), g(t))dm(t)(A(t, \bar{w}), B(t, \bar{w}))^-$$

for all complex w. Then $F(z)$ is an entire function, it belongs to $\mathcal{H}(E(c))$, and

$$\pi \int_{-\infty}^{+\infty} |F(t)/E(c, t)|^2 dt = \int_{s_-}^{s_+} (f(t), g(t))dm(t)(f(t), g(t))^-.$$

If $G(z)$ is in $\mathcal{H}(E(c))$, then $G(z)$ is equal to $F(z)$ for some such choice of $(f(t), g(t))$ in $L^2(m)$.

Proof of Theorem 44. If (a, b) is an interval of singular points, then

$$(A(t, z), B(t, z)) = (A(a, z), B(a, z))M(a, t, z)$$

for $a < t < b$, where

$$M(a, t, z) = \begin{pmatrix} 1 - [\beta(t) - \beta(a)]z & [\alpha(t) - \alpha(a)]z \\ -[\gamma(t) - \gamma(a)]z & 1 + [\beta(t) - \beta(a)]z \end{pmatrix}.$$

It follows that

$$A(t, z) = A(a, z) - A(a, z)[\beta(t) - \beta(a)]z - B(a, z)[\gamma(t) - \gamma(a)]z,$$

$$B(t, z) = B(a, z) + A(a, z)[\alpha(t) - \alpha(a)]z + B(a, z)[\beta(t) - \beta(a)]z.$$

If $h(t) = [\alpha(t) + \gamma(t)] - [\alpha(a) + \gamma(a)]$, then

$$\alpha(t) - \alpha(a) = ph(t), \qquad \beta(t) - \beta(a) = qh(t), \qquad \gamma(t) - \gamma(a) = rh(t)$$

for $a < t < b$, where p, q, and r are real numbers such that $p \geq 0$, $r \geq 0$, and $pr = q^2$. It follows that

$$\int_a^b (\beta(a) - \beta(t), \alpha(t) - \alpha(a))dm(t)(\beta(a) - \beta(t), \alpha(t) - \alpha(a))^-$$
$$= (-q \quad p)\begin{pmatrix} p & q \\ q & r \end{pmatrix}(-q \quad p)^- \int_a^b h(t)^2 dh(t) = 0,$$

$$\int_a^b (\gamma(a) - \gamma(t), \beta(t) - \beta(a))dm(t)(\gamma(a) - \gamma(t), \beta(t) - \beta(a))^-$$
$$= (-r \quad q)\begin{pmatrix} p & q \\ q & r \end{pmatrix}(-r \quad q)^- \int_a^b h(t)^2 dh(t) = 0.$$

So $(A(t, z), B(t, z))$ is equivalent to a constant in any interval of singular points. By Problem 139,

$$\pi K(b, w, z) - \pi K(a, w, z) = \int_a^b (A(t, z), B(t, z))dm(t)(A(t, w), B(t, w))^-$$

whenever $s_- < a < b < s_+$. Since we assume that $\alpha(t) > 0$ for $t > 0$ and that $\lim \alpha(t) = 0$ as $t \searrow s_-$, we can conclude that $\lim K(t, w, z) = 0$ for all complex z and w as $t \searrow s_-$, by the proof of Theorem 40. It follows that

$$\pi K(c, w, z) = \int_{s_-}^c (A(t, z), B(t, z))dm(t)(A(t, w), B(t, w))^-$$

for all complex z and w if $s_- < c < s_+$. So $\chi(c, t)(A(t, w), B(t, w))$ belongs to $L^2(m)$ for every complex number w if c is regular with respect to $m(t)$. Its transform $K(c, \bar{w}, z)$ is an entire function which belongs to $\mathcal{H}(E(c))$, and the identity

$$\pi \int_{-\infty}^{+\infty} K(c, \bar{\alpha}, t)\bar{K}(c, \bar{\beta}, t) \, |E(c, t)|^{-2}dt = \int_{s_-}^c (A(t, \alpha), B(t, \alpha))dm(t)(A(t, \beta), B(t, \beta))^-$$

holds for all complex numbers α and β. Therefore if $(f(t), g(t))$ is a finite linear combination of elements $\chi(c, t)(A(t, w), B(t, w))$ of $L^2(m)$, the corresponding entire function $F(z)$ belongs to $\mathcal{H}(E(c))$ and

$$\pi \int_{-\infty}^{+\infty} |F(t)/E(c, t)|^2dt = \int_{s_-}^c (f(t), g(t))dm(t)(f(t), g(t))^-.$$

The same conclusion follows by continuity if $(f(t), g(t))$ belongs to the closed span $\mathcal{M}(c)$ of the functions $\chi(c, t)(A(t, w), B(t, w))$ in $L^2(m)$. The set of elements of $\mathcal{H}(E(c))$ which are transforms of elements of $\mathcal{M}(c)$ is a closed subspace of $\mathcal{H}(E(c))$ which contains $K(c, w, z)$ for all complex numbers w. It is therefore the full space, and every element of $\mathcal{H}(E(c))$ is the eigentransform of an element of $\mathcal{M}(c)$.

We show that $\mathcal{M}(a)$ is contained in $\mathcal{M}(b)$ when a and b are regular points, $a < b$. Let $(f(t), g(t))$ be in $\mathcal{M}(a)$ and let $(f_1(t), g_1(t))$ be its projection in $\mathcal{M}(b)$. If $F(z)$ is the eigentransform of $(f(t), g(t))$, then

$$\pi F(w) = \int (f(t), g(t))dm(t)(A(t, \bar{w}), B(t, \bar{w}))^-$$
$$= \int (f_1(t), g_1(t))dm(t)(A(t, \bar{w}), B(t, \bar{w}))^-$$

for all complex w. By what we have already shown,

$$\pi \int |F(t)/E(a, t)|^2dt = \int (f(t), g(t))dm(t)(f(t), g(t))^-,$$

$$\pi \int |F(t)/E(b, t)|^2dt = \int (f_1(t), g_1(t))dm(t)(f_1(t), g_1(t))^-.$$

Since $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ by Theorem 40,

$$\int (f(t), g(t))dm(t)(f(t), g(t))^- = \int (f_1(t), g_1(t))dm(t)(f_1(t), g_1(t))^-.$$

Since $(f_1(t), g_1(t))$ is the projection of $(f(t), g(t))$ in $\mathcal{M}(b)$, it follows that $(f(t), g(t)) = (f_1(t), g_1(t))$ in $L^2(m)$. This completes the proof that $\mathcal{M}(a)$ is contained in $\mathcal{M}(b)$.

If a and b are regular points, if $a < b$, and if $(f(t), g(t))$ is in $\mathcal{M}(b)$, then its projection in $\mathcal{M}(a)$ is $\chi(a, t)(f(t), g(t))$. This is clear if

$$(f(t), g(t)) = \chi(b, t)(A(t, w), B(t, w))$$

for some complex number w. The general case follows by linearity and continuity.

If c is a regular point, we show that $\mathcal{M}(c)$ contains every element of $L^2(m)$ which vanishes in (c, s_+). It is sufficient to show that there is no nonzero element $(f(t), g(t))$ of $L^2(m)$ which vanishes in (c, s_+) and which is orthogonal to $\mathcal{M}(c)$. In this case $\chi(a, t)(f(t), g(t))$ is orthogonal to $\mathcal{M}(c)$ for every regular number $a < c$. Therefore

$$\int_a^b (f(t), g(t))dm(t)(A(t, w), B(t, w))^- = 0$$

for all complex numbers w if a and b are regular numbers and if $a < b$. If (a, b) is an interval of singular points, $(f(t), g(t))$ and $(A(t, w), B(t, w))$ are equivalent to constants in (a, b). It follows that the same formula holds also when a or b is a singular point. Let $\sigma(t) = \alpha(t) + \gamma(t)$ and $p(t)$, $q(t)$, and $r(t)$ be the Borel measurable functions such that

$$\alpha(b) - \alpha(a) = \int_a^b p(t)d\sigma(t),$$

$$\beta(b) - \beta(a) = \int_a^b q(t)d\sigma(t),$$

$$\gamma(b) - \gamma(a) = \int_a^b r(t)d\sigma(t)$$

whenever $s_- < a < b < s_+$. Then

$$\int_a^b (f(t), g(t))\begin{pmatrix} p(t) & q(t) \\ q(t) & r(t) \end{pmatrix}(A(t, w), B(t, w))^- d\sigma(t) = 0$$

whenever $s_- < a < b < s_+$. It follows that

$$(f(t), g(t))\begin{pmatrix} p(t) & q(t) \\ q(t) & r(t) \end{pmatrix}(A(t, w), B(t, w))^- = 0$$

except in a set of zero σ-measure. But for each fixed t,

$$(A(t, w), B(t, w)) \quad \text{and} \quad (A(t, \bar{w}), B(t, \bar{w}))$$

are linearly independent row vectors if w is a nonreal number. It follows that

$$(f(t), g(t))\begin{pmatrix} p(t) & q(t) \\ q(t) & r(t) \end{pmatrix} = 0$$

except in a set of zero σ-measure. So

$$(f(t), g(t))\begin{pmatrix} p(t) & q(t) \\ q(t) & r(t) \end{pmatrix}(f(t), g(t))^- = 0$$

except in a set of zero σ-measure. This implies that

$$\int_{s_-}^{s_+} (f(t), g(t))dm(t)(f(t), g(t))^- = 0.$$

So $\mathcal{M}(c)$ contains every element of $L^2(m)$ which vanishes in (c, s_+). The theorem follows.

PROBLEM 166. In Theorem 44 let $(f(t), g(t))$ be an element of $L^2(m)$ which vanishes in (c, s_+) and let $F(z)$ be the corresponding element of $\mathcal{H}(E(c))$. Show that $(\bar{f}(t), \bar{g}(t))$ belongs to $L^2(m)$ and that $F^*(z)$ is the corresponding element of $\mathcal{H}(E(c))$.

45. EXPANSIONS AND INTEGRAL TRANSFORMATIONS

We now relate the expansion of Theorem 44 to an integral transformation in $L^2(m)$.

THEOREM 45. In Theorem 44 let $(f_2(t), g_2(t))$ be an element of $L^2(m)$ which vanishes in (c, s_+). If

$$\int_{s_-}^{s_+} (f_2(t), g_2(t))dm(t)(1, 0)^- = 0,$$

then there exists an element $(f_1(t), g_1(t))$ of $L^2(m)$, which vanishes in (c, s_+), such that

$$(f_1(b), g_1(b))I - (f_1(a), g_1(a))I = \int_a^b (f_2(t), g_2(t))dm(t)$$

whenever $s_- < a < b < s_+$. If $F_1(z)$ and $F_2(z)$ are elements of $\mathcal{H}(E(c))$ such that

$$\pi F_k(w) = \int_{s_-}^{s_+} (f_k(t), g_k(t))dm(t)(A(t, \bar{w}), B(t, \bar{w}))^-$$

for all complex w, then $F_2(z) = zF_1(z)$.

Proof of Theorem 45. If $(f_2(t), g_2(t))$ is a given element of $L^2(m)$ which vanishes in (c, s_+) and if $\int_{s_-}^{s_+} (f_2(t), g_2(t))dm(t)(1, 0)^- = 0$, then the corresponding entire function $F_2(z)$ has a zero at the origin. Since $E(c, z)$ has a nonzero value at the origin by hypothesis, there is an element $F_1(z)$ of $\mathcal{H}(E(c))$ such that $F_2(z) = zF_1(z)$. By Theorem 44 there exists an element $(f_1(t), g_1(t))$ of $L^2(m)$ which vanishes in (c, s_+), such that $F_1(z)$ is the corresponding element of $\mathcal{H}(E(c))$. To prove the theorem we must show that the identity

$$(f_1(b), g_1(b))I - (f_1(a), g_1(a))I = \int_a^b (f_2(t), g_2(t))dm(t)$$

holds whenever $s_- < a < b < s_+$. Since the transformation

$$(f_2(t), g_2(t)) \to (f_1(t), g_1(t))$$

so defined is continuous, it is sufficient to obtain the identity for special choices of $(f_2(t), g_2(t))$ whose span is dense in the set of elements of $L^2(m)$ which satisfy the hypotheses of the theorem. By Problem 51 the function

$$L(c, w, z) = 2\pi i(\bar{w} - z)K(c, w, z)$$

satisfies the identity

$$L(c, \alpha, \alpha)L(c, w, z) = L(c, \alpha, z)L(c, w, \alpha) - L(c, \bar{\alpha}, z)L(c, w, \bar{\alpha})$$

for every nonreal number α. It follows that

$$\begin{aligned} z(\alpha - \bar{\alpha})K(c, \alpha, \alpha)K(c, w, z) &- z(\alpha - \bar{w})K(c, w, \alpha)K(c, \alpha, z) \\ &+ z(\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})K(c, \bar{\alpha}, z) \\ = \bar{w}(\alpha - \bar{\alpha})K(c, \alpha, \alpha)K(c, w, z) &- \bar{\alpha}(\alpha - \bar{w})K(c, w, \alpha)K(c, \alpha, z) \\ &+ \alpha(\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})K(c, \bar{\alpha}, z). \end{aligned}$$

If w is held fixed and if

$$\begin{aligned} (f_2(t), g_2(t)) = {} & \bar{w}(\alpha - \bar{\alpha})K(c, \alpha, \alpha)\chi(c, t)(A(t, \bar{w}), B(t, \bar{w})) \\ & - \bar{\alpha}(\alpha - \bar{w})K(c, w, \alpha)\chi(c, t)(A(t, \bar{\alpha}), B(t, \bar{\alpha})) \\ & + \alpha(\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})\chi(c, t)(A(t, \alpha), B(t, \alpha)), \end{aligned}$$

then

$$\begin{aligned} F_2(z) &= \bar{w}(\alpha - \bar{\alpha})K(c, \alpha, \alpha)K(c, w, z) \\ &\quad - \bar{\alpha}(\alpha - \bar{w})K(c, w, \alpha)K(c, \alpha, z) \\ &\quad + \alpha(\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})K(c, \bar{\alpha}, z), \\ F_1(z) &= (\alpha - \bar{\alpha})K(c, \alpha, \alpha)K(c, w, z) \\ &\quad - (\alpha - \bar{w})K(c, w, \alpha)K(c, \alpha, z) \\ &\quad + (\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})K(c, \bar{\alpha}, z), \end{aligned}$$

and

$$\begin{aligned} (f_1(t), g_1(t)) &= (\alpha - \bar{\alpha})K(c, \alpha, \alpha)\chi(c, t)(A(t, \bar{w}), B(t, \bar{w})) \\ &\quad - (\alpha - \bar{w})K(c, w, \alpha)\chi(c, t)(A(t, \bar{\alpha}), B(t, \bar{\alpha})) \\ &\quad + (\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})\chi(c, t)(A(t, \alpha), B(t, \alpha)). \end{aligned}$$

The required identity now follows from the hypothesis that

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $s_- < a < b < s_+$. To complete the proof of the theorem, we must show that the closed span of the functions

$$\bar{w}(\alpha - \bar{\alpha})K(c, \alpha, \alpha)K(c, w, z) - \bar{\alpha}(\alpha - \bar{w})K(c, w, \alpha)K(c, \alpha, z) + \alpha(\bar{\alpha} - \bar{w})K(c, w, \bar{\alpha})K(c, \bar{\alpha}, z)$$

in $\mathcal{H}(E(c))$ contains every function which vanishes at the origin. For this we must show that there is no nonzero element $F_2(z)$ of $\mathcal{H}(E(c))$ which vanishes at the origin and which is orthogonal to such special functions. In this case

$$w(\bar{\alpha} - \alpha)\bar{K}(c, \alpha, \alpha)F_2(w) - \alpha(\bar{\alpha} - w)\bar{K}(c, w, \alpha)F_2(\alpha) + \bar{\alpha}(\alpha - w)\bar{K}(c, w, \bar{\alpha})F_2(\bar{\alpha}) = 0$$

for all complex w. Since $zF_2(z)$ is then a linear combination of $A(c, z)$ and $B(c, z)$, $F_2(z)$ is a constant multiple of $K(c, 0, z)$. Since $F_2(z)$ vanishes at the origin, it vanishes identically. The theorem follows.

PROBLEM 167. Let $\{\mathcal{H}(E_n)\}$ be a sequence of spaces and let $\varphi_n(x)$ be a phase function associated with $E_n(z)$ for every n. Assume that $\varphi(x) = \lim \varphi_n(x)$ exists as a finite limit as $n \to \infty$ and that $\varphi(x+) - \varphi(x-) < \pi$ for all real x. If $\varphi(x)$ is not a constant, show that there exists a space $\mathcal{H}(E)$ such that $\varphi(x)$ is a phase function associated with $E(z)$. *Hint:* See Problem 89.

PROBLEM 168. In Problem 154 show that $\lim \varphi(a, x)/x = 0$ for all real x as $a \searrow s_-$. *Hint:* See Problem 93. When $x = 0$, $\varphi(a, x)/x$ is equal to $\alpha(a) - \alpha(s_-)$.

PROBLEM 169. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t \geq 0$ whose entries are continuous, real valued functions of t. Let $M(t, w)$ be the unique, continuous, matrix valued function of t for every w such that

$$M(a, w)I - I = w \int_0^a M(t, w)dm(t)$$

for $a \geq 0$. For each $n = 1, 2, 3, \cdots$, let $m_n(t) = \begin{pmatrix} \alpha_n(t) & \beta_n(t) \\ \beta_n(t) & \gamma_n(t) \end{pmatrix}$ be the matrix valued function of $t \geq 0$ defined by

$$\alpha_n(t) = \alpha(t) + t/n, \qquad \beta_n(t) = \beta(t), \qquad \gamma_n(t) = \gamma(t) + t/n.$$

Let $M_n(t, w)$ be the unique, continuous, matrix valued function of t for every w such that

$$M_n(a, w)I - I = w \int_0^a M_n(t, w)dm_n(t)$$

for $a \geq 0$. Show that

$$M(a, z) = \lim_{n \to \infty} M_n(a, z)$$

for $a > 0$, uniformly for z in any bounded set. Let $\varphi(a, x)$ be the phase function associated with $A(a, z) - iB(a, z)$ which is zero at the origin. Let $\varphi_n(a, x)$ be the phase function associated with $A_n(a, z) - iB_n(a, z)$ which is zero at the origin. Show that

$$\varphi(a, x) = \lim_{n \to \infty} \varphi_n(a, x)$$

for $a \geq 0$ and all real x.

PROBLEM 170. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t \geq 0$ whose entries are continuous, real valued functions of t. Let $M(t, w)$ be the unique, continuous, matrix valued function of t for every w such that

$$M(a, w)I - I = w \int_0^a M(t, w)dm(t)$$

for $a \geq 0$. For each $n = 1, 2, 3, \cdots$, let $m_n(t) = \begin{pmatrix} \alpha_n(t) & \beta_n(t) \\ \beta_n(t) & \gamma_n(t) \end{pmatrix}$ be the unique, matrix valued function of t which is linear in each interval

$$[k/n - 1/n, k/n], \qquad k = 1, 2, 3, \cdots,$$

and which agrees with $m(t)$ at the points k/n, $k = 0, 1, 2, \cdots$. Let $M_n(t, w)$ be the unique, continuous, matrix valued function of t for every w such that

$$M_n(a, w)I - I = w \int_0^a M_n(t, w)dm_n(t)$$

for $a \geq 0$. Show that

$$M(a, z) = \lim_{n \to \infty} M_n(a, z)$$

for $a \geq 0$, uniformly for z in any bounded set. Let $\varphi(a, x)$ be the phase function associated with $A(a, z) - iB(a, z)$ which is zero at the origin. Let $\varphi_n(a, x)$ be the phase function associated with $A_n(a, z) - iB_n(a, z)$ which is zero at the origin. Show that

$$\varphi(a, x) = \lim_{n \to \infty} \varphi_n(a, x)$$

for $a \geq 0$ and for all real x.

PROBLEM 171. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t > 0$ whose entries are continuous, real valued functions of $t > 0$. Assume that $\lim \alpha(t) > -\infty$ and $\lim \gamma(t) > -\infty$ as $t \searrow 0$. Show that $\lim \beta(t)$ exists as $t \searrow 0$ and that the limit is finite.

46. AN ESTIMATE OF PHASE FUNCTIONS

An estimate of phase functions can be given from a knowledge of $m(t)$.

THEOREM 46. For $k = 1, 2$, let $(E_k(t, z))$ be a family of entire functions, $t > 0$, and let $m_k(t) = \begin{pmatrix} \alpha_k(t) & \beta_k(t) \\ \beta_k(t) & \gamma_k(t) \end{pmatrix}$ be a nondecreasing, matrix valued function whose entries are continuous, real valued functions of $t > 0$. Assume that each function $E_k(a, z)$ has no zeros on or above the real axis, that it satisfies the inequality $|E_k(a, x - iy)| \leq |E_k(a, x + iy)|$ for $y > 0$, and that it has value 1 at the origin. Assume that $E_k(t, w)$ is a continuous function of t for every w, that

$$(A_k(b, w), B_k(b, w))I - (A_k(a, w), B_k(a, w))I = w \int_a^b (A_k(t, w), B_k(t, w))\, dm_k(t)$$

whenever $0 < a < b < \infty$, and that

$$\lim_{t \searrow 0} [B_k(t, w)\bar{A}_k(t, w) - A_k(t, w)\bar{B}_k(t, w)]/(w - \bar{w}) = 0$$

for all complex w. Let $\varphi_k(a, x)$ be the phase function associated with $E_k(a, z)$ which is zero at the origin. If

$$m_1(b) - m_1(a) \leq m_2(b) - m_2(a)$$

whenever $0 < a < b < \infty$, then

$$\varphi_1(a, x)/x \leq \varphi_2(a, x)/x$$

for all real x when $a > 0$.

Proof of Theorem 46. Consider first the special case in which each function $m_k(t)$ is piecewise linear and the nonnegative matrix $[m_2(b) - m_2(a)] - [m_1(b) - m_1(a)]$ has no nonzero kernel when $a < b$. If (a, b) is a common interval of linearity for $m_1(t)$ and $m_2(t)$ and if we can always deduce that $\varphi_1(b, x)/x \leq \varphi_2(b, x)/x$ from the hypothesis that $\varphi_1(a, x)/x \leq \varphi_2(a, x)/x$, then the desired inequality can be obtained by induction from the trivial inequality $\varphi_1(0, x)/x \leq \varphi_2(0, x)/x$. To see that this conclusion is valid, note that $\varphi_k(t, x)$ is a differentiable function of t in (a, b) for each fixed x by Problem 154 and that

$$\begin{aligned}\varphi_k'(t, x)/x = \alpha_k'(t) \cos^2 \varphi_k(t, x) \\ + 2\beta_k'(t) \cos \varphi_k(t, x) \sin \varphi_k(t, x) \\ + \gamma_k'(t) \sin^2 \varphi_k(t, x)\end{aligned}$$

where $\alpha_k'(t)$, $\beta_k'(t)$, and $\gamma_k'(t)$ are constant in (a, b). Then $\varphi_2(t, x)/x - \varphi_1(t, x)/x$ has a continuous derivative which is strictly positive at all zeros. If this difference is nonnegative for $t = a$, it must remain so for $t = b$. The desired inequality follows in this case. The general case in which $m_1(t)$ and $m_2(t)$ are piecewise linear follows on applying Problem 169 to $m_2(t)$. More generally, if $\lim \gamma_2(t) > -\infty$ as $t \searrow 0$, then $\lim \gamma_1(t) > -\infty$ as $t \searrow 0$, and $\lim \beta_1(t)$ and $\lim \beta_2(t)$ exist as finite limits as $t \searrow 0$ by Problem 171. The desired inequality follows in this case on applying Problem 170 to $m_1(t)$ and $m_2(t)$. In the remainder of the proof we assume that $\lim \gamma_2(t) = -\infty$ as $t \searrow 0$.

If $m_1(t)$ is constant in some interval $(0, c)$, then $B_1(t, z) = 0$ and $\varphi_1(t, x) = 0$ for $0 < t \leq c$. Since $\varphi_2(t, x)/x \geq 0$ for all t and x, $\varphi_2(t, x)/x \geq \varphi_1(t, x)/x$ for $0 < t \leq c$. The inequality now follows for $t > c$ by an argument used earlier in the proof. In what follows we assume that $m_1(t)$ is not constant in any neighborhood of the origin.

If $a \geq 0$ let $M_1(a, t, w)$ be the unique, continuous, matrix valued function of $t \geq a$ for every w such that

$$M_1(a, b, w)I - I = w \int_a^b M_1(a, t, w)dm_1(t)$$

when $b > a$. Define $M_1(a, b, w) = 1$ for $0 < b < a$, and let $\varphi_1(a, b, x)$ be the phase function associated with $A_1(a, b, z) - iB_1(a, b, z)$ which is zero at the origin. By what we have already shown, $\varphi_1(a, b, x)/x \leq \varphi_2(b, x)/x$ for all b and x when $a > 0$. The theorem follows as soon as we show that

$$\varphi_1(b, x)/x = \lim_{a \searrow 0} \varphi_1(a, b, x)/x.$$

Since we work only with the first family of entire functions in obtaining this formula, we can drop the subscript 1 from the notation.

From the first part of the proof we know that $\varphi(a, b, x)/x \leq \varphi(b, x)/x$ and that $\varphi(a, b, x)/x$ increases as a decreases for each fixed x. Therefore $\psi(b, x) = \lim \varphi(a, b, x)$ exists as $a \searrow 0$. When $0 < a < b < c$,

$$\begin{aligned}\tan \varphi(a, c, x) &= B(a, c, x)/A(a, c, x) \\ &= \frac{A(a, b, x)B(b, c, x) + B(a, b, x)D(b, c, x)}{A(a, b, x)A(b, c, x) + B(a, b, x)C(b, c, x)} \\ &= \frac{B(b, c, x) + \tan \varphi(a, b, x)D(b, c, x)}{A(b, c, x) + \tan \varphi(a, b, x)C(b, c, x)}\,.\end{aligned}$$

In the limit as $a \searrow 0$ we obtain

$$\tan \psi(c, x) = \frac{B(b, c, x) + \tan \psi(b, x)D(b, c, x)}{A(b, c, x) + \tan \psi(b, x)C(b, c, x)}\,.$$

It follows that $\psi(t, x)$ is a continuous function of $t > 0$ for each fixed x. Since $\psi(t, x)/x \leq \varphi(t, x)/x$ and since $\lim \varphi(t, x)/x = 0$ as $t \searrow 0$ by Problem 168, $\lim \psi(t, x)/x = 0$ as $t \searrow 0$. We show that $\psi(t, x+) - \psi(t, x-) < \pi$ for all t and x. Since $\psi(t, x+) - \psi(t, x-)$ is a continuous function of $t > 0$ for each fixed x and since it has limit 0 as $t \searrow 0$, it is sufficient to show that $\psi(t, x+) - \psi(t, x-) \neq \pi$ for all indices t. If $\psi(c, x+) - \psi(c, x-) = \pi$ for some index c, then $\tan \psi(c, x+) = \tan \psi(c, x-)$. It follows that $\tan \psi(t, x+) = \tan \psi(t, x-)$ for every index t and that $\psi(t, x+) - \psi(t, x-) \equiv 0$ modulo π. Since $\psi(t, x+)$ and $\psi(t, x-)$ are continuous functions of t, $\psi(t, x+) - \psi(t, x-) = \pi$ for all indices t. This is impossible because $\lim \psi(t, x+) = 0$ and $\lim \psi(t, x-) = 0$ as $t \searrow 0$. So we can conclude that $\psi(t, x+) - \psi(t, x-) < \pi$ for all t and x.

If $\psi(b, x)$ vanishes identically for some index b, then $\varphi(a, b, x)$ vanishes identically and $\alpha(a) = \alpha(b)$ for $0 < a < b$. In this case $B(b, z)$ and $\varphi(b, x)$ vanish identically, and the desired limit follows trivially. If $\psi(b, x)$ does not vanish identically, then by Problem 167 there exists a space $\mathcal{H}(E_0(b))$ such that $\psi(b, x)$ is a phase function associated with $E_0(b, z)$. Since $\psi(b, 0) = 0$ we can choose $E_0(b, z)$ so as to have no real zeros and so as to have value 1 at the origin. Since

$$\tan \psi(b, x) = \frac{B(a, b, x) + \tan \psi(a, x)D(a, b, x)}{A(a, b, x) + \tan \psi(a, x)C(a, b, x)}$$

when $a < b$. These spaces can be chosen so that

$$(A_0(b, z), B_0(b, z)) = (A_0(a, z), B_0(a, z))M(a, b, z)$$

when $0 < a < b$. It follows that $\psi(b, x)/x - \psi(a, x)/x = \alpha(b) - \alpha(a)$ when $x = 0$ and $0 < a < b$. Since $\lim \psi(a, x)/x = 0$ and $\lim \alpha(a) = 0$ as $a \searrow 0$, $\alpha(b) = \psi(b, x)/x$ when $x = 0$ and so $\alpha(b) = \pi K(b, 0, 0)$. By Problem 164,

$E_0(t, z) = S(z)E(t, z)$ for some entire function $S(z)$ which is real for real z and which has no zeros. This implies that $\psi(t, x) = \varphi(t, x)$, and the desired inequality follows in all cases.

PROBLEM 172. Let $(E_2(t, z))$ be a family of entire functions, $t > 0$, and let $m_2(t) = \begin{pmatrix} \alpha_2(t) & \beta_2(t) \\ \beta_2(t) & \gamma_2(t) \end{pmatrix}$ be a nondecreasing, matrix valued function whose entries are continuous, real valued functions of $t > 0$. Assume that each function $E_2(a, z)$ has no zeros on or above the real axis, that it satisfies the inequality $|E_2(a, x - iy)| \leq |E_2(a, x + iy)|$ for $y > 0$, and that it has value 1 at the origin. Assume that $E_2(t, w)$ is a continuous function of t for every w, that

$$(A_2(b, w), B_2(b, w))I - (A_2(a, w), B_2(a, w))I = w \int_a^b (A_2(t, w), B_2(t, w))dm_2(t)$$

whenever $0 < a < b < \infty$, and that

$$\lim_{t \searrow 0} [B_2(t, w)\bar{A}_2(t, w) - A_2(t, w)\bar{B}_2(t, w)]/(w - \bar{w}) = 0$$

for all complex w. Let $m_1(t)$ be a nondecreasing, matrix valued function whose entries are continuous, real valued functions of $t > 0$, such that

$$m_1(b) - m_1(a) \leq m_2(b) - m_2(a)$$

whenever $0 < a < b < \infty$. Show that there exists a family $(E_1(t, z))$ of entire functions, $t > 0$, such that each function $E_1(a, z)$ has no zeros on or above the real axis, satisfies the inequality $|E_1(a, x - iy)| \leq |E_1(a, x + iy)|$ for $y > 0$, and has value 1 at the origin, such that $E_1(t, w)$ is a continuous function of t for every w, such that

$$(A_1(b, w), B_1(b, w))I - (A_1(a, w), B_1(a, w))I = w \int_a^b (A_1(t, w), B_1(t, w))dm_1(t)$$

whenever $0 < a < b < \infty$, and such that

$$\lim_{t \searrow 0} [B_1(t, w)\bar{A}_1(t, w) - A_1(t, w)\bar{B}_1(t, w)]/(w - \bar{w}) = 0$$

for all complex w.

PROBLEM 173. Let P be a 2×2-matrix having real entries and determinant -1. If $P^2 = 1$, show that $P = Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}Q^{-1}$ for some 2×2-matrix Q having real entries and determinant 1.

CHAPTER 3

Special Spaces

47. SYMMETRY IN SPACES $\mathcal{H}(E)$

A simple example of a space $\mathcal{H}(E(a))$ is obtained with $E(a, z) = \exp(-iaz)$ for any number $a > 0$. The space $\mathcal{H}(E(a))$ is the Paley-Wiener space of entire functions of exponential type at most a which are square integrable on the real axis. These spaces are contained isometrically in $L^2(-\infty, +\infty)$, and $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ when $a > b$. The integral equation is

$$(A(a, z), B(a, z))I - (1, 0)I = z \int_0^a (A(t, z), B(t, z))dm(t)$$

with

$$m(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

All points $t > 0$ are regular with respect to $m(t)$. The space $L^2(m)$ is the space of pairs $(f(t), g(t))$ of elements of $L^2(0, \infty)$ in the norm

$$\|(f(t), g(t))\|_m^2 = \int_0^\infty |f(t)|^2 dt + \int_0^\infty |g(t)|^2 dt.$$

If $(f(t), g(t))$ belongs to $L^2(m)$ and vanishes outside of $(0, a)$, then

$$\pi F(z) = \int_0^\infty f(t) \cos(tz)dt + \int_0^\infty g(t) \sin(tz)dt$$

belongs to $\mathcal{H}(E(a))$ and

$$\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_0^\infty |f(t)|^2 dt + \int_0^\infty |g(t)|^2 dt.$$

Every element of $\mathcal{H}(E(a))$ is of this form. In this case the expansion theorem for Hilbert spaces of entire functions is equivalent to the Fourier transformation. The usual form of the Fourier integral is obtained on extending

$f(x)$ so as to be an even function of real x and extending $g(x)$ so as to be an odd function of real x. The expansion then reads

$$2\pi F(z) = \int_{-\infty}^{+\infty} [f(t) + ig(t)]e^{-itz}dt$$

and

$$2\pi \int_{-\infty}^{+\infty} |F(t)|^2 dt = \int_{-\infty}^{+\infty} |f(t) + ig(t)|^2 dt.$$

The Paley-Wiener spaces have remarkable special properties. A first important property of the Paley-Wiener spaces is symmetry about the origin. By this we mean that the function $F(-z)$ belongs to the space whenever $F(z)$ belongs to the space and that it always has the same norm as $F(z)$. Any space $\mathcal{H}(E)$ such that $E^*(z) = E(-z)$ has the same symmetry property, as is easily verified from the definition of the space. Note that the condition $E^*(z) = E(-z)$ is equivalent to the pair of conditions $A(z) = A(-z)$ and $B(z) = -B(-z)$. These are satisfied when $E(z) = e^{-iaz}$, since then $A(z) = \cos(az)$ is an even function of z and $B(z) = \sin(az)$ is an odd function of z.

THEOREM 47. Let $\mathcal{H}$ be a Hilbert space of entire functions which satisfies (H1), (H2), and (H3) and which contains a nonzero element. A necessary and sufficient condition that $\mathcal{H}$ be symmetric about the origin is that it be equal isometrically to a space $\mathcal{H}(E)$ such that $E^*(z) = E(-z)$. If $\mathcal{H}$ contains an element which has a nonzero value at the origin, then $E(z)$ can be chosen so that $E(0) = 1$.

Proof of Theorem 47. By Theorem 23 the space $\mathcal{H}$ is equal isometrically to a space $\mathcal{H}(E_1)$. If $E_2(z) = E_1^*(-z)$, then a space $\mathcal{H}(E_2)$ exists and $F(z) \to F(-z)$ is an isometric transformation of $\mathcal{H}(E_1)$ onto $\mathcal{H}(E_2)$, as is easily verified from the definitions of the spaces. Since we assume that $\mathcal{H}$ is symmetric about the origin, $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ are isometrically equal. It follows that

$$(A_2(z), B_2(z)) = (A_1(z), B_1(z))P$$

where P is a 2×2-matrix having real entries and determinant 1. Replacing z by $-z$ we obtain

$$\begin{aligned}(A_1(z), B_1(z))\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} &= (A_2(z), B_2(z))\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}P\\ &= (A_1(z), B_1(z))P\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}P.\end{aligned}$$

Since $A_1(z)$ and $B_1(z)$ are linearly independent,

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} = P\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}P.$$

By Problem 173,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P = Q \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q^{-1}$$

where Q is a 2×2-matrix having real entries and determinant 1. If we define $(A(z), B(z)) = (A_2(z), B_2(z))Q$, then $A(z)$ and $B(z)$ are entire functions which are real for real z, a space $\mathcal{H}(E)$ exists, and it is equal isometrically to $\mathcal{H}$. Since $A(z) = A(-z)$ and $B(z) = -B(-z)$, we obtain the required condition $E^*(z) = E(-z)$. If $\mathcal{H}$ contains an element which has a nonzero value at the origin, then $E(z)$ has a nonzero value at the origin. Since $B(z)$ is an odd function of z, it is zero at the origin. It follows that $A(z)$ has a nonzero value at the origin. If $A_0(z) = A(z)/A(0)$ and $B_0(z) = A(0)B(z)$, then a space $\mathcal{H}(E_0)$ exists, it is equal isometrically to $\mathcal{H}$, $E_0^*(z) = E_0(-z)$, and $E_0(0) = 1$.

PROBLEM 174. Let $\mathcal{H}(E_1)$ and $\mathcal{H}(E_2)$ be spaces which are isometrically equal. Show that $E_1(z) = E_2(z)$ if $E_k^*(z) = E_k(-z)$ and if $E_k(0) = 1$ for $k = 1, 2$.

PROBLEM 175. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $E^*(-z)/E(z)$ is of bounded type in the upper half-plane. Let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. If $\mu(x)$ is an odd function of x, show that $\mathcal{H}(E)$ is symmetric about the origin. *Hint:* Use Theorem 26.

PROBLEM 176. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and such that $E(a, z)$ and $E(b, z)$ have no real zeros. Show that $\mathcal{H}(E(a))$ is symmetric about the origin if $\mathcal{H}(E(b))$ is symmetric about the origin.

PROBLEM 177. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$ such that $M(a, b, 0) = 1$. Let

$$\begin{aligned} A_1(a, z) &= A(a, -z), & B_1(a, z) &= -B(a, -z), \\ A_1(a, b, z) &= A(a, b, -z), & B_1(a, b, z) &= -B(a, b, -z), \\ C_1(a, b, z) &= -C(a, b, -z), & D_1(a, b, z) &= D(a, b, -z). \end{aligned}$$

If $E^*(b, z) = E(b, -z)$, show that spaces $\mathcal{H}(E_1(a))$ and $\mathcal{H}(M_1(a, b))$ exist and that

$$(A(b, z), B(b, z)) = (A_1(a, z), B_1(a, z))M_1(a, b, z).$$

Show that $E_1(a, z) = E^*(a, -z) = E(a, z)$ and that $M_1(a, b, z) = M(a, b, z)$.

PROBLEM 178. If $E^*(z) = E(-z)$ and if $\mu(x) = -\mu(-x)$ in Theorem 40, show that $\beta(t)$ is a constant and that $E^*(a, z) = E(a, -z)$ for all indices a.

PROBLEM 179. If $\beta(t)$ is a constant in Theorem 40, show that $E^*(-z) = S(z)E(z)$ for some entire function $S(z)$ which is real for real z and has no zeros. Show that $S(z)S^*(-z) = 1$.

PROBLEM 180. If $\beta(t) = 0$ in Theorem 41, show that $E^*(a, z) = E(a, -z)$ for $a > 0$.

PROBLEM 181. If $\beta(t) = 0$ in Theorem 42, show that $\mu(x)$ can be chosen so that $\mu(x) = -\mu(-x)$.

PROBLEM 182. Let $\mathcal{H}(E)$ be a given space such that $E^*(z) = E(-z)$. Show that there exists a unique Hilbert space $\mathcal{H}_+$ of entire functions such that $F(z) \to F(z^2)$ is an isometric transformation of $\mathcal{H}_+$ onto the even elements of $\mathcal{H}(E)$. Show that the space $\mathcal{H}_+$ satisfies the axioms (H1), (H2), and (H3). If $\mathcal{H}_+$ contains a nonzero element and if γ is a real number, show that $\mathcal{H}_+$ is equal isometrically to a space $\mathcal{H}(E_+)$ such that

$$A(z) + \gamma z B(z) = A_+(z^2) \quad \text{and} \quad zB(z) = B_+(z^2).$$

PROBLEM 183. Let $\mathcal{H}(E)$ be a given space such that $E^*(z) = E(-z)$. Show that there exists a unique Hilbert space $\mathcal{H}_-$ of entire functions such that $F(z) \to zF(z^2)$ is an isometric transformation of $\mathcal{H}_-$ onto the odd elements of $\mathcal{H}(E)$. Show that the space $\mathcal{H}_-$ satisfies the axioms (H1), (H2), and (H3). If $\mathcal{H}_-$ contains a nonzero element and if α is a given real number, show that $\mathcal{H}_-$ is equal isometrically to a space $\mathcal{H}(E_-)$ such that

$$A(z) = A_-(z^2) \quad \text{and} \quad B(z)/z - \alpha A(z) = B_-(z^2).$$

PROBLEM 184. Assume that $E^*(z) = E(-z)$ and that $\mu(x) = -\mu(-x)$ in Theorem 40, so that $\beta(t)$ is a constant and $E^*(a, z) = E(a, -z)$ for all indices a. For each index a, let $A_+(a, z)$ and $B_+(a, z)$ be the unique entire functions such that

$$A(a, z) + \gamma(a) z B(a, z) = A_+(a, z^2) \quad \text{and} \quad zB(a, z) = B_+(a, z^2)$$

for all complex z. Let

$$\alpha_+(a) = \alpha(a), \qquad \beta_+(a) = -\int^a \gamma(t)\, d\alpha(t), \qquad \gamma_+(a) = \int^a \gamma(t)^2 d\alpha(t).$$

Show that $m_+(t) = \begin{pmatrix} \alpha_+(t) & \beta_+(t) \\ \beta_+(t) & \gamma_+(t) \end{pmatrix}$ is a nondecreasing, matrix valued function of t, that $E_+(t, w)$ is a continuous function of t for every w, and that

$$(A_+(b, w), B_+(b, w))I - (A_+(a, w), B_+(a, w))I = w \int_a^b (A_+(t, w), B_+(t, w))dm_+(t)$$

whenever $s_- < a < b < s_+$.

PROBLEM 185. Assume that $E^*(z) = E(-z)$ and that $\mu(x) = -\mu(-x)$ in Theorem 40, so that $\beta(t)$ is a constant and $E^*(a, z) = E(a, -z)$ for all indices a. For each index a, let $A_-(a, z)$ and $B_-(a, z)$ be the unique entire functions such that

$$A(a, z) = A_-(a, z^2) \quad \text{and} \quad B(a, z)/z - \alpha(a)A(a, z) = B_-(a, z^2)$$

for all complex z. Let

$$\alpha_-(a) = \int^a \alpha(t)^2 d\gamma(t), \qquad \beta_-(a) = \int^a \alpha(t)d\gamma(t), \qquad \gamma_-(a) = \gamma(a).$$

Show that $m_-(t) = \begin{pmatrix} \alpha_-(t) & \beta_-(t) \\ \beta_-(t) & \gamma_-(t) \end{pmatrix}$ is a nondecreasing, matrix valued function of t, that $E_-(t, w)$ is a continuous function of t for every w, and that

$$(A_-(b, w), B_-(b, w))I - (A_-(a, w), B_-(a, w))I = w \int_a^b (A_-(t, w), B_-(t, w))dm_-(t)$$

whenever $s_- < a < b < s_+$.

48. PERIODIC SPACES AND SUBSPACES

Another fundamental property of Paley-Wiener spaces is periodicity. A space $\mathcal{H}(E)$ is said to be periodic of period h, $h > 0$, if $F(z) \to F(z - h)$ is an isometric transformation of the space onto itself. This property is hereditary in subspaces.

THEOREM 48. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. Then $\mathcal{H}(E(a))$ is periodic of period h if $\mathcal{H}(E(b))$ is periodic of period h.

LEMMA 10. Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. If $F(z)$ has no zeros in the half-plane, then

$$\lim_{y \to +\infty} |F(h + iy)/F(iy)| = 1$$

for every $h > 0$.

Proof of Lemma 10. By Problem 24 we can write $F(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are functions which are analytic and bounded by 1 in the upper half-plane and which have no zeros in the half-plane. Therefore, $P(z) = 1/\exp U(z)$ where $U(z)$ is analytic and has a nonnegative real part in the half-plane. There exists a number $p \geq 0$ and a nondecreasing function $\mu(x)$ of real x such that

$$-\log |P(x + iy)| = \operatorname{Re} U(x + iy) = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. It follows that

$$\log |P(h + iy)/P(iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t^2 + y^2} - \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - h)^2 + y^2}$$

$$= \frac{h}{\pi} \int_{-\infty}^{+\infty} \frac{(h - 2t)y}{t^2 + y^2} \frac{d\mu(t)}{(t - h)^2 + y^2}.$$

Since $|2ty| \leq t^2 + y^2$, we obtain

$$\big| \log |P(h + iy)/P(iy)| \big| \leq \frac{h}{2\pi} \int_{-\infty}^{+\infty} \frac{(1 + h/y)\, d\mu(t)}{(t - x)^2 + y^2}.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{y \to +\infty} |P(h + iy)/P(iy)| = 1.$$

The same formula holds with $P(z)$ replaced by $Q(z)$, and the lemma follows.

Proof of Theorem 48. If $E_h(b, z) = E(b, z - h)$, then a space $\mathcal{H}(E_h(b))$ exists, and the transformation $F(z) \to F(z - h)$ takes $\mathcal{H}(E(b))$ isometrically onto $\mathcal{H}(E_h(b))$, as is easily verified from the definitions of the spaces. Since we assume that $\mathcal{H}(E(b))$ is periodic of period h, $\mathcal{H}(E_h(b))$ is equal isometrically to $\mathcal{H}(E(b))$. It follows that

$$(A(z), B(z)) = (A(z - h), B(z - h))P$$

for some matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ which has real entries and determinant 1. Since we then have

$$E(b, z + h)/E(b, z)$$
$$= \tfrac{1}{2}(p + s - iq + ir) + \tfrac{1}{2}(p - s - iq - ir)E^*(b, z)/E(b, z),$$

where $E^*(b, z)/E(b, z)$ is bounded by 1 in the upper half-plane, we obtain

$$|E(b, z + h)/E(b, z)| \leq |\tfrac{1}{2}(p + s - iq + ir)| + |\tfrac{1}{2}(p - s - iq - ir)|$$

for $y > 0$. Since $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$, $E(a, z)/E(b, z)$ is of bounded type in the upper half-plane. By Lemma 10,

$$\lim_{y \to +\infty} |[E(b, h + iy)/E(a, h + iy)]/[E(b, iy)/E(a, iy)]| = 1.$$

Since we know that

$$\limsup_{y \to +\infty} |E(b, h + iy)/E(b, iy)| \leq |\tfrac{1}{2}(p + s - iq + ir)| + |\tfrac{1}{2}(p - s - iq - ir)|,$$

we can conclude that

$$\limsup_{y \to +\infty} |E(a, h + iy)/E(a, iy)| \leq |\tfrac{1}{2}(p + s - iq + ir)| + |\tfrac{1}{2}(p - s - iq - ir)|.$$

Since $E(b, z)/E(a, z)$, $E(b, z + h)/E(b, z)$, and $E(b, z + h)/E(a, z + h)$ are of bounded type in the upper half-plane, $E(a, z + h)/E(a, z)$ is of bounded type in the half-plane. Since $E^*(a, z + h)/E(a, z + h)$ is bounded by 1 in the upper half-plane, $E^*(a, z + h)/E(a, z)$ is of bounded type in the half-plane and

$$\limsup_{y \to +\infty} |E(a, h - iy)/E(a, iy)| \leq |\tfrac{1}{2}(p + s - iq + ir)| + |\tfrac{1}{2}(p - s - iq - ir)|.$$

Since $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$,

$$\int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/E(b, t)|^2 dt$$

for every $F(z)$ in $\mathcal{H}(E(a))$. Since $[F(z)E(a, w) - E(a, z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$, it belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. Since $\mathcal{H}(E(b))$ is periodic of period h, $[F(z + h)E(a, w) - E(a, z + h)F(w)]/(z + h - w)$ belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. Since $\mathcal{H}(E(b))$ is periodic of period h, every element $G(z)$ of $\mathcal{H}(E(b))$ is of the form $G(z) = F(z + h)$ for some $F(z)$ in $\mathcal{H}(E(b))$. It follows that $[G(z)E(a, w) - E(a, z + h)G(w - h)]/(z + h - w)$ belongs to $\mathcal{H}(E(b))$ whenever $G(z)$ belongs to $\mathcal{H}(E(b))$. Equivalently, $[F(z)E(a, w + h) - E(a, z + h)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(b))$ whenever $F(z)$ belongs to $\mathcal{H}(E(b))$. By Theorem 25,

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |E(a, t + h)/E(b, t)|^2 dt < \infty.$$

By Theorem 26, $[F(z)E(a, w + h) - E(a, z + h)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$. Since $G^*(z)$ belongs to $\mathcal{H}(E(a))$ whenever $G(z)$ belongs to $\mathcal{H}(E(a))$, $[F(z)E^*(a, w + h) - E^*(a, z + h)F(w)]/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$. By linearity we obtain

$$[E(a, z+h)E^*(a, w+h) - E^*(a, z+h)E(a, w+h)]/(z-w)$$

in $\mathcal{H}(E(a))$ for every complex number w. Therefore $F(z+h)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ is a finite linear combination of functions $K(a, w, z)$. Since such linear combinations are dense in $\mathcal{H}(E(a))$, since the transformation $F(z) \to F(z+h)$ is isometric in $\mathcal{H}(E(b))$, and since $\mathcal{H}(E(a))$ is a closed subspace of $\mathcal{H}(E(b))$, $F(z+h)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$. A similar argument will show that $F(z-h)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$. The theorem follows since $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and since $\mathcal{H}(E(b))$ is periodic of period h.

PROBLEM 186. Let $\mu(x)$ be a nondecreasing function of real x and let $h > 0$ be a given number such that $\mu(b+h) - \mu(a+h) = \mu(b) - \mu(a)$ for all a and b. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$. Show that the space $\mathcal{H}(E)$ is periodic of period h if $E(z)$ is of bounded type in the upper half-plane.

PROBLEM 187. Let $\mu(x)$ be a nondecreasing function of real x and let $h > 0$ be a given number such that $\mu(b+h) - \mu(a+h) = \mu(b) - \mu(a)$ for all a and b. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $E(a, z)$ and $E(b, z)$ have no real zeros, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and $\mathcal{H}(E(b))$ is contained isometrically in $L^2(\mu)$. Show that $\mathcal{H}(E(b))$ is periodic of period h if $\mathcal{H}(E(a))$ is periodic of period h.

PROBLEM 188. Let $\mathcal{H}(E(b))$ be a one-dimensional space which is periodic of period h. Let

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

where $\mathcal{H}(E(a))$ and $\mathcal{H}(M(a, b))$ exist. Show that $\mathcal{H}(E(a))$ is periodic of period h.

PROBLEM 189. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $E(a, z)$ and $E(b, z)$ have no real zeros, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. If $\mathcal{H}(E(b))$ is periodic of period h and if $S(z)$ is an element of $\mathcal{H}(E(b))$ which spans the orthogonal complement of $\mathcal{H}(E(a))$, show that either $S(z-h) = S(z)$ or $S(z-h) = -S(z)$.

PROBLEM 190. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $E(a, z)$ and $E(b, z)$ have no real zeros, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. Assume that $\mathcal{H}(E(b))$, and hence $\mathcal{H}(E(a))$, is periodic of period h. Let $P(a)$ and $P(b)$ be the 2×2-matrices having real entries and determinant 1 such that

$$(A(b, z - h), B(b, z - h)) = (A(b, z), B(b, z))P(b),$$
$$(A(a, z - h), B(a, z - h)) = (A(a, z), B(a, z))P(a).$$

Show that 1 or -1 is an eigenvalue of $P(a)$ and of $P(b)$. Show that $P(a) = 1$ or $P(a) = -1$ and that $P(b) \neq 1$ and $P(b) \neq -1$.

PROBLEM 191. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, $E(a, z)$ and $E(b, z)$ have no real zeros, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. If $\mathcal{H}(E(b))$ is periodic of period h, show that the domain of multiplication by z in $\mathcal{H}(E(a))$ is dense in $\mathcal{H}(E(a))$.

PROBLEM 192. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(c))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(c))$, $E(a, z)$ and $E(c, z)$ have no real zeros, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(c))$ is one-dimensional. Let $\mathcal{H}(E(b))$ be a space such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z),$$
$$(A(c, z), B(c, z)) = (A(b, z), B(b, z))M(b, c, z)$$

for some spaces $\mathcal{H}(M(a, b))$ and $\mathcal{H}(M(b, c))$. Show that $\mathcal{H}(E(b))$ is periodic of period h if $\mathcal{H}(E(c))$ is periodic of period h.

PROBLEM 193. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $E(a, z)$ and $E(b, z)$ have no real zeros, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. Assume that $\mathcal{H}(E(a))$ is periodic of period h and that it is contained isometrically in $L^2(\mu)$ where $\mu(x)$ is a nondecreasing function of real x such that $\mu(b + h) - \mu(a + h) = \mu(b) - \mu(a)$ for all real numbers a and b. Assume that there exists a number $p \geq 0$ and a function $W(b, z)$, analytic and bounded by 1 in the upper half-plane, such that

$$\operatorname{Re} \frac{E(b, z) + E^*(b, z)W(b, z)}{E(b, z) - E^*(b, z)W(b, z)} = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E(b, t)|^2 d\mu(t)}{(t - x)^2 + y^2}$$

for $y > 0$. Show that $\mathcal{H}(E(b))$ is periodic of period h.

PROBLEM 194. Let P and Q be 2×2-matrices such that P has determinant 1, Q has real entries, $Q \geq 0$ and $Q \neq 0$. If P commutes with QI, show that P is a linear combination of QI and the identity matrix. If P has real entries, show that $|\text{spur } P| \leq 2$.

49. STRUCTURE OF PERIODIC SPACES

The solution of the structure problem is known for periodic spaces.

THEOREM 49. In Theorem 40 assume that $\mathcal{H}(E)$ is periodic of period h, $h > 0$, and that $\mu(b + h) - \mu(a + h) = \mu(b) - \mu(a)$ for all a and b. Let $\tau(t)$ be the largest nondecreasing function of t such that $m(t) - iI\tau(t)$ is nondecreasing. Then $\tau(t)$ is bounded below. Choose the arbitrary constant in $\tau(t)$ so that $\lim \sin [h\tau(t)] = 0$ as $t \searrow s_-$. Then $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are linear functions of $\tau(t)$ in any interval in which $\sin [h\tau(t)] \neq 0$, and they are constant in any interval of regular points in which $\sin [h\tau(t)] = 0$. If b is a regular point and if $a < b$, then a is a regular point.

Proof of Theorem 49. By Theorem 48, $\mathcal{H}(E(a))$ is periodic of period h when a is regular and $a < c$. By Problem 187, $\mathcal{H}(E(a))$ is periodic of period h when a is regular and $a > c$. By Problems 188, 192, and 193, $\mathcal{H}(E(a))$ is periodic of period h for all a. By the proof of Theorem 48, there exists a 2×2-matrix

$$P(a) = \begin{pmatrix} p(a) & q(a) \\ r(a) & s(a) \end{pmatrix}$$

having real entries and determinant 1 such that

$$(A(a, z - h), B(a, z - h)) = (A(a, z), B(a, z))P(a).$$

If $a < b$ we have also

$$(A(b, z - h), B(b, z - h)) = (A(b, z), B(b, z))P(b)$$

and

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

It follows that

$$(A(a, z), B(a, z))P(a)M(a, b, z - h) = (A(a, z), B(a, z))M(a, b, z)P(b).$$

By Problem 100,

$$P(a)M(a, b, z - h) = M(a, b, z)P(b).$$

When $z = 0$ we obtain

$$P(b) = P(a)M(a, b, -h).$$

Therefore the entries of $P(t)$ are continuous functions of t. Since

$$M(a, b, z)I - I = z \int_a^b M(a, t, z)dm(t)$$

when $a < b$, we obtain

$$P(b) - P(a) = h \int_a^b P(t)dm(t)I$$

when $z = -h$. We also have

$$M(a, b, z - h)I - I = (z - h) \int_a^b M(a, t, z - h)dm(t).$$

Multiplying on the left by $P(a)$, we obtain

$$M(a, b, z)P(b)I - P(a)I = (z - h) \int_a^b M(a, t, z)P(t)dm(t).$$

We now use the identity

$$M(a, b, z)I\bar{M}(a, b, w) - I = (z - \bar{w}) \int_a^b M(a, t, z)dm(t)\bar{M}(a, t, w)$$

when $w = h$. Since $P(a) = M(a, b, h)P(b)$ where $P(b)I\bar{P}(b) = I$ and $\bar{P}(a)IP(a) = I$, it can be written

$$M(a, b, z)P(b) - P(a) = -(z - h) \int_a^b M(a, t, z)dm(t)IP(t).$$

Comparing with the previous identity we obtain

$$\int_a^b M(a, t, z)P(t)dm(t)I = \int_a^b M(a, t, z)dm(t)IP(t).$$

When $z = 0$ the equation reads

$$\int_a^b P(t)dm(t)I = \int_a^b dm(t)IP(t).$$

If $m(t)$ is chosen as in the proof of Theorem 40 so that $\alpha(t) + \gamma(t) = t$, then $\tau(t)$ and the entries of $m(t)$ are absolutely continuous functions of t and $m'(t)$ exists for almost all t. Since $\alpha'(t) + \gamma'(t) = 1$, $m'(t) \neq 0$ whenever it exists. When $m'(t)$ exists, $P'(t)$ and $\tau'(t)$ exist, $P(t)$ commutes with $m'(t)I$, and

$$P'(t) = hP(t)m'(t)I.$$

By Problem 194, $|p(t) + s(t)| \leq 2$ whenever $m'(t)$ exists. By continuity the inequality holds for all t. Since $p(t)$ and $s(t)$ are continuous functions of t, we can write $p(t) + s(t) = 2 \cos \theta(t)$ for some continuous function $\theta(t)$, and we can choose it so that $\sin \theta(t)$ has a given sign in any connected component of the set on which $\sin \theta(t) \neq 0$. By Problem 194, $P(t)$ is a linear combination of $m'(t)I$ and the identity matrix when $m'(t)$ exists. Since $m'(t)I$ has zero trace and $\alpha'(t) + \gamma'(t) = 1$,

$$P(t) - \cos \theta(t) = [r(t) - q(t)]m'(t)I$$

whenever $m'(t)$ exists. Since the entries of $m(t)$ are absolutely continuous functions of t, $m'(t)$ exists whenever $r(t) \neq q(t)$. Since $P(t)$ has determinant 1, $r(t) \neq q(t)$ whenever $P(t) \neq 1$ and $P(t) \neq -1$. So $m(t)$ is differentiable on the set where $P(t) \neq 1$ and $P(t) \neq -1$, and $m'(t)$ is continuous on this set. Since $P(t)$ is differentiable on this set, $m'(t)$ is also differentiable on this set. Since $P(t)$ has determinant 1,

$$\tau'(t)^2[r(t) - q(t)]^2 = \sin^2 \theta(t)$$

whenever $m'(t)$ exists. We choose $\theta(t)$ so that

$$\tau'(t)[r(t) - q(t)] = \sin \theta(t).$$

Since

$$m'(t)Im'(t) = I\tau'(t)^2,$$

we obtain the equation

$$P'(t) = hm'(t)I \cos \theta(t) - h \sin \theta(t)\tau'(t).$$

Since the trace of $m'(t)I$ is zero,

$$p'(t) + s'(t) = -2h \sin \theta(t)\tau'(t).$$

Since $p(t) + s(t) = 2 \cos \theta(t)$ and since $p(t)$ and $s(t)$ are differentiable on the set where $\sin \theta(t) \neq 0$, $\theta(t)$ is differentiable on this set and $\theta'(t) = h\tau'(t)$. It follows that

$$\theta(b) - \theta(a) = h\tau(b) - h\tau(a)$$

if $\sin \theta(t) \neq 0$ in (a, b). In such an interval $\tau'(t) > 0$,

$$P(t) = \cos \theta(t) + \frac{dm}{d\tau} I \sin \theta(t)$$

$$\frac{dP}{d\tau} = -h \sin \theta(t) + h \frac{dm}{d\tau} I \cos \theta(t).$$

It follows that $dm/d\tau$ is constant in (a, b) and that the entries of $m(t)$ are linear functions of $\tau(t)$ in the interval. Note that $P(a) = 1$ or $P(a) = -1$ if $\sin \theta(a) = 0$.

If (a, b) is an interval in which $\sin \theta(t) = 0$, $P(t) \neq 1$, and $P(t) \neq -1$, then $q(t) \neq r(t)$ and $\tau'(t) = 0$ in the interval. Since

$$[r(t) - q(t)][P(t) - \cos \theta(t)]' = h[P(t) - \cos \theta(t)] \cos \theta(t)$$

where $\cos \theta(t) = 1$ or $\cos \theta(t) = -1$ in (a, b), the entries of $P(t) - \cos \theta(t)$ are solutions of the same first order differential equation. Since $r(t) - q(t)$ is continuous and nonzero in (a, b), any two solutions of the equation are linearly dependent. It follows that any two entries of $P(t) - \cos \theta(t)$ are linearly dependent functions of t in (a, b). This implies that any two entries of $m'(t)$ are linearly dependent functions of t in (a, b). Since $\alpha(t) + \gamma(t) = t$, the entries of $m(t)$ are linear functions of t in (a, b). Since $\tau'(t) = 0$ in the interval,

$$[\alpha(b) - \alpha(a)][\gamma(b) - \gamma(a)] = [\beta(b) - \beta(a)]^2,$$

and the interval contains only singular points.

If a is a point such that $\sin\theta(a) = 0$ and if a is not the left end point of an interval (a, b) of singular points, then $\tau(t) > \tau(a)$ when $t > a$. It follows that there exists a sequence $\{(a_n, b_n)\}$ of intervals such that $\sin\theta(t) \neq 0$ in each interval, $\sin\theta(a_n) = 0$ for every n, and $a = \lim a_n$. We have seen that $P(a_n) = 1$ or $P(a_n) = -1$ for every n. Since $P(t)$ is a continuous function of t, $P(a) = 1$ or $P(a) = -1$. By Problem 190 we know that $P(a) = 1$ or $P(a) = -1$ if a is a regular point and if the interval (a, b) contains only singular points with respect to $m(t)$. Therefore $P(a) = 1$ or $P(a) = -1$ if a is any regular point such that $\sin\theta(a) = 0$. By Problem 190, $P(b) \neq 1$ and $P(b) \neq -1$ if (a, b) is an interval of singular points, and also $\sin\theta(b) = 0$ since $P(b)$ has determinant 1 and has 1 or -1 as an eigenvalue. From this we see that b is not a regular point with respect to $m(t)$. By the arbitrariness of b there are no regular points to the right of any singular point.

Let $\varphi(t, x)$ be the phase function associated with $E(t, z)$ which is zero at the origin. If $P(t) = 1$, then $E(t, z - h) = E(t, z)$ and $\varphi(t, \pi) \equiv 0$ modulo 2π. If $P(t) = -1$, then $E(t, z - h) = -E(t, z)$ and $\varphi(t, \pi) \equiv \pi$ modulo 2π. Since $\varphi(t, \pi)$ increases as t increases, $\varphi(b, \pi) \geq \varphi(a, \pi) + \pi$ if $a < b$ and $P(a)$ and $P(b)$ are multiples of the identity matrix. Since $\varphi(t, \pi)$ is nonnegative for all t, there are at most $\varphi(c, \pi)/\pi$ points t such that $t \leq c$ and $P(t) = 1$ or $P(t) = -1$. It follows that there are only a finite number of regular points t such that $t < c$ and $\sin\varphi(t, c) = 0$ since $P(t) = 1$ or $P(t) = -1$ at such points. Since $\theta(t) - h\tau(t)$ is continuous and is constant in each interval where $\sin\theta(t) \neq 0$, it is constant on the set of regular points. Since $\theta(t)$ and $\tau(t)$ are constant on the set of singular points, $\theta(t) - h\tau(t)$ is a constant. We choose the arbitrary constant in $\tau(t)$ so that $\theta(t) = h\tau(t)$. Since $\theta(t)$ is a nondecreasing, continuous function and since $\theta(t) \equiv 0$ modulo π for only a finite number of regular points $t < c$, $\theta(t)$, and hence $\tau(t)$, has a lower bound. By construction there is no interval of regular points in which $\tau(t)$ is constant. Since $\tau(t)$ has a lower bound and since $m(t)$ is a linear function of $\tau(t)$ in each interval in which $\sin\theta(t) \neq 0$, $\gamma(t)$ has a lower bound. By Problem 171, $\lim m(t)$ exists as $t \searrow s_-$ and

$$(A(t, z), B(t, z)) = (A(s_-, z), B(s_-, z))M(s_-, t, z)$$

where $\mathcal{H}(M(s_-, t))$ exists and $M(s_-, t, z) = \lim M(a, t, z)$ as $a \searrow s_-$. It follows that $P(s_-) = \lim P(t)$ exists as $t \searrow s_-$ and that

$$(A(s_-, z - h), B(s_-, z - h)) = (A(s_-, z), B(s_-, z))P(s_-).$$

Since $B(s_-, z) = 0$ identically, $P(s_-)$ has a real eigenvalue. Since $P(s_-)$ has determinant 1 and since the absolute value of its trace is at most 2, its trace is 2 or -2. It follows that $\sin \theta(s_-) = \lim \sin \theta(t) = 0$ as $t \searrow s_-$.

PROBLEM 195. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$, and let P be the 2×2-matrix with real entries and determinant 1 such that

$$(A(z-h), B(z-h)) = (A(z), B(z))P.$$

Show that the absolute value of the trace of P is at most 2.

PROBLEM 196. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$. Show that there exists an entire function $S(z)$ which is real for real z such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and such that $E(z)/S(z)$ is entire. Show that either $S(z-h) = S(z)$ or $S(z-h) = -S(z)$.

PROBLEM 197. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$. Show that the domain of multiplication by z in $\mathcal{H}(E)$ is dense in $\mathcal{H}(E)$ if $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$ where $\mu(x)$ is a nondecreasing function of real x such that $\mu(b+h) - \mu(a+h) = \mu(b) - \mu(a)$ for all real numbers a and b.

PROBLEM 198. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t \geq 0$ whose entries are linear, real valued functions of t. If $\alpha'(t)\gamma'(t) - \beta'(t)^2 = 1$, show that

$$M(t, z) = \cos(tz) - m'(t)I \sin(tz)$$

is the unique matrix valued function of t such that $M(t, w)$ is a continuous function of $t \geq 0$ for every w and

$$M(a, w)I - I = w \int_0^a M(t, w)\, dm(t)$$

for $a \geq 0$. Show that a space $\mathcal{H}(E(a))$ exists for every $a > 0$, $E(a, z) = A(a, z) - iB(a, z)$, and that it is periodic of period h for every $h > 0$. Show that

$$M(a, z-h) = M(a, z)M(a, -h).$$

PROBLEM 199. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ is of bounded type in the upper half-plane and has no real zeros. If $\mathcal{H}(E)$ is periodic of period h for every $h > 0$, if the domain of multiplication by z is dense in $\mathcal{H}(E)$, and if $E(0) = 1$, show that $E(z) = E(a, z)$ for some choice of $m(t)$ as in Problem 198 and some number $a > 0$.

PROBLEM 200. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ is of bounded type in the upper half-plane and has no real zeros. Let τ be the mean type of $E(z)$ in the upper half-plane. Assume that the domain of multiplication by z in $\mathcal{H}(E)$ is dense in $\mathcal{H}(E)$. If $\mathcal{H}(E)$ is periodic of period h for some number h, $0 < h \leq \pi/\tau$, show that $\mathcal{H}(E)$ is periodic of period h for every number $h > 0$.

PROBLEM 201. Let $h > 0$ and let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t \geq 0$ whose entries are continuous, real valued functions of t. Assume that the entries of $m(t)$ are linear in each interval $n\pi - \pi < ht < n\pi$, $n = 1, 2, 3, \cdots$, and that $\alpha'(t)\gamma'(t) - \beta'(t)^2 = 1$ when $ht \neq n\pi$, $n = 0, 1, 2, \cdots$. Let $M(t, w)$ be the unique, continuous, matrix valued function of t for every w such that

$$M(a, w)I - I = w \int_0^a M(t, w)\,dm(t)$$

for $a > 0$. Show that

$$(A(a, z - h), B(a, z - h)) = (A(a, z), B(a, z))P(a)$$

where $P(a) = (-1)^n$ if $ha = n\pi$, $n = 1, 2, 3, \cdots$, and

$$P(a) = \cos a + m'(a)I \sin a$$

otherwise.

PROBLEM 202. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$. If the domain of multiplication by z is dense in $\mathcal{H}(E)$ and if $E(z)$ has no real zeros, show that there exists a nondecreasing function $\mu(x)$ of real x such that $\mu(b + h) - \mu(a + h) = \mu(b) - \mu(a)$ for all real numbers a and b and such that $\mathcal{H}(E)$ is contained isometrically in $L^2(\mu)$.

PROBLEM 203. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$. If the domain of multiplication by z is dense in $\mathcal{H}(E)$, show that

$$(A(z - h), B(z - h))Q = (A(z), B(z))Q \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some real number θ and for some 2×2-matrix Q having real entries and determinant 1.

PROBLEM 204. Let $\mathcal{H}(E)$ be a given space which is periodic of period h, $h > 0$, and let $\varphi(x)$ be a phase function associated with $\mathcal{H}(E)$. Assume that

$$(A(z - h), B(z - h)) = (A(z), B(z)) \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some real number θ. Show that $\varphi(b+h) - \varphi(a+h) = \varphi(b) - \varphi(a)$ for all real numbers a and b.

PROBLEM 205. Let $h > 0$ and let $\mu(x)$ be a nondecreasing function of real x such that $\mu(b+h) - \mu(a+h) = \mu(b) - \mu(a)$ for all real numbers a and b. Assume that $\mu(x)$ has a finite number r of points of increase in each interval of length h. Show that there exists a space $\mathcal{H}(E)$ contained isometrically in $L^2(\mu)$ such that $E(z)$ is of bounded type in the upper half-plane and has no real zeros and such that $\mathcal{H}(E)$ fills $L^2(\mu)$. Show that $E(z)$ can be chosen so that $E(z-h) = E(z)$ if r is even and so that $E(z-h) = -E(z)$ if r is odd. Show that the mean type of $E(z)$ in the upper half-plane is $\pi r/h$.

PROBLEM 206. Let $h > 0$ and let $\mu(x)$ be a nondecreasing function of real x such that $\mu(b+h) - \mu(a+h) = \mu(b) - \mu(a)$ for all real numbers a and b. Assume that $\mu(x)$ has an infinite number of points of increase in each interval of length h. Show that for each $a > 0$ there exists a space $\mathcal{H}(E(a))$ contained isometrically in $L^2(\mu)$ such that $E(a, z)$ is of bounded type in the upper half-plane and has no real zeros, and such that the mean type of $E(a, z)$ in the half-plane is a.

PROBLEM 207. A space $\mathcal{H}(E)$ is said to be symmetric about a point h if $F(2h - z)$ belongs to the space whenever $F(z)$ belongs to the space and if the identity

$$\langle F(2h - t), G(t)\rangle = \langle F(t), G(2\bar{h} - t)\rangle$$

holds whenever $F(z)$ and $G(z)$ belong to the space. Note that $G(2\bar{h} - z)$ belongs to the space whenever $G(z)$ belongs to the space since it is obtained by conjugating $G^*(2h - z)$. If a space $\mathcal{H}(E)$ is symmetric about a point h, show that

$$(A(2h - z), B(2h - z)) = (A(z), B(z))P$$

for some matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ which has trace 0 and determinant -1. Show that P is uniquely determined by $E(z)$. Show that the entries of P are real if h is a real number.

PROBLEM 208. Let $f(z)$ be a function which is analytic and of bounded type in the upper half-plane. If $f(z)$ has no zeros in the half-plane and has zero mean type, show that

$$\lim_{y\to+\infty} |f(h + iy)/f(iy)| = 1$$

for every complex number h.

PROBLEM 209. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. If $\mathcal{H}(E(b))$ is symmetric about a point h, show that $\mathcal{H}(E(a))$ is symmetric about the point h.

PROBLEM 210. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, $E(a, z)/E(b, z)$ has no real zeros, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. If $\mathcal{H}(E(b))$ is symmetric about a point h and if $S(z)$ is an element of $\mathcal{H}(E(b))$ which spans the orthogonal complement of $\mathcal{H}(E(a))$, show that h is real and that either $S(2h - z) = S(z)$ or $S(2h - z) = -S(z)$.

PROBLEM 211. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(M(a, b))$ be spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

If $\mathcal{H}(E(b))$ is symmetric about a point h, show that $\mathcal{H}(E(a))$ is symmetric about the point h.

PROBLEM 212. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of t whose entries are continuous, real valued functions of t in an interval $[a, c]$. Let $M(a, t, w)$ be the unique, continuous, matrix valued function of t in $[a, c]$ such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)\,dm(t)$$

for $a \le b \le c$. Assume that $\mathcal{H}(E(a))$ and $\mathcal{H}(E(c))$ are given spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z)$$

and that $\mathcal{H}(E(c))$ is symmetric about a point h. Let $\mathcal{H}(E(b))$ be the space defined by $E(b, z) = A(b, z) - iB(b, z)$ where

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

$a \le b \le c$. Show that there exists a unique matrix

$$P(b) = \begin{pmatrix} p(b) & q(b) \\ r(b) & s(b) \end{pmatrix}$$

having trace 0 and determinant -1 such that

$$(A(b, 2h - z), B(b, 2h - z)) = (A(b, z), B(b, z))P(b).$$

Show that the entries of $P(b)$ are continuous functions of b and that

$$P(b)I - P(a)I = 2h \int_a^b P(t)dm(t),$$

$$\int_a^b P(t)dm(t)I = -\int_a^b dm(t)IP(t)$$

for $a \leq b \leq c$.

PROBLEM 213. Show that the Paley-Wiener spaces are symmetric about every point h in the complex plane.

PROBLEM 214. A space $\mathcal{H}(E)$ is said to be periodic of period h, h a complex number, if $F(z-h)$ and $F(z+\bar{h})$ belong to the space whenever $F(z)$ belongs to the space and if the identity

$$\langle F(t-h), G(t)\rangle = \langle F(t), G(t+\bar{h})\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space. Show that a space $\mathcal{H}(E)$ is periodic of period $2h$ if it is symmetric about the point h and about the origin.

PROBLEM 215. If a space $\mathcal{H}(E)$ is periodic of period h, show that

$$(A(z-h), B(z-h)) = (A(z), B(z))P$$

for some matrix $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ having determinant 1.

PROBLEM 216. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. If $\mathcal{H}(E(b))$ is periodic of period h, show that $\mathcal{H}(E(a))$ is periodic of period h.

PROBLEM 217. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$, $E(a, z)/E(b, z)$ has no real zeros, and the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional. Let $S(z)$ be an element of $\mathcal{H}(E(b))$ which spans the orthogonal complement of $\mathcal{H}(E(a))$. If $\mathcal{H}(E(b))$ is periodic of period h, show that h is real and that either $S(z-h) = S(z)$ or $S(z-h) = -S(z)$.

PROBLEM 218. Let $\mathcal{H}(E(a))$, $\mathcal{H}(E(b))$, and $\mathcal{H}(M(a, b))$ be given spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

Show that $\mathcal{H}(E(a))$ is periodic of period h if $\mathcal{H}(E(b))$ is periodic of period h.

PROBLEM 219. Show that a space $\mathcal{H}(E)$ which is periodic of period h is also periodic of period $\bar{h}$ and of period $-h$.

PROBLEM 220. Show that a space $\mathcal{H}(E)$ which is periodic of period h and of period k is periodic of period $h + k$.

PROBLEM 221. Let $\mathcal{H}(E)$ be a given space such that $F(z + i)$ belongs to the space whenever $F(z)$ belongs to the space. Show that $E(z) = S(z)E_0(z)$ where $S(z)$ is an entire function which is real for real z and periodic of period $2i$, and where $E_0(z)$ is an entire function of Pólya class which has no real zeros.

PROBLEM 222. Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of t whose entries are continuous, real valued functions of t defined in an interval $[a, c]$. For each complex number w, let $M(a, t, w)$ be the unique, continuous, matrix valued function of t in $[a, c]$ such that

$$M(a, b, w)I - I = w \int_a^b M(a, t, w)dm(t)$$

for $a \leq b \leq c$. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(c))$ be given spaces such that

$$(A(c, z), B(c, z)) = (A(a, z), B(a, z))M(a, c, z).$$

For each number b, $a \leq b \leq c$, let $\mathcal{H}(E(b))$ be the space defined by $E(b, z) = A(b, z) - iB(b, z)$ and

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z).$$

If $\mathcal{H}(E(c))$ is periodic of period h, show that there exists a unique matrix

$$P(b) = \begin{pmatrix} p(b) & q(b) \\ r(b) & s(b) \end{pmatrix}$$

with determinant 1 such that

$$(A(b, z - h), B(b, z - h)) = (A(b, z), B(b, z))P(b).$$

Show that the entries of $P(b)$ are continuous functions of b and that

$$\int_a^b P(t)dm(t)I = \int_a^b dm(t)IP(t),$$

$$P(b) - P(a) = h \int_a^b P(t)dm(t)I$$

for $a \leq b \leq c$.

PROBLEM 223. Assume that h is not real in Problem 222 and that $\mathcal{H}(E(b))$ is not one-dimensional. Let $\tau(t)$ be the largest nondecreasing function of t such that $m(t) - iI\tau(t)$ is nondecreasing. Show that the entries of $m(t)$ are linear functions of $\tau(t)$ in (a, b).

PROBLEM 224. Let $\mathcal{H}(E)$ be a given space which is periodic of period h where h is not real. Show that there exists an entire function $S(z)$, which is real for real z and has no zeros, such that $[F(z)S(w) - S(z)F(w)]/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Show that $S(z - h) = S(z)$ or $S(z - h) = -S(z)$. If $E_1(z) = E(z)/S(z)$, show that a space $\mathcal{H}(E_1)$ exists, that $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E_1)$ onto $\mathcal{H}(E)$, that $\mathcal{H}(E_1)$ is periodic of period h, and $E_1(z)$ is of bounded type in the upper half-plane.

PROBLEM 225. Let $\mathcal{H}(E)$ be a given space which is periodic for a nonreal period. If $E(z)$ is of bounded type in the upper half-plane, show that $\mathcal{H}(E)$ is periodic of period h for every complex number h.

50. STRUCTURE OF HOMOGENEOUS SPACES

Paley-Wiener spaces have another special property which enters also in the theory of the Hankel transformation. A space $\mathcal{H}(E)$ is said to be homogeneous of order ν, ν real, if $a^{1+\nu}F(az)$ belongs to the space whenever $F(z)$ belongs to the space and if it always has the same norm as $F(z)$, $0 < a < 1$. Paley-Wiener spaces are homogeneous spaces of order $-\frac{1}{2}$. We now determine the structure of the most general homogeneous space. If a space $\mathcal{H}(E)$ is homogeneous of order ν and if $E(0) = 0$, then a space $\mathcal{H}(E_1)$ exists, $E_1(z) = E(z)/z$, it is homogeneous of order $1 + \nu$, and $F(z) \to zF(z)$ is an isometric transformation of $\mathcal{H}(E_1)$ onto $\mathcal{H}(E)$. It is therefore sufficient to study homogeneous spaces such that $E(0) \neq 0$.

THEOREM 50. Let $\mathcal{H}(E)$ be a given space which is homogeneous of order ν. If $E(0) \neq 0$ and if $\mathcal{H}(E)$ contains a function which is not a constant, then $\nu > -1$. There exists a family $\{\mathcal{H}(E(a))\}$ of spaces, $a > 0$, a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix},$$

and a nondecreasing function $\mu(x)$ with the following properties:

(1) $\mathcal{H}(E)$ is equal isometrically to $\mathcal{H}(E(1))$.

(2) $E(t, w)$ is a continuous function of t for every complex number w, and

$$(A(b, w), B(b, w))I - (A(a, w), B(a, w))I = w \int_a^b (A(t, w), B(t, w))dm(t)$$

whenever $0 < a < b < \infty$.

(3) $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$ for every $a > 0$.

(4) $\mathcal{H}(E(a))$ is contained in $\mathcal{H}(E(b))$ when $a < b$.

(5) The intersection of the spaces $\mathcal{H}(E(a))$ contains no nonzero element.

(6) The union of the spaces $\mathcal{H}(E(a))$ is dense in $L^2(\mu)$.

(7) When $t > 0$,

$$\alpha(t) = \alpha(1)t^{2\nu+2}, \qquad \beta(t) = \beta(1)t, \qquad \gamma(t) = \gamma(1)t^{-2\nu}.$$

(8) When $x > 0$,

$$\mu(x) = \mu(1)x^{2\nu+2} \quad \text{and} \quad \mu(-x) = \mu(-1)x^{2\nu+2}.$$

Proof of Theorem 50. We first show that $E(z)$ has no real zeros. Since we assume that $E(0) \neq 0$, it is sufficient to show that $E(h) \neq 0$ when $h \neq 0$. Argue by contradiction, assuming that $E(h) = 0$. Then h is a zero of every element of $\mathcal{H}(E)$. Since $a^{1+\nu}F(az)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$, $0 < a < 1$, it follows that every element of $\mathcal{H}(E)$ vanishes at the points ah, $0 < a < 1$. Since the elements of $\mathcal{H}(E)$ are entire functions, $\mathcal{H}(E)$ contains no nonzero element, in contradiction of Theorem 19. We can therefore conclude that $E(z)$ has no real zeros.

If $a > 0$ let $\mathcal{M}(a)$ be the set of entire functions $G(z)$ of the form $a^{1+\nu}F(az)$ for some corresponding element $F(z)$ of $\mathcal{H}(E)$. Define a norm in $\mathcal{M}(a)$ so as to make the transformation $F(z) \to a^{1+\nu}F(az)$ an isometry of $\mathcal{H}(E)$ into $\mathcal{M}(a)$. It is easily verified that $\mathcal{M}(a)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element. By Theorem 23, $\mathcal{M}(a)$ is equal isometrically to a space $\mathcal{H}(E(a))$. Since $\mathcal{H}(E)$ is homogeneous of order ν, $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ when $a < b$. Since $E(z)$ has no real zeros, $F(z)/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and $F(w) = 0$. It follows that $F(z)/(z - w)$ belongs to $\mathcal{H}(E(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E(a))$ and $F(w) = 0$. This condition implies that $E(a, z)$ has no real zeros. By Theorem 33 there exists a space $\mathcal{H}(M(a, b))$ such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

when $a < b$. We can clearly choose the spaces $\mathcal{H}(E(a))$ so that $E(a, z)$ has value 1 at the origin for every a and so that the value of $M(a, b, z)$ at the origin is the identity matrix when $a < b$.

Since $a^{1+\nu}K(aw, z)$ belongs to $\mathcal{H}(E)$ for every complex number w, $a^{2+2\nu}K(aw, az)$ belongs to $\mathcal{H}(E(a))$ for every complex number w. If $G(z) = a^{1+\nu}F(az)$ is in $\mathcal{H}(E(a))$, then

$$\langle G(t), a^{2+2\nu}K(aw, at)\rangle = \langle F(t), a^{1+\nu}K(aw, t)\rangle = a^{1+\nu}F(aw) = G(w).$$

By the arbitrariness of $G(z)$, we can conclude that

$$K(a, w, z) = a^{2+2\nu}K(aw, az).$$

Since $K(a, w, w)$ is a nondecreasing function of a for every w, $\nu \geq -1$.

We argue by contradiction to show that $\nu > -1$. If $\nu = -1$, then $K(a, 0, 0)$ does not depend on a. Since $K(a, 0, z)$ is the projection of $K(b, 0, z)$ in $\mathcal{H}(E(a))$ when $a < b$ and since

$$\|K(a, 0, t)\|^2 = K(a, 0, 0) = K(b, 0, 0) = \|K(b, 0, t)\|^2,$$

we have $K(a, 0, z) = K(b, 0, z)$. This implies that $B(a, z)$ is independent of a. Since $B(a, z)$ clearly does not belong to $\mathcal{H}(E(a))$, it follows from Theorem 22 that $\mathcal{H}(E(a))$ fills $\mathcal{H}(E(b))$ when $a < b$. It follows that $K(aw, az) = K(w, z)$ and that $K(w, z) = K(0, 0)$ is independent of z and w, which contradicts the hypothesis that $\mathcal{H}(E)$ contains a nonconstant element. We can therefore conclude that $\nu > -1$.

The identity $K(a, w, z) = a^{2+2\nu}K(1, aw, az)$ implies that

$$(A(a, z), B(a, z)) = a^{\nu+\frac{1}{2}}(A(1, az), B(1, az))P(a)$$

for a unique matrix $P(a) = \begin{pmatrix} p(a) & q(a) \\ r(a) & s(a) \end{pmatrix}$ having real entries and determinant 1. Since $E(a, z)$ and $E(1, z)$ have value 1 at the origin, $p(a) = a^{-\nu-\frac{1}{2}}$, $q(a) = 0$, and $s(a) = a^{\nu+\frac{1}{2}}$. It follows that

$$b^{\nu+\frac{1}{2}}(A(1, bz), B(1, bz))P(b) = a^{\nu+\frac{1}{2}}(A(1, az), B(1, az))P(a)M(a, b, z)$$

when $a < b$. This identity can be written

$$(A(1, z), B(1, z)) = (A(a/b, z), B(a/b, z))P(a/b)^{-1}P(a)M(a, b, z/b)P(b)^{-1}.$$

By the uniqueness property of the linking matrix $M(a, b, z)$, Problem 100, we can conclude that

$$M(a/b, 1, z) = P(a/b)^{-1}P(a)M(a, b, z/b)P(b)^{-1}.$$

When $z = 0$ the identity reads $P(a/b) = P(a)P(b)^{-1}$. This implies that

$$r(a/b) = r(a)b^{\nu+\frac{1}{2}} - r(b)a^{\nu+\frac{1}{2}}$$

when $a < b$. The solution of the equation is

$$r(a) = h(a)(a^{\nu+\frac{1}{2}} - a^{-\nu-\frac{1}{2}})$$

where $h(a) = h$ is a constant. So we have

$$P(a) = \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \begin{pmatrix} a^{-\nu-\frac{1}{2}} & 0 \\ 0 & a^{\nu+\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}$$

Since we can replace $E(a, z)$ by $E(a, z) - hB(a, z)$ without altering the defining property of the original family of functions, we can suppose the construction made in such a way that $h = 0$. In the notation of Theorem 40, we have

$$M'(a, b, 0)I = m(b) - m(a)$$

where

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

is a nondecreasing function of t. Since $K(a, 0, 0) = a^{2+2\nu}K(1, 0, 0)$, we can choose $\alpha(t)$ so that $\alpha(t) = t^{2+2\nu}\alpha(1)$. By comparing derivatives at the origin in the identity for $M(a, b, z)$, we obtain

$$m(a/b)I - m(1)I = P(b)[m(a)I - m(b)I]P(b)^{-1}b^{-1}$$

when $a < b$. Since $P(b)$ has real entries and determinant 1, the identity $IP(b)^{-1} = \bar{P}(b)I$ holds and we can conclude that

$$bm(a/b) - bm(1) = P(b)[m(a) - m(b)]\bar{P}(b).$$

The identity implies that

$$[\beta(a) - \beta(b)]/(a - b) = [\beta(a/b) - \beta(1)]/[(a/b) - 1]$$

$$[\gamma(a) - \gamma(b)]/(a^{-2\nu} - b^{-2\nu}) = [\gamma(a/b) - \gamma(1)]/[(a/b)^{-2\nu} - 1]$$

when $a < b$. Since $\gamma(t)$ is a nondecreasing function of t, it follows that $[\gamma(a) - \gamma(b)]/(a^{-2\nu} - b^{-2\nu})$ is a constant. Since $m(t)$ is nondecreasing,

$$[\beta(b) - \beta(a)]^2 \le [\alpha(b) - \alpha(a)][\gamma(b) - \gamma(a)]$$

when $a < b$. Since $\alpha(t)$ and $\gamma(t)$ are continuous functions of t, $\beta(t)$ is a continuous function of t. It now follows that $[\beta(a) - \beta(b)]/(a - b)$ is a constant. We can therefore choose the arbitrary constants in $\beta(t)$ and $\gamma(t)$ so that (7) holds, and (2) and (5) follow by the proof of Theorem 40. By Theorem 42, there exists a nondecreasing function $\mu(x)$ such that every space $\mathcal{H}(E(a))$ is contained isometrically in $L^2(\mu)$. The union of the spaces $\mathcal{H}(E(a))$ is dense in $L^2(\mu)$ by Problem 163. If $F(x)$ is an element of $L^2(\mu)$ which belongs to the union of the spaces $\mathcal{H}(E(a))$ and if $b > 0$, then $b^{1+\nu}F(bx)$ belongs to the union of the spaces $\mathcal{H}(E(a))$ and has the same norm as $F(x)$. Since the union of the spaces $\mathcal{H}(E(a))$ is dense in $L^2(\mu)$, $b^{1+\nu}F(bx)$ belongs to $L^2(\mu)$ whenever $F(x)$ belongs to $L^2(\mu)$ and it always has the same norm as $F(x)$, $b > 0$. If $F(x) = 1$ for $0 < x < b$ and if $F(x) = 0$ otherwise, then $b^{1+\nu}F(bx) = b^{1+\nu}$ for $0 < x < 1$ and $b^{1+\nu}F(bx) = 0$ otherwise. It follows that

$$\mu(b-) - \mu(0+) = b^{2+2\nu}[\mu(1-) - \mu(0+)].$$

By the arbitrariness of b, $\mu(x)$ is a continuous function of $x > 0$. By considering the function $F(x) = 1$ for $-b < x < 0$ and $F(x) = 0$ otherwise, we obtain

$$\mu(0-) - \mu(-b+) = b^{2+2\nu}[\mu(0-) - \mu(-1+)]$$

for $b > 0$. This implies that $\mu(x)$ is a continuous function of $x < 0$. By considering the function $F(x) = 1$ for $-b < x < b$ and $F(x) = 0$ otherwise, we obtain

$$\mu(b) - \mu(-b) = b^{2+2\nu}[\mu(1) - \mu(-1)]$$

for $b > 0$. It follows that $\mu(0+) = \mu(0-)$. We can therefore choose $\mu(x)$ so that $\mu(0+) = \mu(0) = \mu(0-) = 0$ and the theorem follows.

PROBLEM 226. If $E(z) = E(1, z)$ in Theorem 50, show that

$$B'(z) + (2\nu + 1)B(z)/z = \alpha'(1)A(z) + \beta'(1)B(z),$$
$$-A'(z) = \beta'(1)A(z) + \gamma'(1)B(z).$$

Show that $A(z) = \Sigma A_n z^n$ and $B(z) = \Sigma B_n z^n$ where $A_0 = 1$, $B_0 = 0$, and

$$-(n + 1)A_{n+1} = \beta'(1)A_n + \gamma'(1)B_n,$$
$$(n + 2\nu + 2)B_{n+1} = \alpha'(1)A_n + \beta'(1)B_n$$

for every n.

PROBLEM 227. Let $\nu > -1$ be given and let $\alpha'(1)$, $\beta'(1)$, $\gamma'(1)$ be real numbers such that $\alpha'(1) > 0$, $\gamma'(1) > 0$, and $\beta'(1)^2 \le \alpha'(1)\gamma'(1)$. Let

$$\alpha(t) = \alpha(1)t^{2\nu+2}, \qquad \beta(t) = \beta(1)t, \qquad \gamma(t) = \gamma(1)t^{-2\nu}$$

where $\alpha(1)$, $\beta(1)$, $\gamma(1)$ are defined by

$$\alpha'(1) = (2\nu + 2)\alpha(1), \qquad \beta'(1) = \beta(1), \qquad \gamma'(1) = -2\nu\gamma(1).$$

Let $A(z) = \Sigma A_n z^n$ and $B(z) = \Sigma B_n z^n$ be the formal power series whose coefficients are defined inductively by $A_0 = 1$, $B_0 = 0$, and

$$-(n + 1)A_{n+1} = \beta'(1)A_n + \gamma'(1)B_n,$$
$$(n + 2\nu + 2)B_{n+1} = \alpha'(1)A_n + \beta'(1)B_n$$

for every n. Show that these series converge in the complex plane, that $A(z)$ and $B(z)$ are entire functions which are real for real z, that a space $\mathcal{H}(E)$ exists, $E(z) = A(z) - iB(z)$, and that $\mathcal{H}(E)$ is homogeneous of order ν.

PROBLEM 228. If $\alpha'(1) = 1$, $\beta'(1) = 0$, $\gamma'(1) = 1$ in Problem 227, show that

$$A(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n}}{n!(\nu+1)(\nu+2)\cdots(\nu+n)},$$

$$B(z) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}z)^{2n+1}}{n!(\nu+1)(\nu+2)\cdots(\nu+n+1)}.$$

Show that $E(z) = A(z) - iB(z)$ is of bounded type in the upper half-plane and has mean type equal to 1 in the half-plane. Show that an entire function $F(z)$ belongs to $\mathcal{H}(E)$ if, and only if, $F(z)$ and $F^*(z)$ are of bounded type in the upper half-plane, the mean types of $F(z)$ and $F^*(z)$ are at most 1 in the half-plane, and

$$\int_{-\infty}^{+\infty} |F(t)|^2 \, |t|^{2\nu+1} dt < \infty.$$

PROBLEM 229. The Hankel transformation is defined in terms of the Bessel function

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2}x)^{\nu+2n}}{n!\Gamma(1+\nu+n)}.$$

If $f(x)$ belongs to $L^2(0, \infty)$, its Hankel transform $g(x)$ of order ν is defined formally by

$$g(x) = \int_0^\infty f(t) J_\nu(xt)\sqrt{xt}\, dt.$$

Show that the integral exists as $\lim \int_0^a$ in the metric of $L^2(0, \infty)$ as $a \to \infty$. Show that the function $x^{\nu+\frac{1}{2}}e^{-\frac{1}{2}x^2}$ is its own Hankel transform of order ν. If $f(x)$ belongs to $L^2(0, \infty)$ and if $g(x)$ is its Hankel transform of order ν, show that

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |g(t)|^2 dt$$

and that $f(x)$ is the Hankel transform of $g(x)$ of the same order.

51. ANALYTIC WEIGHT FUNCTIONS

Many of the most familiar Hilbert spaces of entire functions are associated with a weight function. By a weight function $W(z)$ associated with $\mathcal{H}(E)$, we mean a function analytic in the upper half-plane, continuous in the closed half-plane, and having no zeros in the closed half-plane, such that

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$

for every $F(z)$ in $\mathcal{H}(E)$. A weight function is often convenient in characterizing the elements of a space $\mathcal{H}(E)$.

THEOREM 51. Let $\{\mathcal{H}(E(a))\}$, $a > 0$, be a given family of spaces associated with a differentiable, nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$. Let $\tau(t)$ be the choice of a largest nondecreasing function such that $m(t) - iI\tau(t)$ is nondecreasing. Assume that

$$m'(t) = \begin{pmatrix} p(t) & q(t) \\ r(t) & s(t) \end{pmatrix} \begin{pmatrix} p(t) & r(t) \\ q(t) & s(t) \end{pmatrix} \tau'(t)$$

where $p(t)$, $q(t)$, $r(t)$, $s(t)$ are absolutely continuous, real valued functions of $t > 0$ such that

$$p(t)s(t) - q(t)r(t) = 1,$$

$$\begin{pmatrix} p'(t) & q'(t) \\ r'(t) & s'(t) \end{pmatrix} = \begin{pmatrix} p(t) & q(t) \\ r(t) & s(t) \end{pmatrix} \begin{pmatrix} \rho'(t) & -\sigma'(t) \\ \sigma'(t) & -\rho'(t) \end{pmatrix}$$

for almost all t, $\log \tau'(t)$ is an absolutely continuous function of $t > 0$, and $\rho(t)$ and $\sigma(t)$ are absolutely continuous and of bounded variation on each half-line (a, ∞), $a > 0$. If

$$\Phi(t, z) = [A(t, z)p(t) + B(t, z)r(t)] - i[A(t, z)q(t) + B(t, z)s(t)],$$

then

$$W(z) = \lim_{t \to \infty} \Phi(t, z) \exp [i\tau(t)z]$$

exists for $y \geq 0$. The limit function $W(z)$ is a weight function associated with $\mathcal{H}(E(a))$ for every a. A necessary and sufficient condition that an entire function $F(z)$ belong to $\mathcal{H}(E(a))$ is that

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt < \infty,$$

that $F(z)/W(z)$ and $F^*(z)/W(z)$ are of bounded type in the upper half-plane, and that these ratios are of mean type at most $\tau(a)$ in the half-plane.

Proof of Theorem 51. For each fixed z, $\Phi(t, z)$ is an absolutely continuous function of t such that

$$\Phi'(t, z) = \Phi^*(t, z)\rho'(t) + i\Phi(t, z)\sigma'(t) - iz\Phi(t, z)\tau'(t)$$

for almost all t. It follows that $\Phi(t, z) \exp[i\tau(t)z]$ is an absolutely continuous function of t having derivative equal to

$$\Phi^*(t, z) \exp[i\tau(t)z]\rho'(t) + i\Phi(t, z) \exp[i\tau(t)z]\sigma'(t)$$

almost everywhere. For each fixed a, $\Phi(a, z)$ is an entire function of z and a space $\mathcal{H}(\Phi(a))$ exists. Since we assume that $E(a, z)$ has no real zeros, $\Phi(a, z)$ has no zeros on or above the real axis. Since $\Phi^*(a, z)/\Phi(a, z)$ is bounded by one in the upper half-plane, the absolute value of the logarithmic derivative of $\Phi(t, z) \exp[i\tau(t)z]$ is no more than $|\rho'(t)| + |\sigma'(t)|$ in the upper half-plane. It follows that

$$1\Big/\exp\left[\int_a^b |d\rho(t)| + \int_a^b |d\sigma(t)|\right] \leq \left|\frac{\Phi(b, z)\exp[i\tau(b)z]}{\Phi(a, z)\exp[i\tau(a)z]}\right|$$
$$\leq \exp\left[\int_a^b |d\rho(t)| + \int_a^b |d\sigma(t)|\right]$$

when $a \leq b$ and $y \geq 0$. Since we assume that $\rho(t)$ and $\sigma(t)$ are of bounded variation in $[a, \infty)$,

$$W(z) = \lim_{b\to\infty} \Phi(b, z)\exp[i\tau(b)z]$$

exists when $y \geq 0$ and

$$1\Big/\exp\left[\int_a^\infty |d\rho(t)| + \int_a^\infty |d\sigma(t)|\right] \leq |\Phi(a, z)\exp[i\tau(a)z]/W(z)|$$
$$\leq \exp\left[\int_a^\infty |d\rho(t)| + \int_a^\infty |d\sigma(t)|\right].$$

Since convergence is uniform on any bounded set, the limit function $W(z)$ is analytic in the upper half-plane and continuous in the closed half-plane. Since $E(a, z)$ has no zeros on or above the real axis, $W(z)$ has no zeros in the closed half-plane. Since $\mathcal{H}(\Phi(a))$ is equal isometrically to $\mathcal{H}(E(a))$ for every a and since $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ when $a < b$, the identity

$$\int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/\Phi(b, t)|^2 dt$$

holds for every $F(z)$ in $\mathcal{H}(E(a))$ when $a < b$. The proof of Theorem 32 will now show that the identity

$$\int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt$$

holds for all elements $F(z)$ in the domain of multiplication by z in $\mathcal{H}(E(a))$. Since there are no singular points with respect to $m(t)$, the domain of multiplication by z is dense in $\mathcal{H}(E(a))$, and the same formula holds for every element $F(z)$ of $\mathcal{H}(E(a))$. Since $\Phi(a, z) \exp[i\tau(a)z]/W(z)$ is bounded, and bounded away from zero, in the upper half-plane, the space $\mathcal{H}(E(a))$ is the set of entire functions $F(z)$ such that

$$\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt < \infty,$$

such that $F(z)/W(z)$ and $F^*(z)/W(z)$ are of bounded type in the upper half-plane, and such that these ratios are of mean type at most $\tau(a)$ in the half-plane.

PROBLEM 230. In Theorem 51 let $\varphi(a, x)$ be a phase function associated with $\Phi(a, z)$ for every $a > 0$ so that $\varphi(a, 0)$ is independent of a. Show that

$$|\varphi(b, x) - \tau(b)x - \varphi(a, x) + \tau(a)x| \leq \int_a^b |d\rho(t)| + \int_a^b |d\sigma(t)|$$

for all real x when $a < b$. Show that

$$\psi(x) = \lim_{b \to \infty} [\varphi(b, x) - \tau(b)x]$$

exists for all real x and that

$$|\varphi(a, x) - \tau(a)x - \psi(x)| \leq \int_a^\infty |d\rho(t)| + \int_a^\infty |d\sigma(t)|.$$

PROBLEM 231. In Theorem 51 assume that

$$\lim_{a \searrow 0} [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0.$$

Define $\rho(-t) = \rho(t)$, $\sigma(-t) = -\sigma(t)$, $\tau(-t) = -\tau(t)$, and $\Phi(-t, z) = \Phi^*(t, z)$ for $t > 0$. If $f(t)$ belongs to $L^2(-\infty, +\infty)$ and vanishes outside of some finite interval $(-a, a)$, show that its eigentransform $F(z)$, defined by

$$2\pi F(z) = \int_{-\infty}^{+\infty} f(t)\Phi(t, z)\sqrt{\tau'(t)}\, dt,$$

belongs to $\mathcal{H}(E(a))$ and that

$$2\pi \int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt.$$

Show that every element of $\mathcal{H}(E(a))$ is of this form.

PROBLEM 232. In Problem 231 assume that $\rho(t)$, $\sigma(t)$, and $\log \tau'(t)$ are of bounded variation in a neighborhood of the origin. Let H be the transformation defined by

$$H: f(t) \to g(t) = -\tau'(t)^{-\frac{1}{2}} i [\tau'(t)^{-\frac{1}{2}} f(t)]' \\ + i\rho'(t)\tau'(t)^{-1} f(-t) - \sigma'(t)\tau'(t)^{-1} f(t)$$

whenever $f(t)$ belongs to $L^2(-\infty, +\infty)$, $\tau'(t)^{-\frac{1}{2}}f(t)$ is (equivalent to) an absolutely continuous function, and $g(t)$ belongs to $L^2(-\infty, +\infty)$. Let $f(t)$ and $g(t)$ be elements of $L^2(-\infty, +\infty)$ which vanish outside of $(-a, a)$, and let $F(z)$ and $G(z)$ be their eigentransforms. Show that $G(z) = zF(z)$ is a necessary and sufficient condition that $f(t)$ be in the domain of H and that $H: f(t) \to g(t)$.

PROBLEM 233. Let $\mathcal{H}(E)$ be a given space which is not one-dimensional, and let h be a given nonreal number. Show that there exists a space $\mathcal{H}(E_1)$ such that

$$(1 - z/h)[B_1(z)A(h) - A_1(z)B(h)] = B(z)A(h) - A(z)B(h).$$

Show that the transformation $F(z) \to (1 - z/h)F(z)$ is an isometry of $\mathcal{H}(E_1)$ onto the subspace of $\mathcal{H}(E)$ consisting of those functions which vanish at the point h.

PROBLEM 234. Let $\{\mathcal{H}(E(t))\}$, $t > 0$, be a given family of spaces associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(t, z)$ is a continuous function of t for every z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$. Let h be a given nonreal number, and let $A_1(a, z)$ and $B_1(a, z)$ be the unique entire functions which are real for real z such that

$$(1 - z/h)[B_1(a, z)A(a, h) - A_1(a, z)B(a, h)]$$
$$= B(a, z)A(a, h) - A(a, z)B(a, h)$$

when $a > 0$. Let $m_1(t)$ be a matrix valued function of $t > 0$ such that

$$m_1(b) - m_1(a) = \int_a^b P(t)dm(t)\bar{P}(t)$$

when $0 < a < b < \infty$, where

$$|h| \operatorname{Re}[iB(t, h)\bar{A}(t, h)]P(t)$$
$$= \begin{pmatrix} \operatorname{Re}[ihA(t, h)\bar{B}(t, h)] & \operatorname{Re}[ihB(t, h)\bar{B}(t, h)] \\ -\operatorname{Re}[ihA(t, h)\bar{A}(t, h)] & -\operatorname{Re}[ihB(t, h)\bar{A}(t, h)] \end{pmatrix}.$$

Show that

$$(A_1(b, z), B_1(b, z))I - (A_1(a, z), B_1(a, z))I = z\int_a^b (A_1(t, z), B_1(t, z))dm_1(t)$$

when $0 < a < b < \infty$.

PROBLEM 235. Let $\{\mathcal{H}(E_n(a))\}$ be the spaces defined inductively by $E_0(a, z) = \exp(-iaz)$, $a > 0$, and

$$[1 - iz/(n + \tfrac{1}{2})][B_{n+1}(a, z)\bar{A}_n(a, in + \tfrac{1}{2}i) - A_{n+1}(a, z)\bar{B}_n(a, in + \tfrac{1}{2}i)]$$
$$= B_n(a, z)\bar{A}_n(a, in + \tfrac{1}{2}i) - A_n(a, z)\bar{B}_n(a, in + \tfrac{1}{2}i).$$

Show that $\mathcal{H}(E_n(a))$ is the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ are of bounded type and of mean type at most a in the upper half-plane and such that

$$\int_{-\infty}^{+\infty} |F(t)\Gamma(\tfrac{1}{2} + n - it)/\Gamma(\tfrac{1}{2} - it)|^2 dt < \infty.$$

Show that this integral is equal to

$$\int_{-\infty}^{+\infty} |F(t)/E_n(a, t)|^2 dt\ \Gamma(\tfrac{1}{2} + n)^2/\Gamma(\tfrac{1}{2})^2$$

for every $F(z)$ in $\mathcal{H}(E_n(a))$. Show that $F(z + i) + n[F(z + i) - F(-z)]/(\frac{1}{2} - iz)$ belongs to $\mathcal{H}(E_n(a))$ whenever $F(z)$ belongs to $\mathcal{H}(E_n(a))$. Show that the identity

$$\langle F(t + i) + n[F(t + i) - F(-t)]/(\tfrac{1}{2} - it), G(t)\rangle$$
$$= \langle F(t), G(t + i) + n[G(t + i) - G(-t)]/(\tfrac{1}{2} - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E_n(a))$.

52. SPECIAL GAUSS SPACES

Examples of Hilbert spaces of entire functions appear in the eigenfunction expansions associated with Gauss's hypergeometric function. The Gauss spaces are closely related to the Paley-Wiener spaces and satisfy a similar identity. Two real parameters h and k enter into the statement of the identity. We start with the special case $k = 1$.

THEOREM 52. Let h be a real number, and let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$. Assume that $F(z + i)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and that the identity

$$\langle F(t + i) + (h - \tfrac{1}{2})[F(t + i) - F(-t)]/(\tfrac{1}{2} - it), G(t)\rangle$$
$$= \langle F(t), G(t + i) + (h - \tfrac{1}{2})[G(t + i) - G(-t)]/(\tfrac{1}{2} - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$. Then there exist real numbers p, r, and s such that $1 = s^2 - pr$ and such that

$$A(z + i) + (h - \tfrac{1}{2})[A(z + i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r,$$
$$B(z + i) + (h - \tfrac{1}{2})[B(z + i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)s.$$

Proof of Theorem 52. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$F(\beta + i) + (h - \tfrac{1}{2})[F(\beta + i) - F(-\beta)]/(\tfrac{1}{2} - i\beta) = \bar{G}(\alpha + i) + (h - \tfrac{1}{2})[\bar{G}(\alpha + i) - \bar{G}(-\alpha)]/(\tfrac{1}{2} + i\bar{\alpha}).$$

An equivalent identity is

$$K(w, z + i) + (h - \tfrac{1}{2})[K(w, z + i) - K(w, -z)]/(\tfrac{1}{2} - iz) = K(w + i, z) + (h - \tfrac{1}{2})[K(w + i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w}).$$

But

$$\pi(z + i - \bar{w})K(w, z + i) - \pi(z + i - \bar{w})K(w + i, z) = B(z + i)\bar{A}(w) - A(z + i)\bar{B}(w) - B(z)\bar{A}(w + i) + A(z)\bar{B}(w + i),$$

$$\begin{aligned}\pi(z + i - \bar{w})&[K(w, z + i) - K(w, -z)]/(\tfrac{1}{2} - iz) \\ &- \pi(z + i - \bar{w})[K(w + i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w}) \\ &= \bar{A}(w)[B(z + i) - B(-z)]/(\tfrac{1}{2} - iz) \\ &- \bar{B}(w)[A(z + i) - A(-z)]/(\tfrac{1}{2} - iz) \\ &- B(z)[\bar{A}(w + i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\ &+ A(z)[\bar{B}(w + i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}).\end{aligned}$$

It follows that

$$\begin{aligned}\bar{A}(w)&\{B(z + i) + (h - \tfrac{1}{2})[B(z + i) - B(-z)]/(\tfrac{1}{2} - iz)\} \\ &- \bar{B}(w)\{A(z + i) + (h - \tfrac{1}{2})[A(z + i) - A(-z)]/(\tfrac{1}{2} - iz)\} \\ &- B(z)\{\bar{A}(w + i) + (h - \tfrac{1}{2})[\bar{A}(w + i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w})\} \\ &+ A(z)\{\bar{B}(w + i) + (h - \tfrac{1}{2})[\bar{B}(w + i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w})\} = 0.\end{aligned}$$

Since $A(z)$ and $B(z)$ are linearly independent, there exist numbers p, r, and s, p and r real, such that

$$A(z + i) + (h - \tfrac{1}{2})[A(z + i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r,$$
$$B(z + i) + (h - \tfrac{1}{2})[B(z + i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)\bar{s}.$$

Since $E^*(z) = E(-z)$, the same formulas hold with s replaced by $\bar{s}$. It follows that s is real.

The recurrence relations can now be written

$$(h - iz)A(z + i) = a(z)A(z) + c(z)B(z),$$
$$(h - iz)B(z + i) = b(z)A(z) + d(z)B(z)$$

where

$$a(z) = (h - \tfrac{1}{2}) + s(\tfrac{1}{2} - iz), \qquad b(z) = ip(\tfrac{1}{2} - iz),$$
$$c(z) = -ir(\tfrac{1}{2} - iz), \qquad d(z) = -(h - \tfrac{1}{2}) + s(\tfrac{1}{2} - iz).$$

Starring each side of these equations and replacing z by $z + i$, we obtain

$$(h - 1 + iz)A(z) = a^*(z + i)A(z + i) + c^*(z + i)B(z + i),$$
$$(h - 1 + iz)B(z) = b^*(z + i)A(z + i) + d^*(z + i)B(z + i).$$

It follows that

$$(h - iz)(h - 1 + iz)A(z) = [a^*(z + i)a(z) + c^*(z + i)b(z)]A(z) + [a^*(z + i)c(z) + c^*(z + i)d(z)]B(z),$$
$$(h - iz)(h - 1 + iz)B(z) = [b^*(z + i)a(z) + d^*(z + i)b(z)]A(z) + [b^*(z + i)c(z) + d^*(z + i)d(z)]B(z).$$

Since

$$a^*(z + i) = -d(z), \qquad b^*(z + i) = b(z),$$
$$c^*(z + i) = c(z), \qquad d^*(z + i) = -a(z),$$

these equations reduce to the condition

$$(h - iz)(h - 1 + iz) = -a(z)d(z) + b(z)c(z),$$

which is equivalent to the requirement that $1 = s^2 - pr$.

PROBLEM 236. In Theorem 52 let λ be a solution of the equation $\lambda^2 - 2\lambda s + 1 = 0$, and let u and v be numbers such that $su + ipv = \lambda u$ and $sv - iru = \lambda v$. If $F(z) = A(z)u + B(z)v$, show that

$$F(z + i) + (h - \tfrac{1}{2})[F(z + i) - F(-z)]/(\tfrac{1}{2} - iz) = \lambda F(z).$$

Show that

$$\lim_{y \to +\infty} F(iy + i)/F(iy) = \lambda$$

if u and v are not both zero.

PROBLEM 237. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 52 for some number h, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same choice of h.

PROBLEM 238. In Problem 237 assume that the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ has dimension zero or one and that $E(a, 0) = E(b, 0)$. Show that the orthogonal complement is spanned by a function $F(z)$ of the form $F(z) = A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$ where u and v are real numbers which satisfy the hypotheses of Problem 236 for $E(a, z)$ and for $E(b, z)$. Show that any such function $F(z)$ vanishes identically.

PROBLEM 239. Let $\{\mathcal{H}(E(a))\}$, $a > 0$, be a given family of spaces associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))\,dm(t)$$

when $0 < a < b < \infty$. Assume that $m(t)$ is an absolutely continuous function of t and that there exists a real number h such that the hypotheses of Theorem 52 are satisfied for every index a. Show that the quantities $p(a)$, $r(a)$, and $s(a)$ defined by the theorem are absolutely continuous functions of a and that

$$p(a)\gamma'(a) = s'(a) = r(a)\alpha'(a),$$
$$r(a)p'(a) - s(a)s'(a) = 2(h - \tfrac{1}{2})s'(a) = s(a)s'(a) - p(a)r'(a)$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist. Show that $p(a)r(a) \geq 0$ and that $s(a)^2 \geq 1$.

PROBLEM 240. In Problem 236 show that u and v can be chosen so that $u > 0$ and $iv > 0$. Show that $F(iy) > 0$ for $y > 0$ and that $\lambda > 0$. Show that $s > 1$.

PROBLEM 241. If $E(z)$ satisfies the hypotheses of Theorem 52, show that $E(z) = S(z)E_1(z)$ where $E_1(z)$ is an entire function of Pólya class which satisfies the hypotheses of the theorem and $S(z)$ is an even entire function which is real for real z and periodic of period i. Show that $\lambda > 1$, $p > 0$, and $r > 0$ in Problem 236.

PROBLEM 242. Show that

$$\frac{p(a)}{\sqrt{s(a)^2 - 1}} \left| \frac{s(a) + 1}{s(a) - 1} \right|^{h - \frac{1}{2}} \quad \text{and} \quad \frac{r(a)}{\sqrt{s(a)^2 - 1}} \left| \frac{s(a) - 1}{s(a) + 1} \right|^{h - \frac{1}{2}}$$

are constants in Problem 239. If

$$\lim_{a \searrow 0} \, [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 1$ as $a \searrow 0$.

PROBLEM 243. Show that $|\Gamma(1 - iz)|$ is a nondecreasing function of $y > 0$ for each fixed x, $z = x + iy$.

PROBLEM 244. Show that an entire function is a constant if it is periodic of period i and of bounded type in the upper half-plane.

53. CONSTRUCTION OF SPECIAL GAUSS SPACES

The above analysis allows a direct construction of the special Gauss spaces.

THEOREM 53. Let h be a given positive number, let $\alpha(t)$ and $\gamma(t)$ be differentiable functions of $t > 0$ such that

$$\alpha'(t) = \tanh^{2h-1}(\tfrac{1}{2}t) \quad \text{and} \quad \gamma'(t) = \coth^{2h-1}(\tfrac{1}{2}t)$$

for $t > 0$, and let $\beta(t) = 0$. Then there exists a unique family $(E(t, z))$ of entire functions of Pólya class, $t > 0$, such that $E(t, z)$ is a continuous function of t for every z, such that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$, and such that $\lim E(t, z) = 1$ as $t \searrow 0$ for all complex z. A space $\mathcal{H}(E(a))$ exists for every a and $E^*(a, z) = E(a, -z)$. The space is the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ are of bounded type and of mean type at most a in the upper half-plane, and such that

$$\int_{-\infty}^{+\infty} |F(t)\Gamma(h - it)/\Gamma(\tfrac{1}{2} - it)|^2 dt < \infty.$$

The integral is then equal to

$$\int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt \; \Gamma(h)^2/\Gamma(\tfrac{1}{2})^2$$

for every $F(z)$ in $\mathcal{H}(E(a))$. The recurrence relations

$$A(a, z + i) + (h - \tfrac{1}{2})[A(a, z + i) - A(a, -z)]/(\tfrac{1}{2} - iz) = A(a, z)s(a) - iB(a, z)r(a),$$

$$B(a, z + i) + (h - \tfrac{1}{2})[B(a, z + i) - B(a, -z)]/(\tfrac{1}{2} - iz) = iA(a, z)p(a) + B(a, z)s(a)$$

hold with $s(a) = \cosh a$,

$$p(a) = \sinh a \tanh^{2h-1}(\tfrac{1}{2}a) \quad \text{and} \quad r(a) = \sinh a \coth^{2h-1}(\tfrac{1}{2}a).$$

Proof of Theorem 53. Since $\int_0^1 \alpha(t)d\gamma(t) < \infty$, the existence of the family of functions $E(a, z)$ is given by Theorem 41. By Problem 180, $E^*(a, z) = E(a, -z)$ for every index a. We show that the stated recurrence relations are valid. Let

$$P(a, z) = (\tfrac{1}{2} - iz)[A(a, z)s(a) - iB(a, z)r(a)] + (h - \tfrac{1}{2})A(a, z),$$
$$Q(a, z) = (\tfrac{1}{2} - iz)[iA(a, z)p(a) + B(a, z)s(a)] - (h - \tfrac{1}{2})B(a, z)$$

with $p(a)$, $r(a)$, and $s(a)$ defined as in the statement of the theorem. The equations

$$\partial P(a, z)/\partial a = -(z + i)Q(a, z)\gamma'(a),$$
$$\partial Q(a, z)/\partial a = (z + i)P(a, z)\alpha'(a)$$

are verified by a straightforward calculation since

$$s'(a) = p(a)\gamma'(a) = r(a)\alpha'(a),$$
$$r'(a) = s(a)\gamma'(a) - 2(h - \tfrac{1}{2})\gamma'(a),$$
$$p'(a) = s(a)\alpha'(a) + 2(h - \tfrac{1}{2})\alpha'(a).$$

Since the integral equation for $E(a, z)$ implies that

$$\partial A(a, z + i)/\partial a = -(z + i)B(a, z + i)\gamma'(a),$$
$$\partial B(a, z + i)/\partial a = (z + i)A(a, z + i)\alpha'(a),$$

the expression

$$P(a, z)B(a, z + i) - Q(a, z)A(a, z + i)$$

is independent of a. Since $\lim E(a, z) = 1$ as $a \searrow 0$, $\lim Q(a, z) = 0$ as $a \searrow 0$. Since

$$B(a, z) = z \int_0^a A(t, z)d\alpha(t),$$

we obtain $\lim B(a, z)/\alpha(a) = z$ as $a \searrow 0$. It follows that $\lim P(a, z) = h - iz$ as $a \searrow 0$. Since the expression

$$P(a, z)B(a, z + i) - Q(a, z)A(a, z + i)$$

has limit zero as $a \searrow 0$, it vanishes identically. Since $A(a, z)$ and $B(a, z)$ have no common zeros,

$$S(a, z) = P(a, z)/A(a, z + i) = Q(a, z)/B(a, z + i)$$

is an entire function. Since $\partial S(a, z)/\partial a = 0$, $S(a, z)$ is independent of a. Since $\lim S(a, z) = h - iz$ as $a \searrow 0$, $S(a, z) = h - iz$ for all a. The desired recurrence relations for $A(a, z)$ and $B(a, z)$ follow.

Note that the hypotheses of Theorem 51 are satisfied with $\tau(t) = t$, $q(t) = r(t) = \sigma(t) = 0$, and

$$p(t) = \tanh^{h-\frac{1}{2}}(\tfrac{1}{2}t) = s(t)^{-1}.$$

Therefore if we introduce the function

$$\Phi(a, z) = A(a, z)\tanh^{h-\frac{1}{2}}(\tfrac{1}{2}a) - iB(a, z)\coth^{h-\frac{1}{2}}(\tfrac{1}{2}a)$$

we can conclude that the limit

$$W(z) = \lim_{a\to\infty} e^{iaz}\Phi(a, z)$$

exists for $y \geq 0$. The limit function is analytic in the upper half-plane and has no zeros in the half-plane. The function $e^{iaz}\Phi(a, z)/W(z)$ is bounded, and bounded away from zero, in the upper half-plane, for each fixed a. The recurrence relations for $A(a, z)$ and $B(a, z)$ imply that

$$(h - iz)\Phi(a, z + i) = (\tfrac{1}{2} - iz)\Phi(a, z)e^a + (h - \tfrac{1}{2})\Phi^*(a, z).$$

Since $\Phi^*(a, z)/\Phi(a, z)$ is bounded by one in the upper half-plane,

$$(h - iz)W(z + i) = (\tfrac{1}{2} - iz)W(z).$$

We can now write

$$W(z) = T(z)\Gamma(\tfrac{1}{2} - iz)/\Gamma(h - iz)$$

where $T(z)$ is an entire function which is periodic of period i and has no zeros. Since $\Gamma(\frac{1}{2} - iz)/\Gamma(h - iz)$ is of bounded type in the upper half-plane by Problem 243 and the recurrence relation for the gamma function, $E(a, z)/T(z)$ is of bounded type in the upper half-plane for every index a. Since $E(a, z)$ is of Pólya class, $E(a, z)/E(a, z + i\epsilon)$ is bounded by one in the half-plane for every $\epsilon > 0$. It follows that $T(z + i\epsilon)/T(z)$ is of bounded type in the half-plane when $\epsilon > 0$. Since the function is periodic of period i, it is a constant by Problem 244. Since ϵ is arbitrary, we can conclude that $T(z)$ is the exponential of a linear function. Since $T(z + i) = T(z)$ and since $T^*(z) = T(-z)$, $T(z)$ is a constant. Since

$$W(0) = \lim_{a\to\infty} \tanh^{h-\frac{1}{2}}(\tfrac{1}{2}a) = 1,$$

we obtain

$$T(0) = \Gamma(h)/\Gamma(\tfrac{1}{2}).$$

The theorem now follows from Theorem 51.

PROBLEM 245. Show that

$$A(a, z) = \cosh^{2h}(\tfrac{1}{2}a)F(h - iz, h + iz; h; -\sinh^2(\tfrac{1}{2}a)),$$
$$B(a, z) = \sinh^{2h}(\tfrac{1}{2}a)(z/h)F(h - iz, h + iz; h + 1; -\sinh^2(\tfrac{1}{2}a))$$

in Theorem 53. If $f(t)$ and $g(t)$ are elements of $L^2(-\infty, +\infty)$ which vanish outside of $(-a, a)$, such that $g(t) = e^t f(t)$ for almost all t, show that their eigentransforms $F(z)$ and $G(z)$, defined as in Problem 231, are related by

$$G(z) = F(z+i) + (h-\tfrac{1}{2})[F(z+i) - F(-z)]/(\tfrac{1}{2} - iz).$$

Show that $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 52.

PROBLEM 246. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 52, show that there exists an index a in Theorem 53 such that the transformation $F(z) \to S(z)F(z)$ is an isometry of $\mathcal{H}(E(a))$ onto $\mathcal{H}(E)$ for some even entire function $S(z)$ which is real for real z and periodic of period i.

54. GENERAL GAUSS SPACES

A more complicated recurrence relation holds for the general Gauss spaces.

THEOREM 54. Let h and k be numbers, $(h-1)(k-1) \neq 0$, and let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$. Assume that $F(z+i)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\begin{aligned}&\langle F(t+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[F(t+i) - F(-t)]/(\tfrac{1}{2} - it)\\ &\qquad - 2(h-1)(k-1)F(t+i)/(1-it), G(t)\rangle\\ &= \langle F(t), G(t+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[G(t+i) - G(-t)]/(\tfrac{1}{2} - it)\\ &\qquad - 2(h-1)(k-1)G(t+i)/(1-it)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space which vanish at the origin. Then there exist a real number u and an imaginary number v such that $L(z) = A(z)u + B(z)v$ has value 1 at $-i$ and such that the identity

$$\begin{aligned}&\langle F(t+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[F(t+i) - F(-t)]/(\tfrac{1}{2} - it)\\ &\qquad - 2(h-1)(k-1)[F(t+i) - L(t)F(0)]/(1-it), G(t)\rangle\\ &= \langle F(t), G(t+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[G(t+i) - G(-t)]/(\tfrac{1}{2} - it)\\ &\qquad - 2(h-1)(k-1)[G(t+i) - L(t)G(0)]/(1-it)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space. There exist real numbers p, r, and s such that $1 = s^2 - pr$ and such that

$$\begin{aligned}&A(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz)\\ &\qquad - 2(h-1)(k-1)[A(z+i) - L(z)A(0)]/(1-iz)\\ &\qquad - 2\pi i(h-1)(k-1)vK(0,z) = A(z)s - iB(z)r,\end{aligned}$$

$$\begin{aligned}&B(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz)\\ &\qquad - 2(h-1)(k-1)B(z+i)/(1-iz) - 2\pi i(h-1)(k-1)uK(0,z)\\ &\qquad = iA(z)p + B(z)s.\end{aligned}$$

These numbers are related by the identity

$$2hk - 4(h - \tfrac{1}{2})(k - \tfrac{1}{2})A(0)u + 2(h - 1)(k - 1)A(0)^2u^2 = A(0)(su - ipv).$$

Proof of Theorem 54. Let $P(z)$ be the choice of an entire function which has value 1 at $-i$ such that $[F(z) - P(z)F(-i)]/(1 - iz)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$. Since we assume that $F(z + i)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$, the transformation

$$F(z) \to F(z + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(z + i) - F(-z)]/(\tfrac{1}{2} - iz)$$
$$- 2(h - 1)(k - 1)[F(z + i) - P(z)F(0)]/(1 - iz)$$

is everywhere defined in $\mathcal{H}(E)$. Since the transformation has a closed graph, it is bounded. By hypothesis, the expression

$$\langle F(t + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(t + i) - F(-t)]/(\tfrac{1}{2} - it)$$
$$- 2(h - 1)(k - 1)[F(t + i) - P(t)F(0)]/(1 - it), G(t)\rangle$$
$$- \langle F(t), G(t + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[G(t + i) - G(-t)]/(\tfrac{1}{2} - it)$$
$$- 2(h - 1)(k - 1)[G(t + i) - P(t)G(0)]/(1 - it)\rangle$$

vanishes for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$ which vanish at the origin. The expression depends continuously on $F(z)$ for each fixed $G(z)$. If $F(z)$ and $G(z)$ are interchanged, the expression is conjugated and multiplied by -1. Since we assume that $(h - 1)(k - 1) \neq 0$, the expression is of the form

$$2(h - 1)(k - 1)\langle F(t), Q(t)\rangle\bar{G}(0) - 2(h - 1)(k - 1)F(0)\langle Q(t), G(t)\rangle$$

for some fixed element $Q(z)$ of $\mathcal{H}(E)$. Then

$$L(z) = P(z) + (1 - iz)Q(z)$$

is an entire function which has value 1 at $-i$, such that $[F(z) - L(z)F(-i)]/(1 - iz)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$, and such that the identity

$$\langle F(t + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(t + i) - F(-t)]/(\tfrac{1}{2} - it)$$
$$- 2(h - 1)(k - 1)[F(t + i) - L(t)F(0)]/(1 - it), G(t)\rangle$$
$$= \langle F(t), G(t + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[G(t + i) - G(-t)]/(\tfrac{1}{2} - it)$$
$$- 2(h - 1)(k - 1)[G(t + i) - L(t)G(0)]/(1 - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$F(\beta + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(\beta + i) - F(-\beta)]/(\tfrac{1}{2} - i\beta)$$
$$- 2(h - 1)(k - 1)[F(\beta + i) - L(\beta)F(0)]/(1 - i\beta)$$
$$= \bar{G}(\alpha + i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[\bar{G}(\alpha + i) - \bar{G}(-\alpha)]/(\tfrac{1}{2} + i\bar{\alpha})$$
$$- 2(h - 1)(k - 1)[\bar{G}(\alpha + i) - \bar{L}(\alpha)\bar{G}(0)]/(1 + i\bar{\alpha}).$$

An equivalent identity is

$$K(w, z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[K(w, z+i) - K(w, -z)]/(\tfrac{1}{2} - iz)$$
$$- 2(h-1)(k-1)[K(w, z+i) - L(z)K(w, 0)]/(1 - iz)$$
$$= K(w+i, z) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[K(w+i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w})$$
$$- 2(h-1)(k-1)[K(w+i, z) - \bar{L}(w)K(0, z)]/(1 + i\bar{w}).$$

But

$$\pi(z+i-\bar{w})K(w, z+i) - \pi(z+i-\bar{w})K(w+i, z)$$
$$= B(z+i)\bar{A}(w) - A(z+i)\bar{B}(w) - B(z)\bar{A}(w+i) + A(z)\bar{B}(w+i),$$

$$\pi(z+i-\bar{w})[K(w, z+i) - K(w, -z)]/(\tfrac{1}{2} - iz)$$
$$- \pi(z+i-\bar{w})[K(w+i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w})$$
$$= \bar{A}(w)[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz)$$
$$- \bar{B}(w)[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz)$$
$$- B(z)[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w})$$
$$+ A(z)[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}),$$

$$\pi(z+i-\bar{w})[K(w, z+i) - L(z)K(w, 0)]/(1 - iz)$$
$$- \pi(z+i-\bar{w})[K(w+i, z) - \bar{L}(w)K(0, z)]/(1 + i\bar{w})$$
$$= \bar{A}(w)B(z+i)/(1 - iz) - \bar{B}(w)[A(z+i) - L(z)A(0)]/(1 - iz)$$
$$- B(z)[\bar{A}(w+i) - \bar{L}(w)\bar{A}(0)]/(1 + i\bar{w}) + A(z)\bar{B}(w+i)/(1 + i\bar{w})$$
$$- \pi i L(z)K(w, 0) + \pi i \bar{L}(w)K(0, z).$$

It follows that

$$\bar{A}(w)\{B(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz)$$
$$- 2(h-1)(k-1)B(z+i)/(1 - iz)\}$$
$$- \bar{B}(w)\{A(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz)$$
$$- 2(h-1)(k-1)[A(z+i) - L(z)A(0)]/(1 - iz)\}$$
$$- B(z)\{\bar{A}(w+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w})$$
$$- 2(h-1)(k-1)[\bar{A}(w+i) - \bar{L}(w)\bar{A}(0)]/(1 + i\bar{w})\}$$
$$+ A(z)\{\bar{B}(w+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w})$$
$$- 2(h-1)(k-1)\bar{B}(w+i)/(1 + i\bar{w})\}$$
$$= -2\pi i(h-1)(k-1)L(z)K(w, 0) + 2\pi i(h-1)(k-1)\bar{L}(w)K(0, z).$$

Since $A(z)$ and $B(z)$ are linearly independent, the functions

$$A(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz)$$
$$- 2(h-1)(k-1)[A(z+i) - L(z)A(0)]/(1 - iz)$$

and

$$B(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) \\ - 2(h-1)(k-1)B(z+i)/(1-iz)$$

can be written as linear combinations of $L(z)$, $A(z)$, $B(z)$, and $K(0, z)$. A contradiction is obtained on assuming that these four functions are linearly independent and making corresponding substitutions. We can therefore conclude that the functions are linearly dependent. Since we assume that the space $\mathcal{H}(E)$ is not one-dimensional, the last three of the functions are linearly independent. We can therefore write

$$L(z) = A(z)u + B(z)v + \lambda K(0, z)$$

for some numbers u, v, and λ. Substitution in the main identity will show that λ is real. Since we can add a real multiple of $(1 - iz)K(0, z)$ to $L(z)$ without changing the defining property of the function, we can always choose $L(z)$ so that $\lambda = 0$. The main identity now reads

$$\begin{aligned}
&\bar{A}(w)\{B(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) \\
&\qquad - 2(h-1)(k-1)B(z+i)/(1-iz) - 2\pi i(h-1)(k-1)\bar{u}K(0, z)\} \\
&-\bar{B}(w)\{A(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) \\
&\qquad - 2(h-1)(k-1)[A(z+i) - L(z)A(0)]/(1-iz) \\
&\qquad\qquad + 2\pi i(h-1)(k-1)\bar{v}K(0, z)\} \\
&-B(z)\{\bar{A}(w+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\
&\qquad - 2(h-1)(k-1)[\bar{A}(w+i) - \bar{L}(w)\bar{A}(0)]/(1+i\bar{w}) \\
&\qquad\qquad - 2\pi i(h-1)(k-1)vK(w, 0)\} \\
&+A(z)\{\bar{B}(w+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\
&\qquad - 2(h-1)(k-1)\bar{B}(w+i)/(1+i\bar{w}) \\
&\qquad\qquad + 2\pi i(h-1)(k-1)uK(w, 0)\} = 0.
\end{aligned}$$

Since $A(z)$ and $B(z)$ are linearly independent, there exist numbers p, r, and s, p and r real, such that

$$\begin{aligned}
&A(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) \\
&\qquad - 2(h-1)(k-1)[A(z+i) - L(z)A(0)]/(1-iz) \\
&\qquad\qquad + 2\pi i(h-1)(k-1)\bar{v}K(0, z) = A(z)s - iB(z)r,
\end{aligned}$$

$$\begin{aligned}
&B(z+i) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) \\
&\quad - 2(h-1)(k-1)B(z+i)/(1-iz) - 2\pi i(h-1)(k-1)\bar{u}K(0, z) \\
&\qquad\qquad = iA(z)p + B(z)\bar{s}.
\end{aligned}$$

Since $E^*(z) = E(-z)$, the same formulas hold with u replaced by $\bar{u}$, v replaced by $-\bar{v}$, and s replaced by $\bar{s}$. Since $A(z)$, $B(z)$, and $K(0, z)$ are linearly independent, u and s are real and v is imaginary. The recurrence relations can now be written

$$z(h - iz)(k - iz)A(z + i) = a(z)A(z) + c(z)B(z),$$
$$z(h - iz)(k - iz)B(z + i) = b(z)A(z) + d(z)B(z)$$

where

$$\begin{aligned} a(z) &= 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})z(1 - iz) - 2(h - 1)(k - 1)A(0)uz(\tfrac{1}{2} - iz) \\ &\quad + sz(\tfrac{1}{2} - iz)(1 - iz), \\ b(z) &= ipz(\tfrac{1}{2} - iz)(1 - iz), \\ c(z) &= 2i(h - 1)(k - 1)A(0)v(\tfrac{1}{2} - iz) - irz(\tfrac{1}{2} - iz)(1 - iz), \\ d(z) &= -2(h - \tfrac{1}{2})(k - \tfrac{1}{2})z(1 - iz) \\ &\quad + 2i(h - 1)(k - 1)A(0)u(\tfrac{1}{2} - iz)(1 - iz) + sz(\tfrac{1}{2} - iz)(1 - iz). \end{aligned}$$

Starring each side of the equations and replacing z by $z + i$, we obtain

$$\begin{aligned} &(z + i)(h - 1 + iz)(k - 1 + iz)A(z) \\ &\qquad = a^*(z + i)A(z + i) + c^*(z + i)B(z + i), \\ &(z + i)(h - 1 + iz)(k - 1 + iz)B(z) \\ &\qquad = b^*(z + i)A(z + i) + d^*(z + i)B(z + i). \end{aligned}$$

It follows that

$$\begin{aligned} &z(z + i)(h - iz)(h - 1 + iz)(k - iz)(k - 1 + iz)A(z) \\ &\qquad = [a^*(z + i)a(z) + c^*(z + i)b(z)]A(z) \\ &\qquad\qquad + [a^*(z + i)c(z) + c^*(z + i)d(z)]B(z), \\ &z(z + i)(h - iz)(h - 1 + iz)(k - iz)(k - 1 + iz)B(z) \\ &\qquad = [b^*(z + i)a(z) + d^*(z + i)b(z)]A(z) \\ &\qquad\qquad + [b^*(z + i)c(z) + d^*(z + i)d(z)]B(z). \end{aligned}$$

Since

$$a^*(z + i) = d(z), \qquad b^*(z + i) = -b(z),$$
$$c^*(z + i) = -c(z), \qquad d^*(z + i) = a(z),$$

these equations reduce to the condition

$$z(z + i)(h - iz)(h - 1 + iz)(k - iz)(k - 1 + iz) = a(z)d(z) - b(z)c(z).$$

By comparing the coefficients of the highest power of z on each side of the equation, we obtain $1 = s^2 - pr$. The equation now reduces to the identity stated at the end of the theorem.

PROBLEM 247. In Theorem 54 let λ be a solution of the equation $\lambda^2 - 2\lambda s + 1 = 0$, and let U and V be numbers such that $sU + ipV = \lambda U$ and $sV - irU = \lambda V$. If $F(z) = A(z)U + B(z)V$, show that

$$F(z+i) + 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(z+i) - F(-z)]/(\tfrac{1}{2} - iz) - 2(h-1)(k-1)[F(z+i) - L(z)F(0)]/(1 - iz) = \lambda F(z).$$

Show that

$$\lim_{y \to +\infty} F(iy + i)/F(iy) = \lambda$$

if $F(z)$ does not belong to $\mathcal{H}(E)$ or if $F(z)$ is an odd function which belongs to $\mathcal{H}(E)$ and does not vanish identically.

PROBLEM 248. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 54 for some h and k, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same h and k.

PROBLEM 249. In Problem 248 assume that the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ has dimension zero or one and that $E(a, 0) = E(b, 0)$. Show that the orthogonal complement is spanned by a function $F(z)$ of the form

$$F(z) = A(a, z)U + B(a, z)V = A(b, z)U + B(b, z)V$$

where U and V are real numbers which satisfy the hypotheses of Problem 247 for $E(a, z)$ and for $E(b, z)$. Show that such a function $F(z)$ vanishes identically.

PROBLEM 250. Let $\{\mathcal{H}(E(a))\}$, $a > 0$, be a given family of spaces associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$. Assume that $m(t)$ is an absolutely continuous function of t and that there exist numbers h and k such that the hypotheses of Theorem 54 are satisfied for every index a. Show that the quantities $p(a)$, $r(a)$, $s(a)$, $u(a)$, and $v(a)$ defined by the theorem are absolutely continuous functions of a and that

$$u'(a) = iv(a)\alpha'(a) \quad \text{and} \quad v(a) = -iu(a)\gamma'(a),$$

$$p(a)\gamma'(a) = s'(a) = r(a)\alpha'(a),$$

$$\begin{aligned} r(a)p'(a) - s(a)s'(a) &= s(a)s'(a) - p(a)r'(a) \\ &= 4(h - \tfrac{1}{2})(k - \tfrac{1}{2})s'(a) - 4(h-1)(k-1)A(a, 0)u(a)s'(a) \end{aligned}$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist. Show that $p(a) > 0$, $r(a) > 0$, and $s(a) > 1$. If

$$\lim_{a \searrow 0} \; [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 1$ as $a \searrow 0$.

PROBLEM 251. If $s(a) = \cosh a$ in Problem 250, show that

$$B(a, z)\sqrt{\gamma'(a)} = f(a, z)$$

where $f(a, z)$ is a solution of the equation

$$\begin{aligned} z^2 f(a, z) &= -f''(a, z) \\ &\quad + \frac{2(h - \frac{1}{2})(k - \frac{1}{2})\cosh a + h^2 - h + k^2 - k + \frac{1}{4}}{\sinh^2 a} f(a, z) \end{aligned}$$

for each fixed z. Show that

$$f(a, z) = \sinh^h(a) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}a) g(-\sinh^2(\tfrac{1}{2}a), z)$$

where $g(a, z)$ is a solution of the hypergeometric equation

$$a(1 - a)g''(a, z) + [h + k - (2h + 1)a]g'(a, z) - (h^2 + z^2)g(a, z) = 0$$

for each fixed z. If $h + k \geq 1$, show that $g(a, z)$ is a constant multiple of the hypergeometric function $F(h - iz, h + iz; h + k; a)$ for each fixed z. Show that there exists an even entire function $S(z)$, which is real for real z and periodic of period i, such that

$$\begin{aligned} &B(a, z)\sqrt{\gamma'(a)} \\ &\quad = \sinh^h(a) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}a) zS(z) F(h - iz, h + iz; h + k; -\sinh^2(\tfrac{1}{2}a)) \end{aligned}$$

for all a and z.

PROBLEM 252. Let h and k be given positive numbers, and let

$$W(z) = \frac{\Gamma(\frac{1}{2} - iz)\Gamma(1 - iz)}{\Gamma(h - iz)\Gamma(k - iz)} \frac{2^h \Gamma(h + k)}{\Gamma(\frac{1}{2})}.$$

For every positive number a show that the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ are of bounded type and of mean type at most a in the upper half-plane and such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt < \infty$$

is equal isometrically to a space $\mathcal{H}(E(a))$ such that $E^*(a, z) = E(a, -z)$ and $E(a, 0) = 1$. Show that the hypotheses of Theorem 54 are satisfied for every index a. Show that the hypotheses of Problems 250 and 251 are satisfied for a suitable choice of $m(t)$. Show that

$$\alpha(a)\sqrt{\gamma'(a)} = \sinh^h(a) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}a) F(h, h; h + k; -\sinh^2(\tfrac{1}{2}a)),$$
$$\sqrt{\gamma'(a)} = W(0)^{-1} \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}a) F(h, 1 - k; 1; -\sinh^{-2}(\tfrac{1}{2}a)).$$

If $f(t)$ belongs to $L^2(0, \infty)$ and vanishes outside of $(0, a)$, show that

$$\pi F(z) = z \int_0^\infty f(t) \sinh^h(t) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}t) \times F(h - iz, h + iz; h + k; -\sinh^2(\tfrac{1}{2}t)) dt$$

is an odd element of $\mathcal{H}(E(a))$ and that

$$\pi \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int_0^\infty |f(t)|^2 dt.$$

Show that every odd element of $\mathcal{H}(E(a))$ is of this form.

PROBLEM 253. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 54 for some h and k, show that a space $\mathcal{H}(E_1)$ exists such that

$$(1 + iz/h)[B_1(z)A(ih) - A_1(z)B(ih)] = B(z)A(ih) - A(z)B(ih).$$

Show that it satisfies the hypotheses of Theorem 54 with h replaced by $h + 1$.

55. SPECIAL KUMMER SPACES

The Kummer spaces are a limiting case of the Gauss spaces and satisfy a similar identity. A real parameter k appears in the identity. The special case $k = 1$ is of particular interest.

THEOREM 55. Let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$. Assume that $[F(z + i) - F(-z)]/(\frac{1}{2} - iz)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle [F(t + i) - F(-t)]/(\tfrac{1}{2} - it), G(t)\rangle = \langle F(t), [G(t + i) - G(-t)]/(\tfrac{1}{2} - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space. Then there exist real numbers p, r, and s such that $pr = s^2$ and such that

$$[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r,$$
$$[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)s.$$

Proof of Theorem 55. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$[F(\beta+i) - F(-\beta)]/(\tfrac{1}{2} - i\beta) = [\bar{G}(\alpha+i) - \bar{G}(-\alpha)]/(\tfrac{1}{2} + i\bar{\alpha}).$$

An equivalent identity is

$$[K(w, z+i) - K(w, -z)]/(\tfrac{1}{2} - iz) = [K(w+i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w}).$$

As in the proof of Theorem 52 it follows that

$$\begin{aligned}\bar{A}(w)[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) - \bar{B}(w)[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz)\\ - B(z)[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w})\\ + A(z)[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}) = 0\end{aligned}$$

and that

$$[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) = A(z)s - iB(z)r,$$
$$[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) = iA(z)p + B(z)s$$

for some numbers p, r, and s. These recurrence relations can be written

$$A(z+i) = a(z)A(z) + c(z)B(z)$$
$$B(z+i) = b(z)A(z) + d(z)B(z)$$

where

$$a(z) = 1 + (\tfrac{1}{2} - iz)s, \qquad b(z) = i(\tfrac{1}{2} - iz)p,$$
$$c(z) = -i(\tfrac{1}{2} - iz)r, \qquad d(z) = -1 + (\tfrac{1}{2} - iz)s.$$

As in the proof of Theorem 52 it follows that

$$\begin{aligned}A(z) = {} & [a^*(z+i)a(z) + c^*(z+i)c(z)]A(z)\\ & + [a^*(z+i)b(z) + c^*(z+i)d(z)]B(z),\end{aligned}$$

$$\begin{aligned}B(z) = {} & [c^*(z+i)a(z) + d^*(z+i)c(z)]A(z)\\ & + [c^*(z+i)b(z) + d^*(z+i)d(z)]B(z).\end{aligned}$$

Since

$$a^*(z+i) = -d(z), \qquad b^*(z+i) = b(z),$$
$$c^*(z+i) = c(z), \qquad d^*(z+i) = -a(z),$$

these equations reduce to the condition

$$1 = -a(z)d(z) + b(z)c(z),$$

which implies that $pr = s^2$.

PROBLEM 254. In Theorem 55 let $F(z) = A(z)u + B(z)v$ where u is a real number and v is an imaginary number such that $ipv = su$ and $-iru = sv$. Show that

$$[F(z+i) - F(-z)]/(\tfrac{1}{2} - iz) = 2sF(z)$$

and that

$$\lim_{y\to+\infty} y^{-1}F(iy+i)/F(iy) = 2s$$

if u and v are not both zero. If $G(z) = F^*(z)$, show that $G(z+i) = G(-z)$. Show that p, r, and s are nonzero. Show that all zeros of $G(z)$ lie above the real axis and that u and iv have the same sign. Show that p, r, and s have the same sign. If u and iv are chosen positive, show that $F(iy) > 0$ for $y > 0$. Show that p, r, and s are positive.

It is convenient to parametrize Kummer spaces in decreasing order, so that $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ when $a < b$.

PROBLEM 255. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ isometrically and such that $E(b, z)/E(a, z)$ has no real zeros. Show that $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 55 if $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 55 and $E^*(b, z) = E(b, -z)$. Show that the domain of multiplication by z is dense in any space which satisfies the hypotheses of Theorem 55.

PROBLEM 256. Let $\{\mathcal{H}(E(a))\}$, $a > 0$, be a given family of spaces associated with a nonincreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$. If $m(t)$ is an absolutely continuous function of t and if $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 55 for every index a, show that $p(a)$, $r(a)$, and $s(a)$ are absolutely continuous functions of a and that

$$p(a)\gamma'(a) = s'(a) = r(a)\alpha'(a),$$
$$r(a)p'(a) - s(a)s'(a) = 2s'(a) = s(a)s'(a) - p(a)r'(a)$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist. Show that

$$[p(a)/s(a)] \exp [-2/s(a)] \quad \text{and} \quad [r(a)/s(a)] \exp [2/s(a)]$$

are constants. Show that $s(a) > s(b)$ when $a < b$ and $m(t)$ is not constant in (a, b). If

$$\lim_{a \to \infty} [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 0$ as $a \to \infty$.

56. CONSTRUCTION OF SPECIAL KUMMER SPACES

We now construct the special Kummer spaces

THEOREM 56. Let $\alpha(t)$ and $\gamma(t)$ be differentiable functions of $t > 0$ such that

$$-t\alpha'(t) = \exp(-t) \quad \text{and} \quad -t\gamma'(t) = \exp(t),$$

and let $\beta(t) = 0$. Then there exists a unique family $(E(t, z))$ of entire functions of Pólya class, $t > 0$, such that $E(t, z)$ is a continuous function of t for every z, such that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$, and such that $\lim E(t, z) = 1$ as $t \to \infty$ for all complex z. A space $\mathcal{H}(E(a))$ exists for every $a > 0$ and $E^*(a, z) = E(a, -z)$. The space is the set of entire functions $F(z)$ such that $F(z)/\Gamma(\frac{1}{2} - iz)$ and $F^*(z)/\Gamma(\frac{1}{2} - iz)$ are of bounded type and of mean type at most $\log (4/a)$ in the upper half-plane, and such that

$$\int_{-\infty}^{+\infty} |F(t)/\Gamma(\tfrac{1}{2} - it)|^2 dt < \infty.$$

The integral is then equal to

$$\int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt/\Gamma(\tfrac{1}{2})^2$$

for every $F(z)$ in $\mathcal{H}(E(a))$. The recurrence relations

$$[A(a, z + i) - A(a, -z)]/(\tfrac{1}{2} - iz) = A(a, z)s(a) - iB(a, z)r(a),$$
$$[B(a, z + i) - B(a, -z)]/(\tfrac{1}{2} - iz) = iA(a, z)p(a) + B(a, z)s(a)$$

hold with $s(a) = 2/a$,

$$p(a) = 2a^{-1} \exp(-a) \quad \text{and} \quad r(a) = 2a^{-1} \exp(a).$$

Proof of Theorem 56. Since $\int_1^\infty \alpha(t)d\gamma(t) > -\infty$, the existence of the functions $E(a, z)$ is given by Theorem 41 and Problem 180. To obtain the recurrence relations, define

$$P(a, z) = (\tfrac{1}{2} - iz)[A(a, z)s(a) - iB(a, z)r(a)] + A(a, z),$$
$$Q(a, z) = (\tfrac{1}{2} - iz)[iA(a, z)p(a) + B(a, z)s(a)] - B(a, z)$$

with $p(a)$, $r(a)$, and $s(a)$ defined as in the statement of the theorem. The equations

$$\partial P(a, z)/\partial a = -(z + i)Q(a, z)\gamma'(a),$$
$$\partial Q(a, z)/\partial a = (z + i)P(a, z)\alpha'(a)$$

hold because

$$s'(a) = p(a)\gamma'(a) = r(a)\alpha'(a),$$
$$p'(a) = s(a)\alpha'(a) + 2\alpha'(a),$$
$$r'(a) = s(a)\gamma'(a) - 2\gamma'(a).$$

Since

$$\partial A(a, z + i)/\partial a = -(z + i)B(a, z + i)\gamma'(a),$$
$$\partial B(a, z + i)/\partial a = (z + i)A(a, z + i)\alpha'(a),$$

the expression

$$P(a, z)B(a, z + i) - Q(a, z)A(a, z + i)$$

is independent of a. Since $\lim E(a, z) = 1$ and $\lim B(a, z)/\alpha(a) = z$ as $a \to \infty$, the expression goes to zero as $a \to \infty$, and so vanishes identically. Since $A(a, z)$ and $B(a, z)$ have no common zeros,

$$S(a, z) = P(a, z)/A(a, z + i) = Q(a, z)/B(a, z + i)$$

is an entire function. Since $\partial S(a, z)/\partial a = 0$, $S(a, z)$ is independent of a. Since $\lim S(a, z) = 1$ as $a \to \infty$, $S(a, z) = 1$ identically. The recurrence relations for $A(a, z)$ and $B(a, z)$ now follow.

If we introduce the functions

$$\Phi(a, z) = A(a, z)e^{-\frac{1}{2}a} - iB(a, z)e^{\frac{1}{2}a},$$

it follows from Theorem 51 that

$$W(z) = \lim_{a \searrow 0} (4/a)^{iz}\Phi(a, z)$$

exists for $y \geq 0$. The limit function is analytic and without zeros in the upper half-plane. The recurrence relations for $A(a, z)$ and $B(a, z)$ imply that

$$\Phi(a, z + i) = (\tfrac{1}{2} - iz)\Phi(a, z)(4/a) + \Phi^*(a, z).$$

Since $\Phi^*(a, z)/\Phi(a, z)$ is bounded by one in the upper half-plane, it follows that

$$W(z + i) = (\tfrac{1}{2} - iz)W(z).$$

We can now write

$$W(z) = T(z)\Gamma(\tfrac{1}{2} - iz)$$

where $T(z)$ is an entire function which is periodic of period i and has no zeros. The proof of Theorem 53 will now show that $T(z)$ is a constant and that

$$T(z) = \lim_{a \searrow 0} e^{-\frac{1}{2}a} = 1.$$

The theorem follows from Theorem 51.

PROBLEM 257. In Theorem 56, let

$$\Phi(a, z) = A(a, z)e^{-\frac{1}{2}a} - iB(a, z)e^{\frac{1}{2}a},$$
$$\Phi(-a, z) = A(a, z)e^{-\frac{1}{2}a} + iB(a, z)e^{\frac{1}{2}a}$$

for $a > 0$. If $f(x)$ belongs to $L^2(-\infty, +\infty)$ and vanishes in some interval $(-a, a)$, show that its eigentransforms $F(z)$, defined by

$$2\pi F(z) = \int_{-\infty}^{+\infty} f(t)\Phi(t, z)\,|t|^{-\frac{1}{2}}dt$$

belongs to $\mathcal{H}(E(a))$ and that

$$2\pi \int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

for every $F(z)$ in $\mathcal{H}(E(a))$. Show that every element of $\mathcal{H}(E(a))$ is of this form. Let $f(x)$ and $g(x)$ be elements of $L^2(-\infty, +\infty)$ which vanish in $(-a, a)$, such that $g(x) = 4x^{-1}f(x)$ for $x > 0$ and $g(x) = 0$ for $x < 0$. Show that their eigentransforms are related by

$$G(z) = [F(z + i) - F(-z)]/(\tfrac{1}{2} - iz).$$

Show that $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 55.

PROBLEM 258. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 55, show that there exists an index a in Theorem 56 such that the transformation $F(z) \to S(z)F(z)$ is an isometry of $\mathcal{H}(E(a))$ onto $\mathcal{H}(E)$ for some entire function $S(z)$ which is real for real z and periodic of period i.

57. GENERAL KUMMER SPACES

A more complicated recurrence relation holds for the general Kummer spaces.

THEOREM 57. Let k be a real number, $k \neq 1$, and let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$. Assume that $[F(z+i) - F(-z)]/(\frac{1}{2} - iz)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\begin{aligned}\langle (2k-1)[F(t+i) - F(-t)]/(\tfrac{1}{2} - it) - 2(k-1)F(t+i)/(1-it), G(t)\rangle \\ = \langle F(t), (2k-1)[G(t+i) - G(-t)]/(\tfrac{1}{2} - it) \\ - 2(k-1)G(t+i)/(1-it)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space which vanish at the origin. Then there exists a real number u and an imaginary number v such that $L(z) = A(z)u + B(z)v$ has value 1 at $-i$ and such that the identity

$$\begin{aligned}\langle (2k-1)[F(t+i) - F(-t)]/(\tfrac{1}{2} - it) \\ - 2(k-1)[F(t+i) - L(t)F(0)]/(1-it), G(t)\rangle \\ = \langle F(t), (2k-1)[G(t+i) - G(-t)]/(\tfrac{1}{2} - it) \\ - 2(k-1)[G(t+i) - L(t)G(0)]/(1-it)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space. There exist real numbers p, r, and s such that $pr = s^2$ and such that

$$\begin{aligned}(2k-1)[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) \\ - 2(k-1)[A(z+i) - L(z)A(0)]/(1-iz) \\ - 2\pi i(k-1)vK(0,z) = A(z)s - iB(z)r,\end{aligned}$$

$$\begin{aligned}(2k-1)[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) - 2(k-1)B(z+i)/(1-iz) \\ - 2\pi i(k-1)uK(0,z) = iA(z)p + B(z)s.\end{aligned}$$

These numbers are related by the identity

$$2k - 4(k - \tfrac{1}{2})A(0)u + 2(k-1)A(0)^2u^2 = A(0)(su - ipv).$$

Proof of Theorem 57. By the proof of Theorem 54 there exists an entire function $L(z)$ which has value 1 at $-i$ such that $[F(z) - L(z)F(-i)]/(1-iz)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and such that the identity

$$\begin{aligned}\langle (2k-1)[F(t+i) - F(-t)]/(\tfrac{1}{2} - it) \\ - 2(k-1)[F(t+i) - L(t)F(0)]/(1-it), G(t)\rangle \\ = \langle F(t), (2k-1)[G(t+i) - G(-t)]/(\tfrac{1}{2} - it) \\ - 2(k-1)[G(t+i) - L(t)G(0)]/(1-it)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E)$. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$\begin{aligned}(2k-1)[F(\beta+i) - F(-\beta)]/(\tfrac{1}{2} - i\beta) \\ - 2(k-1)[F(\beta+i) - L(\beta)F(0)]/(1-i\beta) \\ = (2k-1)[\bar{G}(\alpha+i) - \bar{G}(-\alpha)]/(\tfrac{1}{2} + i\bar{\alpha}) \\ - 2(k-1)[\bar{G}(\alpha+i) - \bar{L}(\alpha)\bar{G}(0)]/(1+i\bar{\alpha}).\end{aligned}$$

An equivalent identity is

$$\begin{aligned}(2k-1)[K(w, z+i) - K(w, -z)]/(\tfrac{1}{2} - iz) \\ - 2(k-1)[K(w, z+i) - L(z)K(w, 0)]/(1-iz) \\ = (2k-1)[K(w+i, z) - K(-w, z)]/(\tfrac{1}{2} + i\bar{w}) \\ - 2(k-1)[K(w+i, z) - \bar{L}(w)K(0, z)]/(1+i\bar{w}).\end{aligned}$$

As in the proof of Theorem 54, it follows that

$$\begin{aligned}\bar{A}(w)\{(2k-1)[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) - 2(k-1)B(z+i)/(1-iz)\} \\ - \bar{B}(w)\{(2k-1)[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) \\ - 2(k-1)[A(z+i) - L(z)A(0)]/(1-iz)\} \\ - B(z)\{(2k-1)[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\ - 2(k-1)[\bar{A}(w+i) - \bar{L}(w)\bar{A}(0)]/(1+i\bar{w})\} \\ + A(z)\{(2k-1)[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\ - 2(k-1)\bar{B}(w+i)/(1+i\bar{w})\} \\ = -2\pi i(k-1)L(z)K(w, 0) + 2\pi i(k-1)\bar{L}(w)K(0, z).\end{aligned}$$

As in the proof of Theorem 54 we can choose $L(z)$ of the form

$$L(z) = A(z)u + B(z)v$$

for some numbers u and v. The identity then reads

$$\begin{aligned}\bar{A}(w)\{(2k-1)[B(z+i) - B(-z)]/(\tfrac{1}{2} - iz) \\ - 2(k-1)B(z+i)/(1-iz) - 2\pi i(k-1)\bar{u}K(0, z)\} \\ - \bar{B}(w)\{(2k-1)[A(z+i) - A(-z)]/(\tfrac{1}{2} - iz) \\ - 2(k-1)[A(z+i) - L(z)A(0)]/(1-iz) + 2\pi i(k-1)\bar{v}K(0, z)\} \\ - B(z)\{(2k-1)[\bar{A}(w+i) - \bar{A}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\ - 2(k-1)[\bar{A}(w+i) - \bar{L}(w)\bar{A}(0)]/(1+i\bar{w}) - 2\pi i(k-1)vK(w, 0)\} \\ + A(z)\{(2k-1)[\bar{B}(w+i) - \bar{B}(-w)]/(\tfrac{1}{2} + i\bar{w}) \\ - 2(k-1)\bar{B}(w+i)/(1+i\bar{w}) + 2\pi i(k-1)uK(w, 0)\} = 0.\end{aligned}$$

Since $A(z)$ and $B(z)$ are linearly independent, there exist numbers p, r, and s, p and r real, such that

$$(2k-1)[A(z+i)-A(-z)]/(\tfrac{1}{2}-iz)$$
$$-2(k-1)[A(z+i)-L(z)A(0)]/(1-iz)$$
$$+2\pi i(k-1)\bar{v}K(0,z)=A(z)s-iB(z)r,$$
$$(2k-1)[B(z+i)-B(-z)]/(\tfrac{1}{2}-iz)-2(k-1)B(z+i)/(1-iz)$$
$$+2\pi i(k-1)\bar{u}K(0,z)=iA(z)p+B(z)\bar{s}.$$

Since $E^*(z)=E(-z)$, the formulas remain valid when u is replaced by $\bar{u}$, v is replaced by $-\bar{v}$, and s is replaced by $\bar{s}$. Since $A(z)$, $B(z)$, and $K(0,z)$ are linearly independent, u and s are real and v is imaginary. The recurrence relations can now be written

$$z(k-iz)A(z+i)=a(z)A(z)+c(z)B(z),$$
$$z(k-iz)B(z+i)=b(z)A(z)+d(z)B(z)$$

where

$$a(z)=(2k-1)z(1-iz)-2(k-1)A(0)uz(\tfrac{1}{2}-iz)$$
$$+sz(\tfrac{1}{2}-iz)(1-iz),$$
$$b(z)=ipz(\tfrac{1}{2}-iz)(1-iz),$$
$$c(z)=2i(k-1)A(0)v(\tfrac{1}{2}-iz)-irz(\tfrac{1}{2}-iz)(1-iz),$$
$$d(z)=-(2k-1)z(1-iz)+2i(k-1)A(0)u(\tfrac{1}{2}-iz)(1-iz)$$
$$+sz(\tfrac{1}{2}-iz)(1-iz).$$

Starring each side of the equations and replacing z by $z+i$, we obtain

$$-i(1-iz)(k-1+iz)A(z)=a^*(z+i)A(z+i)+c^*(z+i)B(z+i),$$
$$-i(1-iz)(k-1+iz)B(z)=b^*(z+i)A(z+i)+d^*(z+i)B(z+i).$$

It follows that

$$z(z+i)(k-iz)(k-1+iz)A(z)$$
$$=[a^*(z+i)a(z)+c^*(z+i)b(z)]A(z)$$
$$+[a^*(z+i)c(z)+c^*(z+i)d(z)]B(z),$$
$$z(z+i)(k-iz)(k-1+iz)B(z)$$
$$=[b^*(z+i)a(z)+d^*(z+i)b(z)]A(z)$$
$$+[b^*(z+i)c(z)+d^*(z+i)d(z)]B(z).$$

Since

$$a^*(z+i)=d(z),\qquad b^*(z+i)=-b(z),$$
$$c^*(z+i)=-c(z),\qquad d^*(z+i)=a(z),$$

these equations reduce to the condition

$$z(z+i)(k-iz)(k-1+iz) = a(z)d(z) - b(z)c(z).$$

By comparing the coefficient of the highest power of z on each side of the equation, we obtain $pr = s^2$. The equation is now equivalent to the identity given at the end of the theorem.

PROBLEM 259. Show that p, r, and s are positive in Theorem 57. Show that the domain of multiplication by z is dense in the space.

PROBLEM 260. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ contains $\mathcal{H}(E(b))$ isometrically and such that $E(b, z)/E(a, z)$ has no real zeros. Show that $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 57 if $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 57 and if $E^*(b, z) = E(b, -z)$.

PROBLEM 261. Let $\{\mathcal{H}(E(a))\}$, $a > 0$, be a given family of spaces associated with a nonincreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))\,dm(t)$$

when $0 < a < b < \infty$. If $m(t)$ is an absolutely continuous function of t and if $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 57 for every index a, show that $p(a)$, $r(a)$, $s(a)$, $u(a)$, and $v(a)$ are absolutely continuous functions of a and that

$$u'(a) = iv(a)\alpha'(a) \quad \text{and} \quad v'(a) = -iu(a)\gamma'(a),$$

$$p(a)\gamma'(a) = s'(a) = r(a)\alpha'(a),$$

$$\begin{aligned} r(a)p'(a) - s(a)s'(a) &= s(a)s'(a) - p(a)r'(a) \\ &= 4(k - \tfrac{1}{2})s'(a) - 4(k-1)A(a, 0)u(a)s'(a) \end{aligned}$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist. If

$$\lim_{a\to\infty} [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 0$ as $a \to \infty$.

PROBLEM 262. If $s(a) = 2/a$ in Problem 261, show that

$$B(a, z)\sqrt{-\gamma'(a)} = f(a, z)$$

where $f(a, z)$ is a solution of the equation

$$z^2 f(a, z) = -[a^2 f'(a, z)]' + [\tfrac{1}{4}a^2 + (k - \tfrac{1}{2})a - \tfrac{1}{4}]f(a, z)$$

for each fixed z. Show that $f(a, z) = g(a, z)/a$ where $g(a, z)$ is a solution of the Whittaker equation

$$g''(a, z) + [-\tfrac{1}{4} + (\tfrac{1}{2} - k)a^{-1} + (\tfrac{1}{4} + z^2)a^{-2}]g(a, z) = 0$$

for each fixed z. Show that $g(a, z)$ is a constant multiple of the Whittaker function $W_{\frac{1}{2}-k,iz}(a)$ for each fixed z. Show that there exists an even entire function $S(z)$ which is real for real z and periodic of period i such that

$$B(a, z)\sqrt{-\gamma'(a)} = zS(z)W_{\frac{1}{2}-k,iz}(a)/a$$

for all a and z.

PROBLEM 263. If k is a given positive number, let

$$W(z) = \frac{\Gamma(\tfrac{1}{2} - iz)\Gamma(1 - iz)}{\Gamma(\tfrac{1}{2})\Gamma(k - iz)}.$$

For every positive number a show that the set of entire functions $F(z)$ such that $F(z)/W(z)$ and $F^*(z)/W(z)$ are of bounded type and of mean type at most $\log(4/a)$ in the upper half-plane and such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt < \infty$$

is equal isometrically to a space $\mathcal{H}(E(a))$ such that $E^*(a, z) = E(a, -z)$ and $E(a, 0) = 1$. Show that the hypotheses of Theorem 57 are satisfied for every index a. Show that the hypotheses of Problems 261 and 262 are satisfied with a suitable choice of $m(t)$. Show that

$$\alpha(a)\sqrt{-\gamma'(a)} = W_{\frac{1}{2}-k,0}(a)/a,$$

$$\sqrt{-\gamma'(a)} = W(0)^{-1}M_{\frac{1}{2}-k,0}(a)/a.$$

If $f(t)$ belongs to $L^2(0, \infty)$ and vanishes in $(0, a)$, show that

$$\pi F(z) = z\int_0^\infty f(t)W_{\frac{1}{2}-k,iz}(t)/t\,dt$$

is an odd element of $\mathcal{H}(E(a))$ and that

$$\pi\int_{-\infty}^{+\infty} |F(t)/W(t)|^2 dt = \int_0^\infty |f(t)|^2 dt.$$

Show that every odd element of $\mathcal{H}(E(a))$ is of this form.

PROBLEM 264. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 57 for some index k, show that a space $\mathcal{H}(E_1)$ exists such that

$$(1 + iz/k)[B_1(z)A(ik) - A_1(z)B(ik)] = B(z)A(ik) - A(z)B(ik).$$

Show that it satisfies the hypotheses of Theorem 57 with k replaced by $k + 1$.

An application of the Gauss and Kummer expansions appears in M. Rosenblum's theory of the Hilbert matrix.

PROBLEM 265. If $\nu > -1$, the Hardy space $\mathfrak{D}_\nu$ is defined to be the set of functions $F(z)$, analytic in the upper half-plane, of the form

$$F(z) = \int_0^\infty t^{\frac{1}{2}\nu} e^{izt} f(t)\,dt$$

with $f(x)$ in $L^2(0, \infty)$. Show that $\mathfrak{D}_\nu$ is a Hilbert space in the norm

$$\|F\|_\nu^2 = \int_0^\infty |f(t)|^2 dt.$$

If $\nu > 0$, show that a function $F(z)$, analytic in the upper half-plane, belongs to $\mathfrak{D}_\nu$ if, and only if,

$$\int_0^\infty \int_{-\infty}^{+\infty} |F(x + iy)|^2 y^{\nu-1} dx\, dy < \infty.$$

In this case show that the integral is equal to $2\pi 2^{-\nu}\Gamma(\nu)\, \|F\|_\nu^2$. If $\nu = 0$, show that a function $F(z)$, analytic in the upper half-plane, belongs to $\mathfrak{D}_\nu$ if, and only if,

$$\sup_{y>0} \int_{-\infty}^{+\infty} |F(x + iy)|^2 dx < \infty.$$

In this case show that the integral is equal to $2\pi \|F\|_0^2$.

PROBLEM 266. Show that the function

$$\Gamma(1 + \nu)(i\bar{w} - iz)^{-1-\nu}$$

belongs to $\mathfrak{D}_\nu$ if w is in the upper half-plane. Show that the identity

$$F(w) = \langle F(t), \Gamma(1 + \nu)(i\bar{w} - it)^{-1-\nu}\rangle$$

holds for every $F(z)$ in $\mathfrak{D}_\nu$. (The fractional power is defined so as to be continuous in the upper half-plane and positive at w.) Show that the functions

$$\left(\frac{1 + iz}{1 - iz}\right)^n \left(\frac{2}{1 - iz}\right)^{1+\nu}, \qquad n = 0, 1, 2, \cdots,$$

are a complete orthogonal set in $\mathfrak{D}_\nu$ and that the square of the norm of the nth function is $n!/\Gamma(1 + \nu + n)$.

PROBLEM 267. Let h and k be given positive numbers, and let $\nu = 2h - 1$. If $f(t)$ and $g(t)$ are elements of $L^2(0, \infty)$, define corresponding elements $F(z)$ and $G(z)$ of $\mathfrak{D}_\nu$ by

$$F(z) = \int_0^\infty t^{\frac{1}{2}\nu} e^{\frac{1}{2}izt} f(t)\,dt \quad \text{and} \quad G(z) = \int_0^\infty t^{\frac{1}{2}\nu} e^{\frac{1}{2}izt} g(t)\,dt.$$

Show that the condition

$$G(z) = -(z^2 + 1)F''(z) - 2(h + \tfrac{1}{2})zF'(z) - 2i(k - \tfrac{1}{2})F'(z) - h^2F(z)$$

for $y > 0$ is necessary and sufficient that $f(t)$ be (equivalent to) an absolutely continuous function of t, that $f'(t)$ be absolutely continuous, and that

$$g(t) = -(t^2f'(t))' + [\tfrac{1}{4}t^2 + (k - \tfrac{1}{2})t - \tfrac{1}{4}]f(t)$$

for almost all t.

PROBLEM 268. Let h and k be given positive numbers, and let $\nu = 2h - 1$. If $F(z)$ and $G(z)$ are elements of $\mathfrak{D}_\nu$, define corresponding functions $f(t)$ and $g(t)$ of $t > 0$ by

$$f(t) = \sinh^h(t) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}t)F(i\cosh t)$$
$$g(t) = \sinh^h(t) \tanh^{k-\frac{1}{2}}(\tfrac{1}{2}t)G(i\cosh t).$$

Show that the condition

$$G(z) = -(z^2 + 1)F''(z) - 2(h + \tfrac{1}{2})zF'(z) - 2i(k - \tfrac{1}{2})F'(z) - h^2F(z)$$

for $y > 0$ is equivalent to the condition

$$g(t) = -f''(t) + \frac{2(h - \frac{1}{2})(k - \frac{1}{2})\cosh t + h^2 - h + k^2 - k + \frac{1}{4}}{\sinh^2 t} f(t)$$

for $t > 0$.

PROBLEM 269. Let h and k be given positive numbers, and let $\nu = 2h - 1$. Show that

$$\int_1^\infty (t^2 - 1)^{h-\frac{1}{2}} \left(\frac{t-1}{t+1}\right)^{k-\frac{1}{2}} F(h - iz, h + iz; h + k; \tfrac{1}{2} - \tfrac{1}{2}t)e^{-\frac{1}{2}at}dt$$
$$= 2^{2h}\Gamma(h + k)a^{-\frac{1}{2}\nu}W_{\frac{1}{2}-k, iz}(a)/a$$

when $a > 0$. If $F(z)$ is in $\mathfrak{D}_\nu$ and if

$$\pi F(z) = z\int_1^\infty f(it)(t^2 - 1)^{h-\frac{1}{2}} \left(\frac{t-1}{t+1}\right)^{k-\frac{1}{2}} F(h - iz, h + iz; h + k; \tfrac{1}{2} - \tfrac{1}{2}t)dt,$$

show that

$$\pi \int_{-\infty}^{+\infty} \left| \frac{F(t)\Gamma(h-it)\Gamma(k-it)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-it)\Gamma(1-it)2^h\Gamma(h+k)} \right|^2 dt$$
$$= \int_1^\infty |f(it)|^2(t^2-1)^{h-\frac{1}{2}}\left(\frac{t-1}{t+1}\right)^{k-\frac{1}{2}} dt,$$

$$\pi \int_{-\infty}^{+\infty} \left| \frac{F(t)\Gamma(k-it)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-it)\Gamma(1-it)2^{2h}\Gamma(h+k)} \right|^2 dt = \|f\|_\nu^2.$$

Show that

$$\int_1^\infty |f(it)|^2(t^2-1)^{h-\frac{1}{2}}\left(\frac{t-1}{t+1}\right)^{k-\frac{1}{2}} dt \le 2^{2h}\Gamma(h)^{-2}\|f\|_\nu^2$$

for every element $f(z)$ of $\mathfrak{D}_\nu$.

PROBLEM 270. If h and k are positive numbers and if $\nu = 2h - 1$, show that

$$\int_1^\infty (y^2-1)^{h-\frac{1}{2}}\left(\frac{y-1}{y+1}\right)^{m+n+k-\frac{1}{2}}\left(\frac{2}{y+1}\right)^{2+2\nu} dy$$
$$= 2^{2h}\frac{\Gamma(\nu+1)\Gamma(m+n+h+k)}{\Gamma(\nu+m+n+h+k+1)}$$

for all nonnegative integers m and n. The generalized Hilbert matrix of order k is the infinite matrix with entry $1/(m+n+\frac{1}{2}+k)$ in the mth row and nth column, $m, n = 0, 1, 2, \cdots$. Show that the generalized Hilbert matrix of order k is the matrix of a bounded self-adjoint transformation with bound π if $k > 0$.

58. SPECIAL JACOBI SPACES

Special Hilbert spaces of entire functions occur also in the theory of Jacobi polynomials. The Jacobi spaces are a variant of the Gauss spaces in which the imaginary shift is replaced by a real shift. Two real parameters h and k enter into the statement of the identity satisfied by the spaces. The case $k = 1$ is of particular interest.

THEOREM 58. Let h be a real number, and let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$. Assume that $F(z+1)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle F(t+1) + (h-\tfrac{1}{2})[F(t+1) - F(-t)]/(\tfrac{1}{2}+t), G(t)\rangle$$
$$= \langle F(t), G(t-1) + (h-\tfrac{1}{2})[G(t-1) - G(-t)]/(\tfrac{1}{2}-t)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space. Then there exist real numbers p, r, and s such that $1 = s^2 + pr$ and such that

$$A(z+1) + (h-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z) = A(z)s - B(z)r,$$
$$B(z+1) + (h-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z) = A(z)p + B(z)s.$$

Proof of Theorem 58. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$F(\beta+1) + (h-\tfrac{1}{2})[F(\beta+1) - F(-\beta)]/(\tfrac{1}{2}+\beta)$$
$$= \bar{G}(\alpha-1) + (h-\tfrac{1}{2})[\bar{G}(\alpha-1) - \bar{G}(-\alpha)]/(\tfrac{1}{2}-\bar{\alpha}),$$

or equivalently,

$$K(w, z+1) + (h-\tfrac{1}{2})[K(w, z+1) - K(w, -z)]/(\tfrac{1}{2}+z)$$
$$= K(w-1, z) + (h-\tfrac{1}{2})[K(w-1, z) - K(-w, z)]/(\tfrac{1}{2}-\bar{w}).$$

But

$$\pi(z+1-\bar{w})K(w, z+1) - \pi(z+1-\bar{w})K(w-1, z)$$
$$= B(z+1)\bar{A}(w) - A(z+1)\bar{B}(w) - B(z)\bar{A}(w-1) + A(z)\bar{B}(w-1)$$

and

$$\pi(z+1-\bar{w})[K(w, z+1) - K(w, -z)]/(\tfrac{1}{2}+z)$$
$$- \pi(z+1-\bar{w})[K(w-1, z) - K(-w, z)]/(\tfrac{1}{2}-\bar{w})$$
$$= \bar{A}(w)[B(z+1) - B(-z)]/(\tfrac{1}{2}+z)$$
$$- \bar{B}(w)[A(z+1) - A(-z)]/(\tfrac{1}{2}+z)$$
$$- B(z)[\bar{A}(w-1) - \bar{A}(-w)]/(\tfrac{1}{2}-\bar{w})$$
$$+ A(z)[\bar{B}(w-1) - \bar{B}(-w)]/(\tfrac{1}{2}-\bar{w}).$$

The identity can now be written

$$\bar{A}(w)\{B(z+1) + (h-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z)\}$$
$$- \bar{B}(w)\{A(z+1) + (h-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z)\}$$
$$- B(z)\{\bar{A}(w-1) + (h-\tfrac{1}{2})[\bar{A}(w-1) - \bar{A}(-w)]/(\tfrac{1}{2}-\bar{w})\}$$
$$+ A(z)\{\bar{B}(w-1) + (h-\tfrac{1}{2})[\bar{B}(w-1) - \bar{B}(-w)]/(\tfrac{1}{2}-\bar{w})\} = 0.$$

Since $A(z)$ and $B(z)$ are linearly independent and real for real z, we can write

$$A(z+1) + (h-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z) = A(z)s - B(z)r,$$
$$B(z+1) + (h-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z) = A(z)p + B(z)q$$

for some real numbers p, q, r, and s. The identity now implies that

$$A(z-1) + (h-\tfrac{1}{2})[A(z-1) - A(-z)]/(\tfrac{1}{2}-z) = A(z)q + B(z)r,$$
$$B(z-1) + (h-\tfrac{1}{2})[B(z-1) - B(-z)]/(\tfrac{1}{2}-z) = -A(z)p + B(z)s.$$

Replacing z by $z + 1$ in the pair of equations, we obtain

$$(h-1-z)A(z) = [(h-\tfrac{1}{2}) - (z+\tfrac{1}{2})q]A(z+1) - (z+\tfrac{1}{2})rB(z+1),$$
$$(h-1-z)B(z) = (z+\tfrac{1}{2})pA(z+1) + [-(h-\tfrac{1}{2}) - (z+\tfrac{1}{2})s]B(z+1).$$

The previous pair of equations can be written

$$(h+z)A(z+1) = [(h-\tfrac{1}{2}) + (z+\tfrac{1}{2})s]A(z) - (z+\tfrac{1}{2})rB(z),$$
$$(h+z)B(z+1) = (z+\tfrac{1}{2})pA(z) + [-(h-\tfrac{1}{2}) + (z+\tfrac{1}{2})q]B(z).$$

It follows that

$$(h+z)(h-1-z) = [(h-\tfrac{1}{2}) - (z+\tfrac{1}{2})q][(h-\tfrac{1}{2}) + (z+\tfrac{1}{2})s] - (z+\tfrac{1}{2})r(z+\tfrac{1}{2})p.$$

So $q = s$ and $1 = s^2 + pr$.

PROBLEM 271. In Theorem 58 let λ be a solution of the equation $\lambda^2 - 2\lambda s + 1 = 0$, and let u and v be numbers such that $su + pv = \lambda u$ and $-ru + sv = \lambda v$. If $F(z) = A(z)u + B(z)v$, show that

$$F(z+1) + (h-\tfrac{1}{2})[F(z+1) - F(-z)]/(\tfrac{1}{2}+z) = \lambda F(z)$$

and that

$$\lim_{y\to+\infty} F(1+iy)/F(iy) = \lambda$$

if u and v are not both zero.

PROBLEM 272. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a, z)/E(b, z)$ has no real zeros. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 58 for some number h, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same h.

PROBLEM 273. In Problem 272 assume that the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional and that $E(a, 0) = E(b, 0)$. Show that it is spanned by a function $F(z)$ of the form

$$F(z) = A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$$

where u and v are real numbers which satisfy the hypotheses of Problem 271 for $E(a, z)$ and for $E(b, z)$. Show that $\lambda = s(a) = s(b) = \pm 1$, that $p(a) = r(a) = 0$, and that

$$p(b) = \pi u^2[s(b) + 2h - 1] \quad \text{and} \quad r(b) = \pi v^2[s(b) - 2h + 1].$$

PROBLEM 274. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 58 for some number h, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same choice of h.

PROBLEM 275. Let $\{\mathcal{H}(E(a))\}$, $0 < a < \pi$, be a given family of spaces associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for each fixed z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \pi$. Assume that $m(t)$ is an absolutely continuous function of t and that there exists a real number h such that $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 58 for every index a. Show that $p(a)$, $r(a)$, and $s(a)$ are absolutely continuous functions of a and that

$$p(a)\gamma'(a) = -s'(a) = r(a)\alpha'(a),$$
$$p(a)r'(a) + s(a)s'(a) = (2h - 1)s'(a) = -r(a)p'(a) - s(a)s'(a),$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist and $s(a)^2 \neq 1$. Show that $-1 \leq s(a) \leq 1$ for all a and that the quantities

$$\frac{p(a)}{\sqrt{1 - s(a)^2}}\left(\frac{1 + s(a)}{1 - s(a)}\right)^{h-\frac{1}{2}} \quad \text{and} \quad \frac{r(a)}{\sqrt{1 - s(a)^2}}\left(\frac{1 - s(a)}{1 + s(a)}\right)^{h-\frac{1}{2}}$$

are constant in any interval where $s(a)^2 \neq 1$. If

$$\lim_{a \searrow 0} [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 1$ as $a \searrow 0$. Show that $h > 0$.

PROBLEM 276. In Problem 275, show that $0 < h < 1$ if $s(a)^2 = 1$ for some index a. Show that $p(a) = r(a) = 0$ at any regular point a where

$s(a)^2 = 1$ and (a, b) contains no singular points for some number b such that $m(a) \neq m(b)$. Show that $p(a) = r(a) = 0$ at any regular point where $s(a)^2 = 1$.

PROBLEM 277. In Problem 272, show that the domain of multiplication by z in $\mathcal{H}(E(a))$ is dense in $\mathcal{H}(E(a))$ if $\mathcal{H}(E(a))$ is contained properly in $\mathcal{H}(E(b))$.

PROBLEM 278. In Problem 272, assume that $s(a)^2 = s(b)^2 = 1$. Let $S(a, z) = A(a, z)u(a) + B(a, z)v(a)$ where $u(a)$ and $v(a)$ are numbers, not both zero, which satisfy the hypotheses of Problem 271 for $E(a, z)$. Let $S(b, z) = A(b, z)u(b) + B(b, z)v(b)$ where $u(b)$ and $v(b)$ are numbers, not both zero, which satisfy the hypotheses of Problem 271 for $E(b, z)$. Show that

$$S(a, z + 1) = s(a)S(a, z) \quad \text{and} \quad S(b, z + 1) = s(b)S(b, z).$$

Show that the mean type of $S(b, z)/S(a, z)$ in the upper half-plane is equal to π times the difference between the number of zeros of $S(b, z)$ in $[0, 1)$ and the number of zeros of $S(a, z)$ in the same interval.

59. CONSTRUCTION OF SPECIAL JACOBI SPACES

These results allow a direct construction of the special Jacobi spaces.

THEOREM 59. Let h be a given positive number, and let $\alpha(t)$ and $\gamma(t)$ be differentiable functions of t, $0 < t < \pi$, such that

$$\alpha'(t) = \tan^{2h-1}(\tfrac{1}{2}t) \quad \text{and} \quad \gamma'(t) = \cot^{2h-1}(\tfrac{1}{2}t),$$

and $\beta(t) = 0$. Then there exists a unique family $(E(t, z))$ of entire functions of Pólya class, $0 < t < \pi$, such that $E(t, z)$ is a continuous function of t for every z, such that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z\int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \pi$, and such that $\lim E(t, z) = 1$ as $t \searrow 0$ for all complex z. A space $\mathcal{H}(E(a))$ exists for every a and $E^*(a, z) = E(a, -z)$. The recurrence relations

$$A(a, z + 1) + (h - \tfrac{1}{2})[A(a, z + 1) - A(a, -z)]/(\tfrac{1}{2} + z) = A(a, z)s(a) + B(a, z)r(a),$$

$$B(a, z + 1) + (h - \tfrac{1}{2})[B(a, z + 1) - B(a, -z)]/(\tfrac{1}{2} + z) = A(a, z)p(a) + B(a, z)s(a)$$

hold with $s(a) = \cos a$,

$$p(a) = \sin a \tan^{2h-1}(\tfrac{1}{2}a), \quad \text{and} \quad r(a) = \sin a \cot^{2h-1}(\tfrac{1}{2}a).$$

Proof of Theorem 59. Since $\int_0^1 \alpha(t)d\gamma(t) < \infty$, the functions $E(a, z)$ exist by Theorem 41, and $E^*(a, z) = E(a, -z)$ by Problem 180. To obtain the recurrence relations, we introduce the functions

$$P(a, z) = (\tfrac{1}{2} + z)A(a, z)s(a) + (\tfrac{1}{2} + z)B(a, z)r(a) + (h - \tfrac{1}{2})A(a, z),$$
$$Q(a, z) = (\tfrac{1}{2} + z)A(a, z)p(a) + (\tfrac{1}{2} + z)B(a, z)s(a) - (h - \tfrac{1}{2})B(a, z)$$

with $p(a)$, $r(a)$, and $s(a)$ defined by the theorem. The equations

$$\partial P(a, z)/\partial a = -(z + 1)Q(a, z)\gamma'(a),$$
$$\partial Q(a, z)/\partial a = (z + 1)P(a, z)\alpha'(a)$$

are a consequence of the formulas

$$s'(a) = -p(a)\gamma'(a) = -r(a)\alpha'(a),$$
$$p'(a) = s(a)\alpha'(a) + 2(h - \tfrac{1}{2})\alpha'(a),$$
$$r'(a) = s(a)\gamma'(a) - 2(h - \tfrac{1}{2})\gamma'(a).$$

Since

$$\partial A(a, z + 1)/\partial a = -(z + 1)B(a, z + 1)\gamma'(a),$$
$$\partial B(a, z + 1)/\partial a = (z + 1)A(a, z + 1)\alpha'(a),$$

the expression

$$P(a, z)B(a, z + 1) - Q(a, z)A(a, z + 1)$$

is independent of a. Since $\lim A(a, z) = 1$ and $\lim B(a, z)/\alpha(a) = z$ as $a \searrow 0$, $\lim P(a, z) = h + z$ as $a \searrow 0$. Since the above expression has limit zero as $a \searrow 0$, it vanishes identically. Since $A(a, z)$ and $B(a, z)$ have no common zeros,

$$S(a, z) = P(a, z)/A(a, z + 1) = Q(a, z)/B(a, z + 1)$$

is an entire function. Since $\partial S(a, z)/\partial a = 0$, $S(a, z)$ is independent of a. Since $\lim S(a, z) = h + z$ as $a \searrow 0$, $S(a, z) = h + z$ for all a. The recurrence relations for $A(a, z)$ and $B(a, z)$ follow.

PROBLEM 279. In Theorem 59 let

$$\Phi(a, z) = A(a, z) \tan^{h-\frac{1}{2}}(\tfrac{1}{2}a) - iB(a, z) \cot^{h-\frac{1}{2}}(\tfrac{1}{2}a),$$
$$\Phi(-a, z) = A(a, z) \tan^{h-\frac{1}{2}}(\tfrac{1}{2}a) + iB(a, z) \cot^{h-\frac{1}{2}}(\tfrac{1}{2}a)$$

for $0 < a < \pi$. If $f(x)$ belongs to $L^2(-\pi, \pi)$ and vanishes outside of some interval $(-a, a)$, $a < \pi$, show that its eigentransform $F(z)$, defined by

$$2\pi F(z) = \int_{-\pi}^{\pi} f(t)\Phi(t, z)dt,$$

belongs to $\mathcal{H}(E(a))$ and that

$$2\pi \int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\pi}^{\pi} |f(t)|^2 dt$$

for every $F(z)$ in $\mathcal{H}(E(a))$. Show that every element of $\mathcal{H}(E(a))$ is of this form. Let $f(t)$ and $g(t)$ be elements of $L^2(-\pi, \pi)$ which vanish outside of $(-a, a)$, such that $g(t) = e^{it}f(t)$. Show that their eigentransforms are related by

$$G(z) = F(z + 1) + (h - \tfrac{1}{2})[F(z + 1) - F(-z)]/(\tfrac{1}{2} + z).$$

Show that $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 58.

PROBLEM 280. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 58 for some number h, $h \geq 1$, show that there exists an index a in Theorem 59 such that the transformation $F(z) \to S(z)F(z)$ takes $\mathcal{H}(E(a))$ isometrically onto $\mathcal{H}(E)$ for some even entire function $S(z)$ which is real for real z and periodic of period one.

PROBLEM 281. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Theorem 58 for some number h, $h < 1$, show that there is an even entire function $S(z)$ which is real for real z and periodic of period one such that either $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E(a))$ onto $\mathcal{H}(E)$ for some index a in Theorem 59 or $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E(a))$ into $\mathcal{H}(E)$ for every index a in Theorem 59.

60. GENERAL JACOBI SPACES

The general Jacobi spaces are analogous to the general Gauss spaces.

THEOREM 60. Let h and k be real numbers, $(h - 1)(k - 1) \neq 0$, and let $\mathcal{H}(E)$ be a given space, not one-dimensional, such that $E^*(z) = E(-z)$ and $E(0) \neq 0$. Assume that $F(z + 1)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\begin{aligned}\langle F(t + 1) &+ 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[F(t + 1) - F(-t)]/(\tfrac{1}{2} + t) \\ &\qquad - 2(h - 1)(k - 1)F(t + 1)/(1 + t), G(t)\rangle \\ = \langle F(t), G(t - 1) &+ 2(h - \tfrac{1}{2})(k - \tfrac{1}{2})[G(t - 1) - G(-t)]/(\tfrac{1}{2} - t) \\ &\qquad - 2(h - 1)(k - 1)G(t - 1)/(1 - t)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space which vanish at the origin. Then there exist real numbers u and v such that $L(z) = A(z)u + B(z)v$ has value 1 at -1 and such that the identity

$$\begin{aligned}\langle F(t+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[F(t+1) - F(-t)]/(\tfrac{1}{2}+t) \\ &- 2(h-1)(k-1)[F(t+1) - L(t)F(0)]/(1+t), G(t)\rangle \\ = \langle F(t), G(t-1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[G(t-1) - G(-t)]/(\tfrac{1}{2}-t) \\ &- 2(h-1)(k-1)[G(t-1) - L(-t)G(0)]/(1-t)\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space. There exist real numbers p, r, and s such that $1 = s^2 + pr$ and such that

$$\begin{aligned}A(z+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z) \\ &- 2(h-1)(k-1)[A(z+1) - L(z)A(0)]/(1+z) \\ &- 2\pi(h-1)(k-1)vK(0,z) = A(z)s - B(z)r,\end{aligned}$$

$$\begin{aligned}B(z+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z) \\ &- 2(h-1)(k-1)B(z+1)/(1+z) \\ &- 2\pi(h-1)(k-1)uK(0,z) = A(z)p + B(z)s.\end{aligned}$$

These numbers are related by the identity

$$2hk - 4(h-\tfrac{1}{2})(k-\tfrac{1}{2})A(0)u + 2(h-1)(k-1)A(0)^2u^2 = A(0)(su - pv).$$

Proof of Theorem 60. By the proof of Theorem 54 there exists an entire function $L(z)$, which has value 1 at -1, such that $[F(z) - L(z)F(-1)]/(1+z)$ belongs to the space whenever $F(z)$ belongs to the space and such that the inner-product identity stated in the theorem holds for all elements $F(z)$ and $G(z)$ of the space. When $F(z) = K(\alpha, z)$ and $G(z) = K(\beta, z)$ for some fixed numbers α and β, the identity reads

$$\begin{aligned}F(\beta+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[F(\beta+1) - F(-\beta)]/(\tfrac{1}{2}+\beta) \\ &- 2(h-1)(k-1)[F(\beta+1) - L(\beta)F(0)]/(1+\beta) \\ = \bar{G}(\alpha-1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{G}(\alpha-1) - \bar{G}(-\alpha)]/(\tfrac{1}{2}-\bar{\alpha}) \\ &- 2(h-1)(k-1)[\bar{G}(\alpha-1) - \bar{L}(-\alpha)\bar{G}(0)]/(1-\bar{\alpha}).\end{aligned}$$

An equivalent identity is

$$\begin{aligned}K(w, z+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[K(w, z+1) - K(w, -z)]/(\tfrac{1}{2}+z) \\ &- 2(h-1)(k-1)[K(w, z+1) - L(z)K(w, 0)]/(1+z) \\ = K(w-1, z) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[K(w-1, z) - K(-w, z)]/(\tfrac{1}{2}-\bar{w}) \\ &- 2(h-1)(k-1)[K(w-1, z) - \bar{L}(w)K(0, z)]/(1-\bar{w}).\end{aligned}$$

But

$$\pi(z+1-\bar{w})K(w,z+1)-\pi(z+1-\bar{w})K(w-1,z)$$
$$=B(z+1)\bar{A}(w)-A(z+1)\bar{B}(w)-B(z)\bar{A}(w-1)+A(z)\bar{B}(w-1),$$

$$\pi(z+1-\bar{w})[K(w,z+1)-K(w,-z)]/(\tfrac{1}{2}+z)$$
$$-\pi(z+1-\bar{w})[K(w-1,z)-K(-w,z)]/(\tfrac{1}{2}-\bar{w})$$
$$=\bar{A}(w)[B(z+1)-B(-z)]/(\tfrac{1}{2}+z)$$
$$-\bar{B}(w)[A(z+1)-A(-z)]/(\tfrac{1}{2}+z)$$
$$-B(z)[\bar{A}(w-1)-\bar{A}(-w)]/(\tfrac{1}{2}-\bar{w})$$
$$+A(z)[\bar{B}(w-1)-\bar{B}(-w)]/(\tfrac{1}{2}-\bar{w}),$$

$$\pi(z+1-\bar{w})[K(w,z+1)-L(z)K(w,0)]/(1+z)$$
$$-\pi(z+1-\bar{w})[K(w+1,z)-\bar{L}(-w)K(0,z)]/(1-\bar{w})$$
$$=\bar{A}(w)B(z+1)/(1+z)-\bar{B}(w)[A(z+1)-L(z)A(0)]/(1+z)$$
$$-B(z)[\bar{A}(w-1)-\bar{L}(-w)\bar{A}(0)]/(1-\bar{w})+A(z)\bar{B}(w+1)/(1+\bar{w})$$
$$-\pi L(z)K(w,0)+\pi\bar{L}(-w)K(0,z).$$

It follows that

$$\bar{A}(w)\{B(z+1)+2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+1)-B(-z)]/(\tfrac{1}{2}+z)$$
$$-2(h-1)(k-1)B(z+1)/(1+z)\}$$
$$-\bar{B}(w)\{A(z+1)+2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+1)-A(-z)]/(\tfrac{1}{2}+z)$$
$$-2(h-1)(k-1)[A(z+1)-L(z)A(0)]/(1+z)\}$$
$$-B(z)\{\bar{A}(w-1)+2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{A}(w-1)-\bar{A}(-w)]/(\tfrac{1}{2}-\bar{w})$$
$$-2(h-1)(k-1)[\bar{A}(w-1)-\bar{L}(-w)\bar{A}(0)]/(1-\bar{w})\}$$
$$+A(z)\{\bar{B}(w-1)+2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{B}(w-1)-\bar{B}(-w)]/(\tfrac{1}{2}-\bar{w})$$
$$-2(h-1)(k-1)\bar{B}(w-1)/(1-\bar{w})\}$$
$$=-2\pi(h-1)(k-1)L(z)K(w,0)+2\pi(h-1)(k-1)\bar{L}(-w)K(0,z).$$

As in the proof of Theorem 54, we can choose $L(z)$ of the form

$$L(z)=A(z)u+B(z)v$$

for some numbers u and v. The identity then reads

$$\begin{aligned}
&\bar{A}(w)\{B(z+1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z) \\
&\qquad - 2(h-1)(k-1)B(z+1)/(1+z) - 2\pi(h-1)(k-1)\bar{u}K(0,z)\} \\
&- \bar{B}(w)\{A(z+1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z) \\
&\qquad - 2(h-1)(k-1)[A(z+1) - L(z)A(0)]/(1+z) \\
&\qquad\qquad - 2\pi(h-1)(k-1)\bar{v}K(0,z)\} \\
&- B(z)\{\bar{A}(w-1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{A}(w-1) - \bar{A}(-w)]/(\tfrac{1}{2}-\bar{w}) \\
&\qquad - 2(h-1)(k-1)[\bar{A}(w-1) - \bar{L}(-w)\bar{A}(0)]/(1-\bar{w}) \\
&\qquad\qquad - 2\pi(h-1)(k-1)vK(w,0)\} \\
&+ A(z)\{\bar{B}(w-1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[\bar{B}(w-1) - \bar{B}(-w)]/(\tfrac{1}{2}-\bar{w}) \\
&\quad - 2(h-1)(k-1)\bar{B}(w-1)/(1-\bar{w}) + 2\pi(h-1)(k-1)uK(w,0)\} = 0.
\end{aligned}$$

Since $A(z)$ and $B(z)$ are linearly independent, there exist numbers p, r, and s, p and r real, such that

$$\begin{aligned}
&A(z+1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z+1) - A(-z)]/(\tfrac{1}{2}+z) \\
&\qquad - 2(h-1)(k-1)[A(z+1) - L(z)A(0)]/(1+z) \\
&\qquad\qquad - 2\pi(h-1)(k-1)\bar{v}K(0,z) = A(z)s - B(z)r, \\
&B(z+1) + 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z+1) - B(-z)]/(\tfrac{1}{2}+z) \\
&\qquad - 2(h-1)(k-1)B(z+1)/(1+z) \\
&\qquad\qquad - 2\pi(h-1)(k-1)\bar{u}K(0,z) = A(z)p + B(z)\bar{s}.
\end{aligned}$$

Since $A(z)$ and $B(z)$ are real for real z, the same formulas hold with s replaced by $\bar{s}$, u replaced by $\bar{u}$, and v replaced by $\bar{v}$. Since $A(z)$, $B(z)$, and $K(0, z)$ are linearly independent, s, u, and v are real. These equations can now be written

$$\begin{aligned}
z(h+z)(k+z)A(z+1) &= a(z)A(z) + c(z)B(z), \\
z(h+z)(k+z)B(z+1) &= b(z)A(z) + d(z)B(z)
\end{aligned}$$

where

$$\begin{aligned}
a(z) &= 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})z(1+z) - 2(h-1)(k-1)A(0)uz(\tfrac{1}{2}+z) \\
&\qquad + sz(\tfrac{1}{2}+z)(1+z), \\
b(z) &= pz(\tfrac{1}{2}+z)(1+z), \\
c(z) &= 2(h-1)(k-1)A(0)v(1+z) - rz(\tfrac{1}{2}+z)(1+z), \\
d(z) &= -2(h-\tfrac{1}{2})(k-\tfrac{1}{2})z(1+z) + 2(h-1)(k-1)A(0)u(\tfrac{1}{2}+z)(1+z) \\
&\qquad + sz(\tfrac{1}{2}+z)(1+z).
\end{aligned}$$

The main identity now implies that

$$\begin{aligned} A(z-1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[A(z-1)-A(-z)]/(\tfrac{1}{2}-z) \\ &- 2(h-1)(k-1)[A(z-1)-L(-z)A(0)]/(1-z) \\ &- 2\pi(h-1)(k-1)vK(0,z) = A(z)s + B(z)r, \end{aligned}$$

$$\begin{aligned} B(z-1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[B(z-1)-B(-z)]/(\tfrac{1}{2}-z) \\ &- 2(h-1)(k-1)B(z-1)/(1-z) \\ &+ 2\pi(h-1)(k-1)uK(0,z) = -A(z)p + B(z)s. \end{aligned}$$

When z is replaced by $z+1$, these identities can be written

$$(z+1)(h-1-z)(k-1-z)A(z) = d(z)A(z+1) - c(z)B(z+1),$$
$$(z+1)(h-1-z)(k-1-z)B(z) = -b(z)A(z+1) + a(z)B(z+1).$$

It follows that

$$z(z+1)(h+z)(h-1-z)(k+z)(k-1-z) = a(z)d(z) - b(z)c(z).$$

By comparing the coefficient of the highest power of z on each side, we obtain $1 = s^2 + pr$. The identity now reduces to the identity stated at the end of the theorem.

PROBLEM 282. In Theorem 60 let λ be a solution of the equation $\lambda^2 - 2\lambda s + 1 = 0$, and let U and V be numbers, not both zero, such that $sU + pV = \lambda U$ and $-rU + sV = \lambda V$. If $F(z) = A(z)U + B(z)V$, show that

$$\begin{aligned} F(z+1) &+ 2(h-\tfrac{1}{2})(k-\tfrac{1}{2})[F(z+1)-F(-z)]/(\tfrac{1}{2}+z) \\ &- 2(h-1)(k-1)[F(z+1)-L(z)F(0)]/(1+z) \\ &- 2\pi(h-1)(k-1)(Uv+Vu)K(0,z) = \lambda F(z). \end{aligned}$$

If $s^2 \neq 1$, show that λ can be chosen so that $F(z)$ does not belong to $\mathcal{H}(E)$. If $F(z)$ does not belong to $\mathcal{H}(E)$, show that either

$$\lim_{y\to+\infty} F(1+iy)/F(iy) = \lambda$$

or

$$\lim_{y\to-\infty} F(1+iy)/F(iy) = \lambda.$$

Show that $|\lambda| \leq 1$ and that $-1 \leq s \leq 1$. Show that p and r are positive if $s^2 \neq 1$.

PROBLEM 283. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that $\mathcal{H}(E(a))$ is contained isometrically in $\mathcal{H}(E(b))$ and $E(a,z)/E(b,z)$ has no real zeros. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 60 for some h

and k, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same h and k.

PROBLEM 284. In Problem 283 assume that the orthogonal complement of $\mathcal{H}(E(a))$ in $\mathcal{H}(E(b))$ is one-dimensional and that $E(a, 0) = E(b, 0)$. Show that the orthogonal complement is spanned by a function $F(z)$ of the form

$$F(z) = A(a, z)U + B(a, z)V = A(b, z)U + B(b, z)V$$

where U and V are numbers which satisfy the hypotheses of Problem 282 for $E(a, z)$ and for $E(b, z)$.

PROBLEM 285. Let $\mathcal{H}(E(a))$ and $\mathcal{H}(E(b))$ be given spaces such that

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

for some space $\mathcal{H}(M(a, b))$. If $\mathcal{H}(E(b))$ satisfies the hypotheses of Theorem 60 for some h and k, if $E^*(a, z) = E(a, -z)$, and if $\mathcal{H}(E(a))$ is not one-dimensional, show that $\mathcal{H}(E(a))$ satisfies the hypotheses of the theorem for the same h and k.

PROBLEM 286. Let $\{\mathcal{H}(E(a))\}$, $0 < a < \pi$, be a given family of spaces associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(a, z)$ is a continuous function of a for every z and so that

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \pi$. Assume that the entries of $m(t)$ are absolutely continuous functions of t and that there exist numbers h and k such that the hypotheses of Theorem 60 are satisfied for every index a. Show that the quantities $p(a)$, $r(a)$, $s(a)$, $u(a)$, and $v(a)$ defined by Theorem 60 are absolutely continuous functions of a and that

$$u'(a) = v(a)\alpha'(a) \quad \text{and} \quad v'(a) = -u(a)\gamma'(a),$$

$$p(a)\gamma'(a) = -s'(a) = r(a)\alpha'(a),$$

$$\begin{aligned} p(a)r'(a) + s(a)s'(a) &= -r(a)p'(a) - s(a)s'(a) \\ &= 4(h - \tfrac{1}{2})(k - \tfrac{1}{2})s'(a) - 4(h - 1)(k - 1)A(a, 0)u(a)s'(a) \end{aligned}$$

whenever $\alpha'(a)$ and $\gamma'(a)$ exist and $s(a)^2 \neq 1$. If

$$\lim_{a \searrow 0} [B(a, z)\bar{A}(a, w) - A(a, z)\bar{B}(a, w)] = 0$$

for all complex z and w, show that $\lim s(a) = 1$ as $a \searrow 0$.

PROBLEM 287. If $s(a) = \cos a$ in Problem 286, show that

$$B(a, z)\sqrt{\gamma'(a)} = f(a, z)$$

where $f(a, z)$ is a solution of the equation

$$z^2 f(a, z) = -f''(a, z) + \frac{2(h - \frac{1}{2})(k - \frac{1}{2}) \cos a + h^2 - h + k^2 - k + \frac{1}{4}}{\sin^2 a} f(a, z)$$

for each fixed z. Show that

$$f(a, z) = \sin^h (a) \tan^{k-\frac{1}{2}} (\tfrac{1}{2}a) g(\sin^2 (\tfrac{1}{2}a), z)$$

where $g(a, z)$ is a solution of the hypergeometric equation

$$a(1 - a)g''(a, z) + [h + k - (2h + 1)a]g'(a, z) - (h^2 - z^2)g(a, z) = 0$$

for each fixed z. If $h + k \geq 1$, show that $g(a, z)$ is a constant multiple of the hypergeometric function $F(h - z, h + z; h + k; a)$ for each fixed z. Show that there exists an even entire function $S(z)$, which is real for real z and periodic of period one, such that

$$B(a, z)\sqrt{\gamma'(a)} = \sin^h (a) \tan^{k-\frac{1}{2}} (\tfrac{1}{2}a) zS(z)F(h - z, h + z; h + k; \sin^2 (\tfrac{1}{2}a)).$$

PROBLEM 288. Let h and k be given positive numbers, $h > k - 1$. If $0 < a < \pi$, show that the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ are of bounded type and of mean type at most a in the upper half-plane and such that

$$\|F\|^2 = \frac{\pi}{2^{2h}\Gamma(h + k)^2} \sum_{n=0}^{\infty} \frac{|F(h + n)|^2}{h + n} \frac{\Gamma(2h + n)\Gamma(h + k + n)}{\Gamma(1 + n)\Gamma(h - k + 1 + n)} + \frac{\pi}{2^{2h}\Gamma(h + k)^2} \sum_{n=0}^{\infty} \frac{|F(-h - n)|^2}{h + n} \frac{\Gamma(2h + n)\Gamma(h + k + n)}{\Gamma(1 + n)\Gamma(h - k + 1 + n)} < \infty$$

is equal isometrically to a space $\mathcal{H}(E(a))$ such that $E^*(a, z) = E(a, -z)$ and $E(a, 0) = 1$. Show that $\mathcal{H}(E(a))$ satisfies the hypotheses of Theorem 60 for every index a. Show that the hypotheses of Problem 286 are satisfied for a suitable choice of $m(t)$. Show that

$$B(a, z)\sqrt{\gamma'(a)} = \sin^h (a) \tan^{k-\frac{1}{2}} (\tfrac{1}{2}a) zF(h - z, h + z; h + k; \sin^2 (\tfrac{1}{2}a)).$$

Show that

$$B(a, h + n)\sqrt{\gamma'(a)} = \frac{n!}{(n + h + k - 1) \cdots (h + k)} \sin^h (a) \tan^{k-\frac{1}{2}} (\tfrac{1}{2}a) P_n^{(h+k-1, h-k)}(\cos a)$$

for every $n = 0, 1, 2, \cdots$, where $P_n^{(h+k-1,h-k)}(x)$ is the nth Jacobi polynomial of index $(h + k - 1, h - k)$. If $f(t)$ belongs to $L^2(-1, 1)$ and vanishes outside of $(\cos a, 1)$, show that

$$\pi F(z) = z \int_{-1}^{1} f(t)(1 - t^2)^{h-\frac{1}{2}} \left(\frac{1 - t}{1 + t}\right)^{k-\frac{1}{2}} F(h - z, h + z; h + k; \tfrac{1}{2} - \tfrac{1}{2}t)\,dt$$

is an odd element of $\mathcal{H}(E(a))$ and that

$$\pi \int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-1}^{1} |f(t)|^2 dt.$$

Show that every odd element of $\mathcal{H}(E(a))$ is of this form.

PROBLEM 289. Let ν and σ be real numbers, $\nu \neq \sigma$ and $\nu\sigma > 0$. Let $\mathcal{H}(E)$ be a given space which is not one-dimensional. Assume that $E(z)$ has nonzero values at ν and σ and that the inequality

$$\langle (t - \nu)F(t), (t - \sigma)F(t) \rangle > 0$$

holds whenever $F(z)$ is a nonzero element in the domain of multiplication by z in $\mathcal{H}(E)$. Show that there exists a space $\mathcal{H}(E_1)$ such that

$$(1 - z/\nu)[B_1(z)A(\nu) - A_1(z)B(\nu)] = B(z)A(\nu) - A(z)B(\nu),$$
$$(1 - z/\sigma)[B_1(z)A(\sigma) - A_1(z)B(\sigma)] = B(z)A(\sigma) - A(z)B(\sigma).$$

Show that the space $\mathcal{H}(E_1)$ coincides with the domain of multiplication by z in $\mathcal{H}(E)$ and that the identity

$$\langle F(t), G(t) \rangle_{E_1} = \langle (1 - t/\nu)F(t), (1 - t/\sigma)G(t) \rangle$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E_1)$. For each $t > 0$ let $\mathcal{H}(E(t))$ be a space which satisfies the above hypotheses. Assume that $E(t, z)$ is a continuous function of t for every z and that the integral equation

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))\,dm(t)$$

holds when $0 < a < b < \infty$ for some nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

whose entries are continuous functions of t. For each $t > 0$ let

$$P(t) = \begin{pmatrix} p(t) & q(t) \\ r(t) & s(t) \end{pmatrix}$$

be the matrix with real entries and determinant one given by

$$[B(t,\sigma)A(t,\nu) - A(t,\sigma)B(t,\nu)]P(t)$$
$$= \sqrt{\frac{\nu}{\sigma}}\, I\begin{pmatrix} A(t,\sigma) \\ B(t,\sigma) \end{pmatrix}(A(t,\nu), B(t,\nu)) - \sqrt{\frac{\sigma}{\nu}}\, I\begin{pmatrix} A(t,\nu) \\ B(t,\nu) \end{pmatrix}(A(t,\sigma), B(t,\sigma)).$$

Let

$$m_1(t) = \begin{pmatrix} \alpha_1(t) & \beta_1(t) \\ \beta_1(t) & \gamma_1(t) \end{pmatrix}$$

be a nondecreasing, matrix valued function of $t > 0$ such that

$$m_1(b) - m_1(a) = \int_a^b P(t)\, dm(t) \bar{P}(t)$$

when $0 < a < b < \infty$. For each $t > 0$ let $E_1(t, z)$ be defined for $E(t, z)$ as above. Show that the integral equation

$$(A_1(b,z), B_1(b,z))I - (A_1(a,z), B_1(a,z))I = z \int_a^b (A_1(t,z), B_1(t,z))\, dm_1(t)$$

holds when $0 < a < b < \infty$.

The Jacobi spaces of entire functions are related to Dirac's theory of the hydrogen atom. Dirac's equations of motion are a variant of Maxwell's equations for the propagation of an electromagnetic field. In Maxwell's theory the state of the field at any time t is described by a pair of vector fields, the electric vector E and the magnetic vector H. In standard vector notation

$$E = \vec{i}E_x + \vec{j}E_y + \vec{k}E_z$$

and

$$H = \vec{i}H_x + \vec{j}H_y + \vec{k}H_z$$

where E_x, E_y, E_z and H_x, H_y, H_z are square integrable, complex valued functions of the Cartesian coordinates x, y, z. Maxwell's equations are

$$\frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times H \quad \text{and} \quad \frac{1}{c}\frac{\partial H}{\partial t} = -\nabla \times E$$

where c is the speed of light and

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

These equations are applied to source-free fields:

$$\nabla \cdot E = 0 \quad \text{and} \quad \nabla \cdot H = 0.$$

The energy of the field, given by

$$\tfrac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(E\cdot\bar{E}+H\cdot\bar{H})dx\,dy\,dz,$$

is a constant of motion.

Maxwell's equations have an obvious generalization for fields with sources. For this it is necessary to introduce two new scalar fields, an electric potential Φ and a magnetic potential Ψ, which are square integrable, complex valued functions of x, y, z. The generalized Maxwell equations are

$$\frac{1}{c}\frac{\partial\Phi}{\partial t}=\nabla\cdot E,$$

$$\frac{1}{c}\frac{\partial E}{\partial t}=\nabla\times H+\nabla\Phi,$$

$$\frac{1}{c}\frac{\partial H}{\partial t}=-\nabla\times E+\nabla\Psi,$$

$$\frac{1}{c}\frac{\partial\Psi}{\partial t}=\nabla\cdot H.$$

The energy of the field,

$$\tfrac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\Phi\bar{\Phi}+E\cdot\bar{E}+H\cdot\bar{H}+\Psi\bar{\Psi})dx\,dy\,dz,$$

is a constant of motion. A formal proof of energy conservation is contained in the identity

$$\frac{1}{c}\frac{\partial}{\partial t}(\Phi\bar{\Phi}+E\cdot\bar{E}+H\cdot\bar{H}+\Psi\bar{\Psi})$$
$$=\nabla\cdot(\Phi\bar{E}+E\bar{\Phi}+E\times\bar{H}-H\times\bar{E}+\Psi\bar{H}+H\bar{\Psi}).$$

The wave equation

$$\nabla^2 f-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0$$

is satisfied if f is any component Φ, E_x, E_y, E_z, H_x, H_y, H_z, Ψ of the electromagnetic field.

The Dirac equations of propagation in free space are similar except that they depend on a nonnegative constant m, which is the mass of the associated electron in Dirac's theory of the hydrogen atom. In Dirac's theory the wave equations for a free particle are

$$\nabla^2 f-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=\frac{m^2c^2}{\hbar^2}f$$

where $2\pi\hbar$ is Planck's unit of action. Corresponding first order equations for the field are

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} = \nabla \cdot E + \frac{imc}{\hbar\sqrt{2}}(\Phi + \Psi),$$

$$\frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times H + \nabla\Phi - \frac{imc}{\hbar\sqrt{2}}(E + H),$$

$$\frac{1}{c}\frac{\partial H}{\partial t} = -\nabla \times E + \nabla\Psi - \frac{imc}{\hbar\sqrt{2}}(E - H),$$

$$\frac{1}{c}\frac{\partial\Psi}{\partial t} = \nabla \cdot H + \frac{imc}{\hbar\sqrt{2}}(\Phi - \Psi).$$

The energy of the field is again a constant of motion for this propagation. The same differential equivalent of the energy conservation law is valid.

In Dirac's theory the field represents a particle, the electron. The Dirac equations of propagation are taken to be analogues of Newton's laws of motion for a particle in an external field. In classical mechanics a particle can move under the influence of external forces which arise from a potential energy function $V = V(x, y, z)$. The same function is used in the quantum mechanical analogue of the motion. The Dirac equations in the presence of such external forces are

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} = \nabla \cdot E + \frac{imc}{\hbar\sqrt{2}}(\Phi + \Psi) - \frac{iV\Phi}{\hbar c},$$

$$\frac{1}{c}\frac{\partial E}{\partial t} = \nabla \times H + \nabla\Phi - \frac{imc}{\hbar\sqrt{2}}(E + H) - \frac{iVE}{\hbar c},$$

$$\frac{1}{c}\frac{\partial H}{\partial t} = -\nabla \times E + \nabla\Psi - \frac{imc}{\hbar\sqrt{2}}(E - H) - \frac{iVH}{\hbar c},$$

$$\frac{1}{c}\frac{\partial\Psi}{\partial t} = \nabla \cdot H + \frac{imc}{\hbar\sqrt{2}}(\Phi - \Psi) - \frac{iV\Psi}{\hbar c}.$$

In what follows, m is treated as a variable which has a limiting value at infinity equal to the electron mass. The same energy conservation law is valid for the Dirac propagation in the presence of external forces.

Since the nature of nuclear forces is not known, it is often necessary in quantum mechanics to treat m and V as arbitrary real valued functions, subject only to whatever integrability conditions are needed to make the equations well-behaved. We give a general treatment of forces which possess spherical symmetry. That is, we assume that m and V depend only on distance from the origin of coordinates.

Dirac's equations can be put in a more compact form using the spin operators σ_x, σ_y, σ_z, which are defined by

$$\sigma_x(\Phi, E, H, \Psi) = (-\vec{i}\cdot H, \vec{i}\times E - \vec{i}\Psi, \vec{i}\times H + \vec{i}\Phi, \vec{i}\cdot E),$$
$$\sigma_y(\Phi, E, H, \Psi) = (-\vec{j}\cdot H, \vec{j}\times E - \vec{j}\Psi, \vec{j}\times H + \vec{j}\Phi, \vec{j}\cdot E),$$
$$\sigma_z(\Phi, E, H, \Psi) = (-\vec{k}\cdot H, \vec{k}\times E - \vec{k}\Psi, \vec{k}\times H + \vec{k}\Phi, \vec{k}\cdot E).$$

They satisfy the identities

$$\begin{aligned} \sigma_x &= \sigma_y\sigma_z = -\sigma_z\sigma_y, \\ \sigma_y &= \sigma_z\sigma_x = -\sigma_x\sigma_z, \\ \sigma_z &= \sigma_x\sigma_y = -\sigma_y\sigma_x, \\ -1 &= \sigma_x^2 = \sigma_y^2 = \sigma_z^2. \end{aligned}$$

The operators I and J defined by

$$I(\Phi, E, H, \Psi) = (\Psi, H, -E, -\Phi),$$

$$J(\Phi, E, H, \Psi) = \left(\frac{\Phi+\Psi}{\sqrt{2}}, -\frac{E+H}{\sqrt{2}}, -\frac{E-H}{\sqrt{2}}, \frac{\Phi-\Psi}{\sqrt{2}}\right)$$

commute with the spin operators and anticommute with each other. Dirac's equations can be written

$$\frac{1}{c}\frac{\partial}{\partial t}(\Phi, E, H, \Psi) = K(\Phi, E, H, \Psi)$$

where the operator K is defined by

$$K = I\sigma_x\frac{\partial}{\partial x} + I\sigma_y\frac{\partial}{\partial y} + I\sigma_z\frac{\partial}{\partial z} + \frac{imc}{\hbar}J - \frac{iV}{\hbar c}.$$

The operators

$$\begin{aligned} L_x &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} - \tfrac{1}{2}\sigma_x, \\ L_y &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} - \tfrac{1}{2}\sigma_y, \\ L_z &= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \tfrac{1}{2}\sigma_z \end{aligned}$$

commute with K. If

$$D = 1 + \sigma_x\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \sigma_y\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \sigma_z\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),$$

then the operator JD commutes with L_x, L_y, L_z, and K.

Dirac's equations are soluble in spherical coordinates r, θ, φ, defined by

$$x = r \sin \theta \cos \varphi, \qquad y = r \sin \theta \sin \varphi, \qquad z = r \cos \theta.$$

In this notation

$$\frac{\partial}{\partial r} = \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z},$$

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi},$$

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial\varphi}.$$

We also use the spherical spin operators

$$\begin{aligned}\sigma_r &= \sigma_x \sin\theta\cos\varphi + \sigma_y\sin\theta\sin\varphi + \sigma_z\cos\theta,\\ \sigma_\theta &= \sigma_x\cos\theta\cos\varphi + \sigma_y\cos\theta\sin\varphi - \sigma_z\sin\theta,\\ \sigma_\varphi &= -\sigma_x\sin\varphi + \sigma_y\cos\varphi.\end{aligned}$$

A straightforward calculation will show that

$$D = 1 + \sigma_\varphi\frac{\partial}{\partial\theta} - \frac{\sigma_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}$$

and

$$K = I\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{I\sigma_r D}{r} + \frac{imc}{\hbar}J - \frac{iV}{\hbar c}.$$

The set of all (equivalence classes of) fields of finite energy becomes a Hilbert space if we choose the energy of the field as the square of its norm. A field which is an eigenfunction of D for the eigenvalue k is said to be a spherical harmonic of order k. The eigenvalues of D are integers. Eigenfunctions corresponding to different eigenvalues are orthogonal. The Hilbert space is the closed span of the spherical harmonics it contains.

For the construction of spherical harmonics, observe that every element of the Hilbert space is the orthogonal sum of an eigenfunction of σ_z for the eigenvalue i and an eigenfunction of σ_z for the eigenvalue $-i$. Since σ_y anticommutes with σ_z, it takes every eigenfunction for the eigenvalue i into an eigenfunction for the eigenvalue $-i$. Since $\sigma_x = \sigma_y\sigma_z$, the action of σ_x

coincides with the action of $-i\sigma_y$ on eigenfunctions for the eigenvalue i. So if $(\Phi_+, E_+, H_+, \Psi_+)$ is any eigenfunction for the eigenvalue i, there exists a corresponding eigenfunction $(\Phi_-, E_-, H_-, \Psi_-)$ for the eigenvalue $-i$ such that

$$\sigma_x(\Phi_+, E_+, H_+, \Psi_+) = i(\Phi_-, E_-, H_-, \Psi_-),$$

$$\sigma_y(\Phi_+, E_+, H_+, \Psi_+) = -(\Phi_-, E_-, H_-, \Psi_-).$$

Since $\sigma_x^2 = \sigma_y^2 = -1$, it follows that

$$\sigma_x(\Phi_-, E_-, H_-, \Psi_-) = i(\Phi_+, E_+, H_+, \Psi_+),$$

$$\sigma_y(\Phi_-, E_-, H_-, \Psi_-) = (\Phi_+, E_+, H_+, \Psi_+).$$

We use this information when $(\Phi_+, E_+, H_+, \Psi_+)$, and hence $(\Phi_-, E_-, H_-, \Psi_-)$, is a field which depends only on distance from the origin. Consider any column vector $\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ whose entries are complex valued functions of the angle variables, θ, φ, such that

$$u_+(\Phi_+, E_+, H_+, \Psi_+) + u_-(\Phi_-, E_-, H_-, \Psi_-)$$

belongs to the Hilbert space. The condition for this is

$$\int_0^{2\pi} \int_0^{\pi} [|u_+(\theta, \varphi)|^2 + |u_-(\theta, \varphi)|^2] \sin\theta \, d\theta \, d\varphi < \infty.$$

Then the action of σ_x, σ_y, σ_z on the field produces another field of the same form with coefficient functions which are linear combinations of u_+ and u_-. The action of the operators σ_x, σ_y, σ_z corresponds to the action of the matrices

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

It is convenient for the moment to identify spin operators with their representing matrices. In this notation

$$\sigma_r = \begin{pmatrix} i\cos\theta & ie^{i\varphi}\sin\theta \\ ie^{-i\varphi}\sin\theta & -i\cos\theta \end{pmatrix},$$

$$\sigma_\theta = \begin{pmatrix} -i\sin\theta & ie^{i\varphi}\cos\theta \\ ie^{-i\varphi}\cos\theta & i\sin\theta \end{pmatrix},$$

$$\sigma_\varphi = \begin{pmatrix} 0 & -e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}.$$

We use the eigenvectors of σ_r as a basis for the space of column vectors. It is easily verified that

$$\sigma_r\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = i\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_r\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = -i\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\theta\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = i\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\theta\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = i\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\varphi\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = \begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\varphi\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = -\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\varphi\frac{\partial}{\partial\theta}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = -\frac{1}{2}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} - \frac{\cot\theta}{2}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\sigma_\varphi\frac{\partial}{\partial\theta}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = -\frac{\cot\theta}{2}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\frac{\sigma_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} - \frac{\cot(\frac{1}{2}\theta)}{2}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix},$$

$$\frac{\sigma_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} = -\frac{\tan(\frac{1}{2}\theta)}{2}\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} + \frac{1}{2}\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix}.$$

An eigenfunction of D for the eigenvalue k is obtained by solving the equation

$$(k-1)f_+\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} + (k-1)f_-\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix}$$
$$= \left\{\sigma_\varphi\frac{\partial}{\partial\theta} - \frac{\sigma_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\right\}\left\{f_+\begin{pmatrix} e^{i\varphi}\cot^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \tan^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix} + f_-\begin{pmatrix} -e^{i\varphi}\tan^{\frac{1}{2}}(\frac{1}{2}\theta) \\ \cot^{\frac{1}{2}}(\frac{1}{2}\theta)\end{pmatrix}\right\}.$$

The vector equation is equivalent to the pair of scalar equations

$$kf_+ = -\frac{\partial f_-}{\partial\theta} - i\csc\theta\frac{\partial f_-}{\partial\varphi} + \tfrac{1}{2}\csc\theta f_-,$$

$$kf_- = \frac{\partial f_+}{\partial\theta} - i\csc\theta\frac{\partial f_+}{\partial\varphi} + \tfrac{1}{2}\csc\theta f_+.$$

If we define

$$f(\theta, \varphi) = f_+(\theta, \varphi) + if_-(\theta, \varphi),$$
$$f(-\theta, \varphi) = f_+(\theta, \varphi) - if_-(\theta, \varphi)$$

when $0 \leq \theta \leq \pi$, these equations can be written

$$kf(\theta, \varphi) = i\frac{\partial f(\theta, \varphi)}{\partial \theta} + \csc\theta \frac{\partial f(-\theta, \varphi)}{\partial \varphi} + \tfrac{1}{2}i \csc\theta f(-\theta, \varphi)$$

for $-\pi \leq \theta \leq \pi$. The energy of the corresponding field is a constant multiple of

$$\int_0^{2\pi} \int_{-\pi}^{\pi} |f(\theta, \varphi)|^2 d\theta\, d\varphi.$$

A solution of the equation is obtained of the form

$$f(\theta, \varphi) = g(\theta)e^{-ih\varphi}$$

if $g(\theta)$ is a solution of the equation

$$kg(\theta) = ig'(\theta) - i(h - \tfrac{1}{2}) \csc\theta g(-\theta).$$

This equation forms a connection with the theory of special Jacobi spaces.

PROBLEM 290. Let $(E(t, z))$ be the family of entire functions defined by Theorem 59 for some number $h > 0$. Define $\Phi(a, z)$ as in Problem 279. If $f(x)$ belongs to $L^2(-\pi, \pi)$ and vanishes outside of some interval $(-a, a)$, $a < \pi$, show that its eigentransform $F(z)$, defined by

$$2\pi F(z) = \int_{-\pi}^{\pi} f(t)\Phi(t, z)dt,$$

belongs to $\mathcal{H}(E(a))$ and that

$$2\pi \int_{-\infty}^{+\infty} |F(t)/E(a, t)|^2 dt = \int_{-\pi}^{\pi} |f(t)|^2 dt.$$

Show that every element of $\mathcal{H}(E(a))$ is of this form. Let H be the transformation defined by

$$H: f(x) \to g(x) = if'(x) - i(h - \tfrac{1}{2}) \csc x f(-x)$$

whenever $f(x)$ is (equivalent to) an absolutely continuous function which belongs to $L^2(-\pi, \pi)$ and $g(x)$ belongs to $L^2(-\pi, \pi)$. If $f(x)$ and $g(x)$ are elements of $L^2(-\pi, \pi)$ which vanish outside of $(-a, a)$ and if $F(z)$ and $G(z)$ are their eigentransforms, show that $G(z) = zF(z)$ is a necessary and sufficient condition that $f(x)$ be in the domain of H and that $H: f(x) \to g(x)$.

Show that

$$A(a, z) = \cos^{2h}(\tfrac{1}{2}a)F(h - z, h + z; h; \sin^2(\tfrac{1}{2}a))$$

and

$$B(a, z) = \sin^{2h}(\tfrac{1}{2}a)(z/h)F(h - z, h + z; h + 1; \sin^2(\tfrac{1}{2}a))$$

for $0 < a < \pi$.

PROBLEM 291. Let h be a given positive number. If $0 < a < \pi$, let $\mathcal{M}(a)$ be the set of entire functions $F(z)$ of bounded type and of mean type at most a in the upper and lower half-planes, such that

$$\|F\|^2 = \sum_{n=0}^{\infty} |F(h + n)|^2\Gamma(2h + n)/n! + \sum_{n=0}^{\infty} |F(-h - n)|^2\Gamma(2h + n)/n! < \infty.$$

Show that $\mathcal{M}(a)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) and which contains a nonzero element. Show that $\mathcal{M}(a)$ is equal isometrically to a space $\mathcal{H}(E_1(a))$ for some entire function $E_1(a, z)$ such that $E_1^*(a, z) = E_1(a, -z)$. Show that $E_1(b, z)/E_1(a, z)$ is of bounded type and of mean type $b - a$ in the upper half-plane when $0 < a < b < \pi$. Show that the intersection of the spaces $\mathcal{H}(E_1(a))$, $0 < a < \pi$, contains no nonzero element. Verify that $\mathcal{H}(E_1(a))$ satisfies the hypotheses of Theorem 58 for every index a. If the spaces $\mathcal{H}(E(a))$ are defined as in Theorem 59 for the present choice of h, show that there exists an entire function $S(z)$, which is real for real z and periodic of period one, such that $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{H}(E(a))$ onto $\mathcal{H}(E_1(a))$ for every index a. Show that $E_1(a, z)$ can be chosen so that $E_1(a, z) = S(z)E(a, z)$ for every a. Show that $S(z)$ is an even entire function of Pólya class which has no zeros. Use this to show that $S(z)$ is a constant. Show that

$$\Phi(a, h) = e^{-\frac{1}{2}ia}|\sin(\tfrac{1}{2}a)\cos(\tfrac{1}{2}a)|^{h-\frac{1}{2}}$$

and that

$$\int_{-\pi}^{\pi} |\Phi(t, h)|^2 dt = 2\Gamma(h)^2/\Gamma(2h).$$

Show that

$$S(z) = \sqrt{\pi}/\Gamma(h).$$

Since JD commutes with K, the eigenfunctions of JD are more convenient than the eigenfunctions of D for solving Dirac's equations. The two are closely related since J and D commute. The eigenvalues of J are 1 and -1. Every element of the Hilbert space is the orthogonal sum of an eigenfunction of J for the eigenvalue 1 and an eigenfunction of J for the eigenvalue -1. The eigenvalues of JD are integers. The Hilbert space is the orthogonal sum of eigenfunctions of JD. Eigenfunctions of JD can be obtained in the form

$$(\Phi, E, H, \Psi) = r^{-1}u_+(\theta, \varphi)(\Phi_+, E_+, H_+, \Psi_+) + r^{-1}u_-(\theta, \varphi)(\Phi_-, E_-, H_-, \Psi_-)$$

where $u_+(\theta, \varphi)$ and $u_-(\theta, \varphi)$ are complex valued functions of the angle variables and the fields $(\Phi_+, E_+, H_+, \Psi_+)$ and $(\Phi_-, E_-, H_-, \Psi_-)$ depend only on distance from the origin. The fields can be chosen so that

$$J\sigma_\varphi(\Phi_+, E_+, H_+, \Psi_+) = -e^{i\varphi}(\Phi_-, E_-, H_-, \Psi_-).$$

If such a field is an eigenfunction of JD for the eigenvalue k, the Dirac equations read

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}(\Phi_\pm, E_\pm, H_\pm, \Psi_\pm) \\ = I\sigma_r \frac{\partial}{\partial r}(\Phi_\pm, E_\pm, H_\pm, \Psi_\pm) + k\frac{I\sigma_r J}{r}(\Phi_\pm, E_\pm, H_\pm, \Psi_\pm) \\ + \frac{imcJ}{\hbar}(\Phi_\pm, E_\pm, H_\pm, \Psi_\pm) - \frac{iV}{\hbar c}(\Phi_\pm, E_\pm, H_\pm, \Psi_\pm). \end{aligned}$$

We now drop subscripts in working with the radial equation. Since $I\sigma_r$ and J anticommute, the radial equation has solutions of the form

$$(\Phi, E, H, \Psi) = f_+(r, t)(\Phi_+, E_+, H_+, \Psi_+) + f_-(r, t)(\Phi_-, E_-, H_-, \Psi_-)$$

where $(\Phi_+, E_+, H_+, \Psi_+)$ and $(\Phi_-, E_-, H_-, \Psi_-)$ now denote constant fields such that

$$\begin{aligned} J(\Phi_+, E_+, H_+, \Psi_+) &= (\Phi_+, E_+, H_+, \Psi_+), \\ J(\Phi_-, E_-, H_-, \Psi_-) &= -(\Phi_-, E_-, H_-, \Psi_-), \\ I\sigma_r(\Phi_+, E_+, H_+, \Psi_+) &= i(\Phi_-, E_-, H_-, \Psi_-), \\ I\sigma_r(\Phi_-, E_-, H_-, \Psi_-) &= -i(\Phi_+, E_+, H_+, \Psi_+). \end{aligned}$$

The Dirac equations reduce to the pair of scalar equations

$$\begin{aligned} \frac{1}{c}\frac{\partial f_+}{\partial t} &= -i\frac{\partial f_-}{\partial r} + \frac{ik}{r}f_- + \frac{imc}{\hbar}f_+ - \frac{iV}{\hbar c}f_+, \\ \frac{1}{c}\frac{\partial f_-}{\partial t} &= i\frac{\partial f_+}{\partial r} + \frac{ik}{r}f_+ - \frac{imc}{\hbar}f_- - \frac{iV}{\hbar c}f_-. \end{aligned}$$

The energy of the field is a constant multiple of

$$\int_0^\infty |f_+(r, t)|^2 dr + \int_0^\infty |f_-(r, t)|^2 dr.$$

These equations can be simplified by a change of variable if m is a differentiable function of r. Let

$$\begin{aligned} g_+(r, t) &= f_+(r, t)\cos\omega - f_-(r, t)\sin\omega, \\ g_-(r, t) &= f_+(r, t)\sin\omega + f_-(r, t)\cos\omega \end{aligned}$$

where $\omega = \omega(r)$ is a differentiable function of r such that

$$(mc/\hbar) \cos(2\omega) = (k/r) \sin(2\omega).$$

Define

$$g(r, t) = g_+(r, t) + ig_-(r, t),$$
$$g(-r, t) = g_+(r, t) - ig_-(r, t)$$

for $r > 0$. A straightforward calculation will show that the radial Dirac equations reduce to the single equation

$$\frac{1}{ic}\frac{\partial g(r, t)}{\partial t} = i\frac{\partial g(r, t)}{\partial r} - i\rho'(r)g(-r, t) + \sigma'(r)g(r, t)$$

where

$$\rho'(r)^2 = (mc/\hbar)^2 + (k/r)^2$$

and

$$\sigma'(r) = -\frac{V}{\hbar c} + \frac{(k/r)(m'c/\hbar) + (mc/\hbar)(k/r^2)}{2\rho'(r)^2}.$$

The theory of Jacobi spaces suggests that the radial Dirac equation may be associated with special Hilbert spaces of entire functions. But as yet no interesting special spaces have been obtained by this method.

61. CONSTRUCTION OF LOCAL OPERATORS

Hilbert spaces of entire functions have applications in Fourier analysis. If $K(x)$ is a Borel measurable function of real x, define a corresponding operator $K(H)$ on absolutely convergent Fourier transforms

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

by $K(H): f(x) \to g(x)$ whenever

$$g(x) = \int_{-\infty}^{+\infty} e^{ixt} K(t) d\mu(t)$$

is an absolutely convergent Fourier transform. For example, if $K(x) = x$, the operator $K(H) = H$ is minus i times differentiation. The formula

$$g(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{ih},$$

which is equivalent to the formula

$$\int_{-\infty}^{+\infty} e^{ixt} t\, d\mu(t) = \lim_{h\to 0} \int_{-\infty}^{+\infty} \frac{e^{iht} - 1}{iht} e^{ixt} t\, d\mu(t),$$

is justified by the Lebesgue dominated convergence theorem since

$$\left| \frac{e^{iht} - 1}{iht} \right| \leq 1$$

for all real numbers h and t. If $K(x)$ is any polynomial in x, the operator $K(H)$ has a similar interpretation as an ordinary differential operator. Differential operators are examples of local operators.

An operator $K(H)$ is said to be local if whenever two functions $f_1(x)$ and $f_2(x)$ in its domain agree in a neighborhood of a point x_0, then $K(H): f_1(x) \to g_1(x)$ and $K(H): f_2(x) \to g_2(x)$ where the functions $g_1(x)$ and $g_2(x)$ agree at x_0. The domain of any operator $K(H)$ is a vector space and $K(H)$ is linear. An equivalent condition for $K(H)$ to be local is obtained on setting $f(x) = f_2(x) - f_1(x)$ and $g(x) = g_2(x) - g_1(x)$. Whenever a function $f(x)$ in the domain of $K(H)$ vanishes in a neighborhood of a point x_0, then $K(H): f(x) \to g(x)$ where $g(x)$ vanishes at x_0. Note that an operator $K(H)$ commutes with translations. If $K(H): f(x) \to g(x)$ and if h is a real number, then $f(x - h)$ is in the domain of $K(H)$ and $K(H): f(x - h) \to g(x - h)$. So a simple condition for $K(H)$ to be local can be given: Whenever a function $f(x)$ in the domain of $K(H)$ vanishes in a neighborhood of the origin, then $K(H): f(x) \to g(x)$ where $g(x)$ vanishes at the origin.

Let a be a given positive number. The operator $K(H)$ is said to be a-local if it satisfies this condition: Whenever $f(x)$ is in the domain of $K(H)$ and vanishes in the interval $[-a, a]$, then $K(H): f(x) \to g(x)$ where $g(x)$ vanishes at the origin. An operator $K(H)$ is local if, and only if, it is a-local for every positive number a. The theory of a-local operators is related to the theory of entire functions which are of bounded type in the upper and lower half-planes.

THEOREM 61. If a is a given positive number, if $K(z)$ is an entire function which is of bounded type in the upper and lower half-planes, and if

$$\exp(-a|y|)K(iy)$$

is bounded, $-\infty < y < \infty$, then the operator $K(H)$ is a-local.

Proof of Theorem 61. If $f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$ is in the domain of $K(H)$, consider the function

$$L(z) = \int_{-\infty}^{+\infty} \frac{K(t) - K(z)}{t - z} \, d\mu(t),$$

which is entire by the proof of Theorem 26. Since the functions

$$\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - z} \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{K(t) d\mu(t)}{t - z}$$

are of bounded type in the upper half-plane by Problem 65, and since $K(z)$ is of bounded type in the upper half-plane by hypothesis, $L(z)$ is of bounded type in the upper half-plane. A similar argument will show that $L(z)$ is of bounded type in the lower half-plane. Since

$$\int_0^h e^{iut}e^{-iuz}du = \frac{e^{iht}e^{-ihz} - 1}{i(t-z)}$$

and since $f(x)$ vanishes in the interval $[-a, a]$, we obtain

$$\int_{-\infty}^{+\infty} \frac{e^{iht}e^{-ihz} - 1}{i(t-z)}\, d\mu(t) = \int_0^h \int_{-\infty}^{+\infty} e^{iut}d\mu(t)e^{-iuz}du \doteq 0$$

when $-a \leq h \leq a$. The change in the order of integration is justified by absolute convergence of the double integral. It follows that

$$e^{ihz}L(z) = \int_{-\infty}^{+\infty} \frac{e^{ihz}K(t) - e^{iht}K(z)}{t-z}\, d\mu(t)$$

when $-a \leq h \leq a$. A consequence of the identity is the estimate

$$|L(iy)| \leq |y|^{-1} \int_{-\infty}^{+\infty} |K(t)d\mu(t)| + |y|^{-1}\, |e^{hy}K(iy)| \int_{-\infty}^{+\infty} |d\mu(t)|.$$

Since we assume that $e^{hy}K(iy)$ is bounded, $h = a$ and $h = -a$, we can conclude that $L(z)$ goes to zero at both ends of the imaginary axis. By Problem 39, $L(z)$ vanishes identically. If $K(H): f(x) \rightarrow g(x)$, then

$$\begin{aligned} g(0) &= \int_{-\infty}^{+\infty} K(t)d\mu(t) \\ &= \int_{-\infty}^{+\infty} \frac{tK(t)d\mu(t)}{t-z} - z\int_{-\infty}^{+\infty} \frac{K(t)d\mu(t)}{t-z} \\ &= \int_{-\infty}^{+\infty} \frac{tK(t)d\mu(t)}{t-z} - ze^{iaz}K(z)\int_{-\infty}^{+\infty} \frac{e^{-iat}d\mu(t)}{t-z} \end{aligned}$$

when z is not real. By the Lebesgue dominated convergence theorem,

$$\lim_{y\to+\infty} - iy\int_{-\infty}^{+\infty} \frac{e^{-iat}d\mu(t)}{t-iy} = \int_{-\infty}^{+\infty} e^{-iat}d\mu(t) = 0,$$

$$\lim_{y\to+\infty} \int_{-\infty}^{+\infty} \frac{tK(t)d\mu(t)}{t-iy} = 0.$$

It follows that $g(0) = 0$.

62. DETERMINATION OF LOCAL OPERATORS

A local operator is determined by an entire function if its domain contains a function which vanishes in an interval and which does not vanish identically.

THEOREM 62. Let a be a given positive number and let $K(x)$ be a Borel measurable function of real x such that the operator $K(H)$ is a-local. If the domain of the operator contains a function which vanishes in the interval $[-a, a]$ and which does not vanish identically, then $K(x)$ is the restriction to the real axis of an entire function $K(z)$ which is of bounded type and of mean type at most a in the upper and lower half-planes.

Proof of Theorem 62. By hypothesis there exists a function

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu_1(t)$$

in the domain of $K(H)$ which vanishes in the interval $[-a, a]$ and which does not vanish identically. Let

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} B(t)\, d\mu(t)$$

where μ is a nonnegative measure and $B(x)$ is a Borel measurable function having absolute value 1 everywhere. Since $f(x)$ vanishes in the interval $[-a, a]$ and does not vanish identically, the closed span of the functions e^{ihx}, $-a \leq h \leq a$, is not all of $L^1(\mu)$. Since $f(x)$ is in the domain of $K(H)$, $K(x)$ belongs to $L^1(\mu)$. The hypothesis that $K(H)$ is a-local implies that $\int_{-\infty}^{+\infty} K(t)C(t)\, d\mu(t) = 0$ whenever $C(x)$ is a bounded measurable function such that $\int_{-\infty}^{+\infty} e^{ixt} C(t)\, d\mu(t) = 0$ for $-a \leq x \leq a$. Since every continuous linear functional on $L^1(\mu)$ is of the form $F(x) \to \int_{-\infty}^{+\infty} F(t)C(t)\, d\mu(t)$ for some bounded measurable function $C(x)$, it follows from the Hahn-Banach theorem that $K(x)$ belongs to the closed span in $L^1(\mu)$ of the functions e^{ihx}, $-a \leq h \leq a$. To study the closed span we introduce the majorant

$$M(z) = \sup |L(z)|$$

where the supremum is taken over all finite linear combinations of the functions e^{ihz}, $-a \leq h \leq a$, such that $\int_{-\infty}^{+\infty} |L(t)|\, d\mu(t) \leq 1$. We show that the supremum is finite.

If $L(z)$ is such a linear combination, then

$$\int_{-\infty}^{+\infty} \frac{L(t)B(t)\, d\mu(t)}{t - z} = L(z) \int_{-\infty}^{+\infty} \frac{B(t)\, d\mu(t)}{t - z}$$

when z is not real, by the proof of Theorem 61. It follows that

$$|L(z)|\,|S(z)| \le |y|^{-1}\int_{-\infty}^{+\infty} |L(t)\,d\mu(t)|$$

where

$$S(z) = \int_{-\infty}^{+\infty} \frac{B(t)\,d\mu(t)}{t-z}\,.$$

By the arbitrariness of $L(z)$,

$$M(z)\,|S(z)| \le |y|^{-1}.$$

Since $f(x)$ does not vanish identically, μ is not the zero measure. By the Stieltjes inversion formula,

$$\frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{B(t)\,d\mu(t)}{(t-x)^2+y^2}$$

does not vanish identically for $y > 0$. It follows that $S(z)$ does not vanish identically in the upper and lower half-planes. Assume for definiteness that the function does not vanish identically in the upper half-plane. Let $y_1 > 0$ be chosen so that $S(x+iy_1) \ne 0$ for $-\infty < x < \infty$. Since $S(z)$ is of bounded type in the upper half-plane by Problem 65, we have

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1}\log^+ |1/S(t+iy_1)|\,dt < \infty$$

by Problem 27. Since

$$M(t+iy_1) \le |y_1|^{-1}\,|1/S(t+iy_1)|,$$

we obtain

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1}\log^+ M(t+iy_1)\,dt < \infty.$$

Since $L(z) = 1\Big/\int_{-\infty}^{+\infty} d\mu(t)$ is a finite linear combination of the functions e^{ihz}, $-a \le h \le a$, such that $\int_{-\infty}^{+\infty} |L(t)|\,d\mu(t) \le 1$, we can conclude that

$$M(z) \ge 1\Big/\int_{-\infty}^{+\infty} d\mu(t)$$

and that

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1}\,|\log M(t+iy_1)|\,dt < \infty.$$

But if $L(z)$ is a finite linear combination of the functions e^{ihz}, $-a \le h \le a$, then $L(z+iy_1)$ is a function of bounded type and of mean type at most a in the upper half-plane. By Problem 27,

$$\log|L(x+iy_2)| \le a\,|y_2-y_1| + \frac{|y_2-y_1|}{\pi}\int_{-\infty}^{+\infty} \frac{\log|L(t+iy_1)|\,dt}{(t-x)^2+(y_2-y_1)^2}$$

$$\le a\,|y_2-y_1| + \frac{|y_2-y_1|}{\pi}\int_{-\infty}^{+\infty} \frac{\log M(t+iy_1)\,dt}{(t-x)^2+(y_2-y_1)^2}$$

when $y_2 > y_1$. A similar argument will show that the inequality holds when $y_2 < y_1$. By the arbitrariness of $L(z)$,

$$\log M(x+iy_2) \le a\,|y_2 - y_1| + \frac{|y_2 - y_1|}{\pi}\int_{-\infty}^{+\infty} \frac{\log M(t+iy_1)\,dt}{(t-x)^2 + (y_2-y_1)^2}$$

when $y_2 \ne y_1$. By the semigroup property of Poisson kernels we see that the same estimate holds with y_1 and y_2 interchanged. For the integral then appearing on the right is absolutely convergent and the previous argument applies with y_1 replaced by y_2. From this we see that $M(z)$ is finite and locally bounded in the complex plane.

Let $(L_n(z))$ be a sequence of finite linear combinations of the functions e^{ihz}, $-a \le h \le a$, such that $K(x) = \lim L_n(x)$ in the metric of $L^1(\mu)$. Since the sequence is Cauchy in the metric of $L^1(\mu)$ and since

$$|L_n(z) - L_r(z)| \le M(z)\int_{-\infty}^{+\infty} |L_n(t) - L_r(t)|\,d\mu(t)$$

for all complex z, $H(z) = \lim L_n(z)$ exists uniformly on every bounded set. The limit is an entire function such that $H(t) = K(t)$ almost everywhere with respect to μ and such that

$$|H(z)| \le M(z)\int_{-\infty}^{+\infty} |H(t)|\,d\mu(t)$$

for all complex z. Since

$$\log M(z) \le a|y| + \frac{|y|}{\pi}\int_{-\infty}^{+\infty} \frac{\log^+ M(t)\,dt}{(t-x)^2 + y^2}$$

for $y \ne 0$, we see that $H(z)$ is of bounded type and of mean type at most a in both half-planes.

To see that $H(x) = K(x)$ for all real x, consider any nonnegative measure μ' on the Borel sets of the real line such that $\mu \le \mu'$ and

$$\int_{-\infty}^{+\infty} [1 + |K(t)|]\,d\mu'(t) < \infty.$$

Then there exists a Borel measurable function $B'(x)$, which is bounded by 1, such that

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} B'(t)\,d\mu'(t)$$

for all real x. Let

$$M'(z) = \sup |L(z)|$$

where the supremum is taken over all finite linear combinations of the functions e^{ihz}, $-a \le h \le a$, such that $\int_{-\infty}^{+\infty} |L(t)|\,d\mu'(t) \le 1$. Since $\mu \le \mu'$, $M'(z) \le M(z)$ for all complex z. The above argument will show that there exists an entire function $H'(z)$ such that $K(x) = H'(x)$ almost everywhere with respect to μ' and such that

$$|H'(z)| \leq M'(z) \int_{-\infty}^{+\infty} |K(t)| d\mu'(t)$$

for all complex z. There exists a sequence $(L'_n(z))$ of finite linear combinations of the functions e^{ihz}, $-a \leq h \leq a$, such that $H'(z) = \lim L'_n(z)$ for all complex z and such that $K(x) = \lim L'_n(x)$ in the metric of $L^1(\mu')$. But then $K(x) = \lim L'_n(x)$ in the metric of $L^1(\mu)$ and $0 = \lim [L'_n(x) - L_n(x)]$ in the metric of $L^1(\mu)$. Since

$$|L'_n(z) - L_n(z)| \leq M(z) \int_{-\infty}^{+\infty} |L'_n(t) - L_n(t)| d\mu(t)$$

for all complex z, we obtain $H'(z) = H(z)$ for all complex z in the limit as $n \to \infty$. So $K(x) = H(x)$ almost everywhere with respect to μ'. Since μ' can be chosen with positive mass at any given point, $K(x) = H(x)$ for all real x.

63. NONVANISHING FOURIER TRANSFORMS

There also exist local operators whose domain contains no function which vanishes in an interval and which does not vanish identically.

THEOREM 63. Let $K(x)$ be a continuous function of real x such that $K(x) \geq 1$, $\log K(x)$ is uniformly continuous, and

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log K(t) dt = \infty.$$

Then there is no nonzero measure μ on the Borel sets of the real line such that

$$\int_{-\infty}^{+\infty} K(t) \, |d\mu(t)| < \infty$$

and such that

$$\int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

vanishes in an interval.

Proof of Theorem 63. Let $a > 0$ and let

$$T(x) = T_a(x) = \sup |L(x)|$$

where the supremum is taken over all entire functions $L(z)$, which are of bounded type in the upper and lower half-planes, such that

$$\exp(-a|y|) L(iy)$$

is bounded, $-\infty < y < \infty$. The proof depends on a lower estimate of $T(x)$. By the uniform continuity of $\log K(x)$, there is some $\epsilon > 0$ such that

$$|\log K(x_2) - \log K(x_1)| \le \epsilon$$

whenever $|x_2 - x_1| \le \frac{1}{2}\pi a$. Then

$$|\log K(x_2) - \log K(x_1)| \le n\epsilon$$

when $|x_2 - x_1| \le \frac{1}{2}\pi n a$, $n = 1, 2, 3, \cdots$. The function

$$L(z) = \left[\frac{\pi n}{2a(z - x_0)} \sin \frac{a(z - x_0)}{n}\right]^n$$

is bounded on the real axis with bound $(\frac{1}{2}\pi)^n$, and it is bounded by 1 on the set $|x - x_0| \ge \frac{1}{2}\pi n a$. It follows that $|L(x)| \le T(x)$ for all real x if

$$n\epsilon + n \log (\tfrac{1}{2}\pi) \le \log K(x_0).$$

In this case, $L(z)$ is one of the entire functions in the supremum definition of $T(x)$. Since it has value $(\frac{1}{2}\pi)^n$ at x_0, we obtain

$$\log T(x_0) \ge n \log (\tfrac{1}{2}\pi)$$

when n is so chosen. By the arbitrariness of n and x_0,

$$\log T(x) \ge \frac{\log (\frac{1}{2}\pi)}{\epsilon + \log (\frac{1}{2}\pi)} \log K(x) - \log (\tfrac{1}{2}\pi)$$

for all real x. The hypotheses on $T(x)$ now imply that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log^+ T(t)dt = \infty.$$

To prove the theorem it is sufficient to show that there is no nonzero measure μ on the Borel sets of the real line such that

$$\int_{-\infty}^{+\infty} K(t)\, |d\mu(t)| < \infty$$

and such that

$$\int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

vanishes for $-a \le x \le a$. Argue by contradiction, assuming that such a measure exists. Let

$$M(z) = \sup |L(z)|$$

where the supremum is taken over all finite linear combinations of the functions e^{ihz}, $-a \le h \le a$, such that

$$\int_{-\infty}^{+\infty} |L(t)d\mu(t)| \le 1.$$

More generally, consider any entire function $L(z)$ which is of bounded type in the upper and lower half-planes, such that $\exp(-a|y|)L(iy)$ is bounded, $-\infty < y < \infty$. By Theorem 61 the operator $L(H)$ is a-local. By the proof of Theorem 62,

$$|L(z)| \leq M(z) \int_{-\infty}^{+\infty} |L(t)d\mu(t)|$$

for all complex z. By the arbitrariness of $L(z)$,

$$T(x) \leq M(x)$$

for all real x where

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log^+ M(t)dt < \infty$$

by the proof of Theorem 62. Since we know that $T(x)$ cannot be this small, no such measure μ can exist.

PROBLEM 292. If $u(z)$ is the real part of a function analytic in a region containing the closed rectangle $a \leq x \leq b$, $c \leq y \leq d$, show that

$$\int_c^d \int_a^b \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} dx\, dy$$
$$= -\int_a^b u(c + ix)\frac{\partial u}{\partial y}(c + ix)dx + \int_c^d u(b + iy)\frac{\partial u}{\partial x}(b + iy)dy$$
$$+ \int_a^b u(d + ix)\frac{\partial u}{\partial y}(d + ix)dx - \int_c^d u(a + iy)\frac{\partial u}{\partial x}(a + iy)dy.$$

PROBLEM 293. Let $f(z)$ be a function analytic in a region containing the closed annular section $a \leq r \leq b$, $c \leq \theta \leq d$ where $0 < a < b < \infty$ and $0 \leq c < d < 2\pi$. If $f(z) = u(z) + iv(z)$ where $u(z)$ and $v(z)$ are real valued functions, show that

$$\int_c^d \int_a^b \left\{\left(\frac{\partial u}{\partial r}\right)^2 + r^2\left(\frac{\partial u}{\partial \theta}\right)^2\right\} r\, dr\, d\theta$$
$$= \int_a^b u(re^{ic})\frac{\partial v}{\partial r}(re^{ic})dr + \int_c^d u(be^{i\theta})\frac{\partial v}{\partial \theta}(be^{i\theta})d\theta$$
$$- \int_a^b u(re^{id})\frac{\partial v}{\partial r}(re^{id})dr - \int_c^d u(ae^{i\theta})\frac{\partial v}{\partial \theta}(ae^{i\theta})d\theta.$$

PROBLEM 294. Let $f(z)$ be a function which is analytic and whose real part is nonnegative in the upper half-plane. If

$$\operatorname{Re} f(z) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

is the Poisson representation, show that

$$\lim_{y \searrow 0} \operatorname{Re} f(iy)/y = p + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{t^2}\,.$$

64. BEURLING-MALLIAVIN THEOREM

A theorem of Beurling and Malliavin gives information about the domain of local operators on Fourier transforms.

THEOREM 64. Let $F(z)$ be an entire function which is of bounded type in the upper and lower half-planes. If $a > 0$ is a given number, there exists a nonzero entire function $G(z)$, which is of bounded type in the upper and lower half-planes and of mean type at most a in these half-planes, such that $F(z)G(z)$ and $G(z)$ are bounded on the real axis.

A Hilbert space method is used to prove the theorem.

LEMMA 11. If $E(z)$ is an entire function of Pólya class which has value 1 at the origin and which is bounded away from zero in the upper half-plane, then

$$E(z) = \exp\,[zf(z)]$$

where $f(z)$ is analytic in the upper half-plane and continuous in the closed half-plane, and the derivative $f'(z)$ belongs to the Hardy space $\mathfrak{D}_1$:

$$\pi\|f'(z)\|_1^2 = \int_0^{\infty}\int_{-\infty}^{+\infty} |f'(x+iy)|^2 dx\, dy < \infty.$$

LEMMA 12. Let $f(z)$ be a function which is analytic in the upper half-plane and whose derivative $f'(z)$ belongs to $\mathfrak{D}_1$. If $a > 0$ is a given number, there exists a function $g(z)$, analytic in the upper half-plane and with derivative $g'(z)$ in $\mathfrak{D}_1$, such that

$$\operatorname{Re} zg(z) \le \operatorname{Re} zf(z)$$

and

$$\operatorname{Re} izg'(z) \ge -a$$

for $y > 0$. If

$$\tau = \liminf_{y\to\infty} \operatorname{Re} if(iy) > -\infty$$

and if

$$\liminf_{y\to 0} \operatorname{Re} if(iy) > -\infty,$$

then $\operatorname{Re} ig(z)$ has a lower bound in the upper half-plane and

$$\tau \le \liminf_{r\to\infty} \operatorname{Re} ig(re^{i\theta})$$

uniformly for $0 \le \theta \le \pi$.

Proof of Lemma 11. As in the proof of Theorem 15,

$$\operatorname{Re} i\,\frac{E'(z)}{E(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\varphi(t)}{(t-x)^2+y^2}$$

for $y > 0$, where $\varphi(x)$ is the phase function associated with $E(z)$ which has value zero at the origin. Since $E(z)$ is bounded away from zero in the upper half-plane and has value 1 at the origin, we can write $E(z) = \exp[zf(z)]$ where $f(z)$ is analytic in the upper half-plane and continuous in the closed half-plane. Let $f(z) = u(z) + iv(z)$ where $u(z)$ and $v(z)$ are real valued functions. By Problem 293,

$$\int_0^\pi\int_\alpha^\beta\left\{\left(\frac{\partial u}{\partial r}\right)^2 + r^2\left(\frac{\partial u}{\partial\theta}\right)^2\right\}r\,dr\,d\theta$$

$$= \int_\alpha^\beta u(t)dv(t) + \int_0^\pi u(\beta e^{i\theta})\,\frac{\partial v}{\partial\theta}\,(\beta e^{i\theta})d\theta$$

$$+ \int_{-\beta}^{-\alpha} u(t)dv(t) - \int_0^\pi u(\alpha e^{i\theta})\,\frac{\partial v}{\partial\theta}\,(\alpha e^{i\theta})d\theta$$

when $0 < \alpha < \beta < \infty$. Since $r(\partial u/\partial r) = \partial v/\partial\theta$, the identity can be written

$$\int_0^\pi\int_\alpha^\beta |f'(re^{i\theta})|^2\,r\,dr\,d\theta = \int_\alpha^\beta u(t)dv(t) + \int_{-\beta}^{-\alpha} u(t)dv(t)$$

$$+ \tfrac{1}{2}\beta\,\frac{d}{d\beta}\int_0^\pi u(\beta e^{i\theta})^2 d\theta - \tfrac{1}{2}\alpha\,\frac{d}{d\alpha}\int_0^\pi u(\alpha e^{i\theta})^2 d\theta.$$

Since there exists a number $c \geq 0$ such that $c + \log|E(x)|$ is nonnegative and since $\varphi(x) = -xv(x)$ is nondecreasing,

$$\int_0^\pi\int_\alpha^\beta |f'(re^{i\theta})|^2\,r\,dr\,d\theta \leq \int_\alpha^\beta \frac{c+\log|E(t)|}{t^2}\,\frac{\varphi(t)}{t}\,dt$$

$$+ \int_{-\beta}^{-\alpha}\frac{c+\log|E(t)|}{t^2}\,\frac{\varphi(t)}{t}\,dt + c\int_\alpha^\beta\frac{d\varphi(t)}{t^2} + c\int_{-\beta}^{-\alpha}\frac{d\varphi(t)}{t^2}$$

$$+ \tfrac{1}{2}\beta\,\frac{d}{d\beta}\int_0^\pi u(\beta e^{i\theta})^2 d\theta - \tfrac{1}{2}\alpha\,\frac{d}{d\alpha}\int_0^\pi u(\alpha e^{i\theta})^2 d\theta.$$

We show that

$$\liminf_{r\to\infty} r\,\frac{d}{dr}\int_0^\pi u(re^{i\theta})^2 d\theta \leq 0.$$

If this were not true there would exist some $\epsilon > 0$ such that

$$r\,\frac{d}{dr}\int_0^\pi u(re^{i\theta})^2 d\theta \geq \epsilon$$

for all sufficiently large values of r, a condition which implies that

$$\lim_{r\to\infty} \int_0^\pi u(re^{i\theta})^2 d\theta = \infty.$$

This is impossible because $E(z)$ is of exponential type by Kreĭn's theorem, Problem 37. So as $\beta \to \infty$ we obtain

$$\int_0^\pi \int_\alpha^\infty |f'(re^{i\theta})|^2 r\,dr\,d\theta \le \int_\alpha^\infty \frac{c + \log|E(t)|}{t^2} \frac{\varphi(t)}{t}\,dt$$

$$+ \int_{-\infty}^{-\alpha} \frac{c + \log|E(t)|}{t^2} \frac{\varphi(t)}{t}\,dt + c\int_\alpha^\infty \frac{d\varphi(t)}{t^2} + c\int_{-\infty}^{-\alpha} \frac{d\varphi(t)}{t^2}$$

$$- \tfrac{1}{2}\alpha \frac{d}{d\alpha} \int_0^\pi u(\alpha e^{i\theta})^2 d\theta.$$

The integrals are finite because

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log^+ |E(t)|\,dt < \infty$$

by Problem 27, because

$$\limsup_{|x|\to\infty} \varphi(x)/x < \infty$$

by Theorem 15 and because

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} d\varphi(t) < \infty$$

by the Poisson representation of Re $iE'(z)/E(z)$.

Proof of Lemma 12. Let $\mathcal{M}$ be the set of all functions $g(z)$, analytic in the upper half-plane and with derivative $g'(z)$ in $\mathfrak{D}_1$, such that

$$\text{Re}\; zg(z) \le \text{Re}\; zf(z)$$

for $y > 0$ and such that

$$\lim_{y\searrow 0} \text{Re}\,[if(iy) - ig(iy)] < \infty.$$

The existence of the limit is a consequence of the Poisson representation of Re $[zf(z) - zg(z)]$, Problem 294. The set $\mathcal{M}$ is not empty since it contains $f(z)$. Define

$$m = \inf\{\|g'(z)\|_1^2 + 2a \lim_{y\searrow 0} \text{Re}\,[if(iy) - ig(iy)]\}$$

where the infimum is taken over all elements $g(z)$ of $\mathcal{M}$. We show that the infimum is attained by an element of $\mathcal{M}$. Let $(g_n(z))$ be a sequence of elements of $\mathcal{M}$ such that

$$\|g_n'(z)\|_1^2 + 2a \lim_{y\searrow 0} \operatorname{Re}\,[if(iy) - ig_n(iy)] \leq m + 1/n$$

for every n. By applying the Helly selection principle in the Poisson representation, we can choose the sequence so that

$$\lim_{n\to\infty} \operatorname{Re}\,[zf(z) - zg_n(z)]$$

exists for $y > 0$. Since we can add a real constant to $g_n(z)$ without changing the above conditions on the functions, we can choose the sequence so that $g(z) = \lim g_n(z)$ exists for $y > 0$. Convergence is uniform on any bounded set at a positive distance from the real axis. The limit function is analytic in the upper half-plane, the real part of $zf(z) - zg(z)$ is nonnegative in the half-plane, and

$$\lim_{y\searrow 0} \operatorname{Re}\,[if(iy) - ig(iy)] \leq \liminf_{n\to\infty} \lim_{y\searrow 0} \operatorname{Re}\,[if(iy) - ig_n(iy)].$$

Since $g'(z) = \lim g_n'(z)$ exists uniformly on any bounded set at a positive distance from the real axis,

$$\int_0^\infty \int_{-\infty}^{+\infty} |g'(x+iy)|^2 dx\, dy \leq \liminf_{y\searrow 0} \int_0^\infty \int_{-\infty}^{+\infty} |g_n'(x+iy)|^2 dx\, dy.$$

So $g(z)$ belongs to $\mathcal{M}$ and is the required minimal element. We show that it has the desired properties.

Consider the function $h(z) = i\bar{w}/(\bar{w} - z)$ for any fixed number w in the upper half-plane. Since $\operatorname{Re} zh(z) \leq 0$ and since $h'(z) = i\bar{w}/(\bar{w} - z)^2$ belongs to $\mathfrak{D}_1$, the function $g(z) + \lambda h(z)$ belongs to $\mathcal{M}$ for every positive number λ. Since

$$\lim_{y\searrow 0} \operatorname{Re}\, ih(iy) = -1,$$

the inequality

$$\|g'(z) + \lambda h'(z)\|_1^2 + 2a \lim_{y\searrow 0} \operatorname{Re}\,[if(iy) - ig(iy) - i\lambda h(iy)]$$
$$\geq \|g'(z)\|_1^2 + 2a \lim_{y\searrow 0} \operatorname{Re}\,[if(iy) - ig(iy)]$$

implies that

$$\operatorname{Re}\,\langle g'(z), h'(z)\rangle_1 \geq -a.$$

By Problem 266 the inequality reads

$$\operatorname{Re}\, iwg'(w) \geq -a.$$

A consequence of the Poisson representation is that

$$p = \lim_{y \to \infty} \operatorname{Re} [if(iy) - ig(iy)]$$

exists as a finite limit and that the real part of $zf(z) - zg(z) + ipz$ is nonnegative in the upper half-plane. By the minimal choice of $g(z)$, $p = 0$. It follows that

$$\liminf_{y \to \infty} \operatorname{Re} ig(iy) = \liminf_{y \to \infty} \operatorname{Re} if(iy) = \tau > -\infty.$$

If we define

$$V(r)^2 = r^2 \int_0^\pi |g'(re^{i\theta})|^2 d\theta,$$

then

$$\int_0^\infty V(r)^2 r^{-1} dr = \pi \|g'(z)\|_1^2 < \infty.$$

Since

$$\partial g(re^{i\theta})/\partial\theta = g'(re^{i\theta}) ire^{i\theta},$$

we obtain the estimate

$$|g(ir) - g(re^{i\theta})| \leq V(r)\sqrt{\tfrac{1}{2}\pi}$$

for $0 < \theta < \pi$. Since

$$\operatorname{Re} ir\, \partial g(re^{i\theta})/\partial r = \operatorname{Re} izg'(z) \geq -a,$$

we have also

$$\operatorname{Re} ig(r_1 e^{i\theta}) \geq \operatorname{Re} ig(r_2 e^{i\theta}) + a \log (r_2/r_1)$$

when $0 < r_1 < r_2 < \infty$. If $\epsilon > 0$ is given, choose $r_0 = r_0(\epsilon) > 0$ so large that

$$\tfrac{1}{2}\pi a \int_{r_0}^\infty V(r)^2 r^{-1} dr < \epsilon^3$$

and so that

$$\operatorname{Re} ig(ir) \geq \tau - \epsilon$$

when $r \geq r_0$. If $r_2 e^{i\theta}$ is a point in the upper half-plane such that

$$\operatorname{Re} ig(r_2 e^{i\theta}) \geq \tau - 3\epsilon,$$

then

$$\operatorname{Re} ig(re^{i\theta}) \geq \tau - 2\epsilon$$

when $r_1 \leq r \leq r_2$ if r_1 is chosen so that

$$a \log (r_2/r_1) = \epsilon.$$

Then $r_1 \leq r_0$, for otherwise

$$\epsilon \leq V(r)\sqrt{\tfrac{1}{2}\pi}$$

for $r_1 \leq r \leq r_2$ and

$$\epsilon^3 = \epsilon^2 a \int_{r_1}^{r_2} t^{-1} dt \leq \tfrac{1}{2}\pi a \int_{r_0}^{\infty} V(r)^2 r^{-1} dr,$$

which is contrary to the choice of r_0. Since

$$r_2 \leq r_0 \exp(\epsilon/a),$$

the set of points z such that $\operatorname{Re} ig(z) \geq \tau - 3\epsilon$ is bounded.

A similar argument will show that

$$\liminf_{y \searrow 0} \operatorname{Re} ig(iy) = \liminf_{y \searrow 0} \operatorname{Re} if(iy) > -\infty$$

and that

$$\liminf_{r \searrow 0} \operatorname{Re} ig(re^{i\theta}) \geq \liminf_{r \searrow 0} \operatorname{Re} ig(iy)$$

uniformly for $0 < \theta < \pi$. Since $\operatorname{Re} ig(z)$ has a lower bound on every bounded set which lies at a positive distance from the origin, it is bounded below in the upper half-plane.

Proof of Theorem 64. Since we can multiply $F(z)$ by a nonzero constant without changing the hypotheses or conclusion of the theorem, we can assume without loss of generality that $|F(0)| < 1$. The entire function

$$F(z)F^*(z) + 1 - F(0)\bar{F}(0)$$

is real for real z, it is of bounded type in the upper half-plane, and it is bounded away from zero on the real axis. By Theorems 8 and 13, it is of the form $E(z)E^*(z)$ for some entire function $E(z)$ such that $|E(x - iy)| \leq |E(x + iy)|$ for $y > 0$. Since $E(z)$ can be multiplied by a constant of absolute value 1, it can be chosen so that $E(0) > 0$ and hence so that $E(0) = 1$. Since $E(z)E^*(z)$ is of bounded type in the upper half-plane and since $E^*(z)/E(z)$ is bounded by 1 in the half-plane, $E(z)$ is of bounded type in the half-plane. By Problem 34, $E(z) = E(z)/1$ is of Pólya class. Since $E(z)$ is bounded away from zero on the real axis, it is bounded away from zero in the upper half-plane. Since

$$|E(x)|^2 = 1 + |F(x)|^2 - |F(0)|^2$$

for all real x, it is sufficient to choose $G(z)$ so that $E(z)G(z)$ is bounded on the real axis. By Lemma 11 (with a change of sign)

$$E(z) = \exp[-zf(z)],$$

where $f(z)$ is analytic in the upper half-plane and $f'(z)$ belongs to $\mathfrak{D}_1$. It follows from the Poisson representation of $\log|E(z)|$ that

$$\liminf_{y \to \infty} \operatorname{Re} if(iy) = -\limsup_{y \to \infty} \frac{\log|E(iy)|}{y} = \tau > -\infty$$

and that

$$\liminf_{y\to 0} \operatorname{Re} if(iy) = -\limsup_{y\to 0} \frac{\log |E(iy)|}{y} > -\infty.$$

By Lemma 12 (for a larger choice of a), there exists a function $g(z)$, analytic in the upper half-plane, such that

$$\operatorname{Re} zg(z) \le \operatorname{Re} zf(z)$$

for $y > 0$, such that

$$\operatorname{Re} i[zg(z)]' = \operatorname{Re} izg'(z) + \operatorname{Re} ig(z) \ge \tau - a$$

outside of some bounded set, and such that $\operatorname{Re} i[zg(z)]'$ has a lower bound in the upper half-plane. Since

$$\operatorname{Re} i[\log(1 - iz)]' = \frac{1+y}{|1-iz|^2} > 0$$

for $y > 0$, there is some $r = 0, 1, 2, \cdots$, such that

$$\operatorname{Re} i[zg(z)]' + r \operatorname{Re} i[\log(1-iz)]' \ge \tau - a$$

for $y > 0$. If

$$W(z) = (1 - iz)^r \exp[zg(z)] \exp(i\tau z - iaz),$$

then $W(z)$ is analytic and without zeros in the upper half-plane,

$$|E(z)W(z)| \le |(1 - iz)^r \exp(i\tau z - iaz)|$$

for $y > 0$, $|W(x + iy)|$ is a nondecreasing function of $y > 0$ for each fixed x, and

$$\limsup_{y\to\infty} y^{-1} \log |W(iy)| \le a.$$

It follows that $W(z)$ is of bounded type in the upper half-plane and of mean type at most a.

By the proof of Theorem 15 there exists a function $E_1(z)$ of Pólya class, which is real for real z, such that

$$\operatorname{Re} [E_1(z)/W(z)] \ge 0$$

for $y > 0$. As in Problem 33, it follows that $(1 - iz)^{-2}E_1(z)/W(z)$ is bounded on every horizontal line in the upper half-plane. Since $(1 - iz)^{-r-2}E(z)E_1(z)$ is then bounded on every horizontal line in the upper half-plane and since $E(z)E_1(z)$ is of Pólya class, the ratio is bounded on the real axis. Since $W(z)$ is of bounded type and of mean type at most a in the upper half-plane and since the real part of $E_1(z)/W(z)$ is nonnegative in the upper half-plane, $E_1(z)$ is of bounded type and of mean type at most a

in the upper half-plane. If $E_1(z)$ has $r+2$ zeros $w_1, \cdots, w_{r+2}$, then

$$E_1(z) = G(z)(z-w_1)\cdots(z-w_{r+2})$$

where $G(z)$ is an entire function of Pólya class, which is of bounded type and of mean type at most a in the upper half-plane, such that $E(z)G(z)$ is bounded on the real axis. If $E_1(z)$ does not have this many zeros, then

$$E_1(z) = P(z)\exp(-ihz)$$

where $P(z)$ is a polynomial of Pólya class and $0 \leq h \leq a$. In this case the desired function $G(z)$ is obtained with

$$G(z) = \sin(az)/Q(z)$$

where $Q(z)$ is a polynomial whose zeros are contained in the zeros of $\sin(az)$ and whose degree is $r+2$ minus the degree of $P(z)$.

PROBLEM 295. Let $\mathcal{H}(E(b))$ be a given space such that $E(b, z)$ is of Pólya class and has no real zeros. If $E(b, z)$ is of bounded type and of mean type $\tau(b)$ in the upper half-plane, if $\operatorname{Re} iE'(b, z)/E(b, z)$ is bounded in the upper half-plane, and if h is a given number, $0 < h < \tau(b)$, show that there exists a space $\mathcal{H}(E(a))$ contained isometrically in $\mathcal{H}(E(b))$ such that $E(a, z)$ has no real zeros and such that the mean type $\tau(a)$ of $E(a, z)$ in the upper half-plane is equal to h.

PROBLEM 296. Show that the function $(i/z)^{1+\nu}f(-1/z)$ belongs to $\mathfrak{D}_\nu$ whenever $f(z)$ belongs to $\mathfrak{D}_\nu$ and that it always has the same norm as $f(z)$. (If ν is not an integer, the root is defined so as to be continuous in the upper half-plane and positive on the imaginary axis.) If $f(z)$ belongs to $\mathfrak{D}_\nu$, show that

$$\int_0^\infty |f(it)|^2 t^\nu dt < \infty.$$

PROBLEM 297. If $f(z)$ belongs to $\mathfrak{D}_\nu$, $\nu = 2h-1$, show that

$$F(x) = \lim \int_a^b f(it)t^{h-ix-1}dt$$

exists in the metric of $L^2(-\infty, +\infty)$ as $a \searrow 0$ and $b \nearrow \infty$. If

$$f(z) = \int_0^\infty t^{\frac{1}{2}\nu}e^{izt}g(t)dt$$

where $g(x)$ belongs to $L^2(0, \infty)$, show that

$$G(x) = \lim \int_a^b g(t)t^{ix-\frac{1}{2}}dt$$

exists in the metric of $L^2(-\infty, +\infty)$ as $a \searrow 0$ and $b \nearrow \infty$, and that

$$F(x) = \Gamma(h - ix)G(x)$$

for almost all real x. Show that

$$\|f(z)\|_\nu^2 = \int_0^\infty |g(t)|^2 dt$$
$$= 2\pi \int_{-\infty}^{+\infty} |G(t)|^2 dt$$
$$= 2\pi \int_{-\infty}^{+\infty} |F(t)/\Gamma(h - it)|^2 dt.$$

Show that a given function $F(x)$ in $L^2(-\infty, +\infty)$ corresponds in this way to an element of $\mathfrak{D}_\nu$ if the last integral converges.

PROBLEM 298. If s is an imaginary constant, let

$$(-iz)^s = \exp [s \log (-iz)]$$

where the logarithm is defined continuously in the upper half-plane so as to be real on the imaginary axis. Show that the function is analytic and bounded in the upper half-plane and that it has a continuous extension to the closed half-plane except for a singularity at the origin. Show that

$$\operatorname{Re} (-iz)^s = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\operatorname{Re} (-it)^s dt}{(t - x)^2 + y^2}$$

for $y > 0$. Show that

$$-is(-iz)^{s-1} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-it)^s dt}{(t - z)^2},$$

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-it)^s dt}{(t - \bar{z})^2}$$

for $y > 0$. Show that

$$\frac{1}{2\pi i} \int_0^\infty \frac{t^{ix} dt}{(t - iy)^2} = y^{ix-1} \frac{x e^{\frac{1}{2}\pi x}}{e^{\pi x} - e^{-\pi x}}$$

for all real x when $y > 0$.

PROBLEM 299. If $h(x)$ is a real valued function of real x such that

$$\int_{-\infty}^{+\infty} \frac{|h(t)|^2 dt}{|t|} < \infty,$$

show that there exists a function $f(z)$ analytic in the upper half-plane such that

$$\operatorname{Re} f(z) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{h(t) dt}{(t - x)^2 + y^2}$$

for $y > 0$. Show that

$$f'(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{h(t)dt}{(t-z)^2}$$

for $y > 0$ and that

$$\int_0^\infty |f'(it)|^2\, t\, dt < \infty.$$

Show that

$$\int_0^\infty f'(it)t^{-ix}dt = \frac{xe^{\frac{1}{2}\pi x}}{e^{\pi x}-e^{-\pi x}}\int_0^\infty h(t)t^{ix-1}dt - \frac{xe^{-\frac{1}{2}\pi x}}{e^{\pi x}-e^{-\pi x}}\int_0^\infty h(-t)t^{ix-1}dt$$

for almost all real x where the integrals are taken as mean square limits of $\int_a^b$ in the metric of $L^2(-\infty, +\infty)$ as $a \searrow 0$ and $b \nearrow \infty$.

65. EXISTENCE OF SUBSPACES WITH GIVEN MEAN TYPE

A variant of the Beurling-Malliavin theorem is used in applications of Hilbert spaces of entire functions.

THEOREM 65. Let $\mathcal{H}(E(b))$ be a given space with phase function $\varphi(b, x)$ such that $\varphi'(b, x)$ is bounded and such that

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1}\,|\varphi(b,t) - \tau(b)t|dt < \infty$$

for some number $\tau(b) > 0$. If h is a given number, $0 < h < \tau(b)$, then there exists a space $\mathcal{H}(E(a))$ contained isometrically in $\mathcal{H}(E(b))$ such that $E(a, z)/E(b, z)$ has no real zeros and such that the mean type of $E(a, z)/E(b, z)$ in the upper half-plane is equal to $h - \tau(b)$.

The proof again depends on Lemma 12. A method of Beurling and Malliavin is used to verify the hypotheses of the lemma from a knowledge of boundary values on the real axis.

LEMMA 13. Let $f(z)$ be a function analytic in the upper half-plane whose real part has a boundary value function $h(x)$ on the real axis such that

$$\int_{-\infty}^{+\infty} h(t)^2 \frac{dt}{|t|} < \infty$$

and such that

$$\operatorname{Re} f(z) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{h(t)dt}{(t-x)^2+y^2}$$

for $y > 0$. If

$$\int_1^\infty \int_{-\infty}^{+\infty} |h(st) - h(t)|^2 \frac{dt}{|t|} \frac{ds}{s(\log s)^2} < \infty,$$

then $f'(z)$ belongs to $\mathfrak{D}_1$.

LEMMA 14. Let $f(z)$ be a function analytic in the upper half-plane whose real part has a boundary value function $h(x)$ on the real axis such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |th(t)| dt < \infty$$

and such that

$$\operatorname{Re} f(z) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{h(t)dt}{(t - x)^2 + y^2}$$

for $y > 0$. If $h(x)$ is differentiable and if the derivative of $xh(x)$ is bounded, then $f'(z)$ belongs to $\mathfrak{D}_1$.

Proof of Lemma 13. The results of Problems 297 and 298 are used to prove the lemma. The integrals

$$F_+(x) = \int_0^\infty h(t) t^{ix-1} dt \quad \text{and} \quad F_-(x) = \int_0^\infty h(-t) t^{ix-1} dt$$

make sense as mean square limits of $\int_a^b$ as $a \searrow 0$ and $b \nearrow \infty$. By Plancherel's formula (with a change of variable)

$$2\pi \int_{-\infty}^{+\infty} |F_+(t)|^2 dt = \int_0^\infty h(t)^2 t^{-1} dt,$$

and similarly for $F_-(x)$. But

$$\int_0^\infty h(st) t^{ix-1} dt = s^{-ix} F_+(x)$$

and

$$2\pi \int_{-\infty}^{+\infty} |s^{it} - 1|^2 |F_+(t)|^2 dt = \int_0^\infty |h(st) - h(t)|^2 t^{-1} dt$$

when $s \geq 1$, where

$$\int_1^\infty \frac{|s^{it} - 1|^2 ds}{s(\log s)^2} = \int_0^\infty \frac{|e^{ist} - 1|^2 ds}{s^2} = 4 \int_0^\infty \frac{\sin^2 (st) ds}{s^2} = \pi\, |t|$$

by Plancherel's formula since

$$\int_{-1}^1 e^{ixt} dt = (2 \sin x)/(ix).$$

It follows that

$$2\pi^2 \int_{-\infty}^{+\infty} |F_+(t)|^2 |t| dt = \int_1^\infty \int_0^\infty |h(st) - h(t)|^2 t^{-1} dt\, ds$$

and in the same way

$$2\pi^2 \int_{-\infty}^{+\infty} |F_-(t)|^2 |t| dt = \int_1^\infty \int_{-\infty}^0 |h(st) - h(t)|^2 |t|^{-1} dt\, ds.$$

So if

$$F(x) = \frac{xe^{-\frac{1}{2}\pi x}F_+(x)}{e^{\pi x} - e^{-\pi x}} - \frac{xe^{\frac{1}{2}\pi x}F_-(x)}{e^{\pi x} - e^{-\pi x}},$$

then

$$\int_{-\infty}^{+\infty} \frac{e^{\pi t} - e^{-\pi t}}{t} |F(t)|^2 dt \le 2\int_{-\infty}^{+\infty} \frac{e^{\pi t} - e^{-\pi t}}{t} \left| \frac{te^{-\frac{1}{2}\pi t}F_+(t)}{e^{\pi t} - e^{-\pi t}} \right|^2 dt$$

$$+ 2\int_{-\infty}^{+\infty} \frac{e^{\pi t} - e^{-\pi t}}{t} \left| \frac{te^{\frac{1}{2}\pi t}F_-(t)}{e^{\pi t} - e^{-\pi t}} \right|^2 dt$$

$$\le 2\int_{-\infty}^{+\infty} \frac{te^{-\pi t}}{e^{\pi t} - e^{-\pi t}} |F_+(t)|^2 dt + 2\int_{-\infty}^{+\infty} \frac{te^{\pi t}}{e^{\pi t} - e^{-\pi t}} |F_-(t)|^2 dt$$

$$\le \int_{-\infty}^{+\infty} (\pi^{-1} + 2|t|)[|F_+(t)|^2 + |F_-(t)|^2] dt$$

$$\le \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} |h(t)|^2 |t|^{-1} dt + \frac{1}{\pi^2} \int_1^\infty \int_{-\infty}^{+\infty} |h(st) - h(t)|^2 |t|^{-1} dt\, ds.$$

Euler's identity

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}$$

can be verified from the product representation of the gamma function, Problem 19, and the analogous infinite product for $\sin(\pi z)$. (Compare with Problem 18.) It follows that

$$\int_{-\infty}^{+\infty} \frac{e^{\pi t} - e^{-\pi t}}{t} |F(t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |F(t)/\Gamma(1 - it)|^2 dt.$$

By Problem 297 there exists a function $g(z)$ in $\mathfrak{D}_1$ such that

$$F(x) = \lim \int_a^b g(it) t^{ix} dt$$

in the metric of $L^2(-\infty, +\infty)$ as $a \searrow 0$ and $b \nearrow \infty$, and

$$\|g(z)\|_1^2 = 2\pi \int_{-\infty}^{+\infty} |F(t)/\Gamma(1 - it)|^2 dt$$

$$\le \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} |h(t)|^2 dt + \frac{1}{\pi^2} \int_1^\infty \int_{-\infty}^{+\infty} |h(st) - h(t)|^2 |t|^{-1} dt\, ds.$$

By Problem 298,

$$\int_0^\infty g(it)t^{i\alpha}dt = \frac{xe^{-\frac{1}{2}\pi\alpha}F_+(x)}{e^{\pi x} - e^{-\pi x}} - \frac{xe^{\frac{1}{2}\pi\alpha}F_-(x)}{e^{\pi x} - e^{-\pi x}}$$

$$= -\int_0^\infty \frac{1}{2\pi i}\int_0^\infty \frac{t^{i\alpha}dt}{(t+iy)^2}\, h(y)dy - \int_0^\infty \frac{1}{2\pi i}\int_0^\infty \frac{t^{i\alpha}dt}{(t-iy)^2}\, h(-y)dy$$

$$= \int_0^\infty \frac{1}{2\pi i}\int_0^\infty \frac{h(y)dy}{(y-it)^2}\, t^{i\alpha}dt + \int_0^\infty \frac{1}{2\pi i}\int_0^\infty \frac{h(-y)dy}{(y+it)^2}\, t^{i\alpha}dt$$

$$= \int_0^\infty \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{h(t)dt}{(t-iy)^2}\, y^{i\alpha}dy$$

$$= \int_0^\infty f(it)t^{i\alpha}dt.$$

The change in the order of integration is justified by absolute convergence if $h(x)$ vanishes in a neighborhood of the origin and outside of some bounded set. In this case we can conclude that $f(z) = g(z)$ belongs to $\mathfrak{D}_1$. The same conclusion is true by continuity in the general case.

Proof of Lemma 14. Consider first the special case in which $h(0) = 0$. Since $h'(0)$ exists, $h(x)/x$ is bounded in a neighborhood of the origin, and the hypotheses imply that

$$\int_{-\infty}^{+\infty} |h(t)/t|dt < \infty.$$

By hypothesis there exists a finite $M > 0$ such that

$$|h(x) + xh'(x)| \le M$$

for all real x. Since

$$|h(x)^2/x + h(x)h'(x)| \le M\,|h(x)/x|$$

and since $h(0) = 0$,

$$\int_0^x h(t)^2t^{-1}dt + \tfrac{1}{2}h(x)^2 \le M\int_0^x |h(t)/t|dt$$

when $x > 0$ and

$$\int_x^0 h(t)^2(-t)^{-1}dt + \tfrac{1}{2}h(x)^2 \le M\int_x^0 |h(t)/t|dt$$

when $x < 0$. It follows that

$$\int_{-\infty}^{+\infty} h(t)^2\,|t|^{-1}dt \le M\int_{-\infty}^{+\infty} |h(t)/t|dt$$

and that

$$h(x)^2 \le 2M\int_{-\infty}^{+\infty} |h(t)/t|dt$$

for all real x. So

$$|xh'(x)| \leq B$$

where

$$B = M + [2M \int_{-\infty}^{+\infty} |h(t)/t| dt]^{\frac{1}{2}}.$$

We verify the hypotheses of Lemma 13. If

$$m(x) = - \int_{|h(t)| \geq x} |t|^{-1} dt,$$

then $m(x)$ is a nondecreasing function of $x > 0$ such that

$$\int_{-\infty}^{+\infty} h(t)^2 |t|^{-1} dt = \int_0^{\infty} x^2 dm(x)$$

and

$$\int_{-\infty}^{+\infty} |h(t)/t| dt = \int_0^{\infty} x \, dm(x).$$

For any fixed $s > 1$, the integral

$$\int_{-\infty}^{+\infty} |h(st) - h(t)|^2 |t|^{-1} dt$$

can be estimated by considering the set on which $|h(t)|$ or $|h(st)|$ exceeds $\log s$. The contribution to the integral from this set is at most

$$-2(B \log s)^2 m(\log s)$$

because

$$|h(st) - h(t)| \leq B \log s$$

for all nonzero t. The contribution to the integral from the complementary set is at most

$$4 \int_{|h(t)| \leq \log s} h(t)^2 |t|^{-1} dt = 4 \int_0^{\log s} x^2 dm(x).$$

It follows that

$$\begin{aligned}
\int_1^{\infty} \int_{-\infty}^{+\infty} |h(st) - h(t)|^2 \frac{dt}{|t|} \frac{ds}{s(\log s)^2}
&\leq -2 \int_1^{\infty} (B \log s)^2 m(\log s) \frac{ds}{s(\log s)^2} \\
&\quad + 4 \int_1^{\infty} \int_0^{\log s} x^2 dm(x) \frac{ds}{s(\log s)^2} \\
&\leq -2B^2 \int_0^{\infty} m(t) dt + 4 \int_0^{\infty} \int_0^t x^2 dm(x) t^{-2} dt \\
&\leq (2B^2 + 4) \int_0^{\infty} t \, dm(t) = (2B^2 + 4) \int_{-\infty}^{+\infty} |h(t)/t| dt < \infty.
\end{aligned}$$

By Lemma 13, $f'(z)$ belongs to $\mathfrak{D}_1$ if $h(0) = 0$.

The same conclusion can also be obtained if $h(0) \neq 0$. In this case consider the function

$$h_1(x) = h(x) - x^{-1} \log |1 - x/w|$$

where w is a number in the lower half-plane chosen so that $h_1(0) = 0$. Then

$$\operatorname{Re} f_1(z) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{h_1(t)\,dt}{(t-x)^2 + y^2}$$

for $y > 0$ where

$$f_1(z) = f(z) - z^{-1} \log (1 - z/w).$$

The previous argument will show that $f_1'(z)$ belongs to $\mathfrak{D}_1$. Since

$$f_1'(z) - f'(z) = -\frac{1}{z(z-w)} + \frac{\log (1 - z/w)}{z^2}$$

belongs to $\mathfrak{D}_1$, $f'(z)$ belongs to $\mathfrak{D}_1$.

Proof of Theorem 65. The hypotheses imply that

$$\lim_{|x| \to \infty} \varphi(b, x)/x = \tau(b)$$

and that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} d\varphi(b, t) < \infty.$$

By Problem 64 we can write $E(b, z) = S(z) E_1(b, z)$ where $S(z)$ is an entire function which is real for real z and where $E_1(b, z)$ is an entire function of Pólya class which has no real zeros and which has value 1 at the origin. With no loss of generality, we can assume that $E(b, z) = E_1(b, z)$, that $\varphi(b, 0) = 0$, and that

$$\frac{\partial}{\partial y} \log |E(b, x + iy)| = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\varphi(b, t)}{(t-x)^2 + y^2}$$

for $y > 0$. Since

$$1 = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(t-x)^2 + y^2},$$

we have

$$\frac{\partial}{\partial y} [\log |E(b, x + iy)| - \tau(b) y] = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d[\varphi(b, t) - \tau(b) t]}{(t-x)^2 + y^2}$$

for $y > 0$. By the proof of Theorem 15, the difference between

$$\log |E(b, x + iy)| - \tau(b) y \quad \text{and} \quad \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(b, t) - \tau(b) t}{(t-x)^2 + y^2},$$

for $y > 0$, is a constant. The constant is seen to be zero by comparing the limiting values when $x = 0$ and $y \searrow 0$. If $\log E(b, z)$ is defined continuously in the closed half-plane so as to have value zero at the origin,

$$\log E(b, z) + i\tau(b)z + \log E(b, w) - i\tau(b)\bar{w}$$
$$= \frac{i(\bar{w} - z)}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(b, t) - \tau(b)t}{(t - z)(t - \bar{w})}\, dt$$

when z and w are in the upper half-plane and

$$\frac{\log E(b, z)}{z} + i\tau(b) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(b, t)t^{-1} - \tau(b)}{t - z}\, dt$$

for $y > 0$. By Lemma 14,

$$\frac{d}{dz}\frac{\log E(b, z)}{z}$$

belongs to $\mathfrak{D}_1$.

By the Lebesgue dominated convergence theorem,

$$\lim_{y\to\infty} \frac{\log |E(b, iy)|}{y} = \tau(b) + \lim_{y\to\infty} \operatorname{Re} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(b, t)t^{-1} - \tau(b)}{t - iy}\, dt$$
$$= \tau(b) + \lim_{y\to\infty} \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(b, t) - \tau(b)t}{t^2 + y^2}\, dt$$
$$= \tau(b).$$

On the other hand,

$$\lim_{y\searrow 0} \frac{\log |E(b, iy)|}{y} = \varphi'(b, 0).$$

By the proof of Theorem 64 there exists a function $W(z)$, analytic and without zeros in the upper half-plane, and there exists a number $r = 0, 1, 2, \cdots$, such that

$$\operatorname{Re} iW'(z)/W(z) \geq 0,$$
$$|W(z)/E(b, z)| \leq |1 - iz|^r$$

for $y > 0$ and such that

$$\lim_{y\to\infty} \frac{\log |W(iy)|}{y} = h.$$

Also by the proof of Theorem 64 there exists an entire function $E_1(z)$ of Pólya class such that $\operatorname{Re} [E_1(z)/W(z)] \geq 0$ for $y > 0$. The functions $(1 - iz)^{-2}E_1(z)/W(z)$ and $(1 - iz)^{-r-2}E_1(z)/E(b, z)$ are bounded on every horizontal line in the upper half-plane. If $E_1(z)$ has $r + 3$ zeros $w_1, \cdots, w_{r+3}$, then

$$E_1(z) = G(z)(z - w_1) \cdots (z - w_{r+3})$$

where $G(z)$ is an entire function of Pólya class such that $(1 - iz)G(z)/E(b, z)$ is bounded on every horizontal line in the upper half-plane. Since we assume that $\varphi'(b, x)$ is bounded, $\operatorname{Re} iE'(b, z)/E(b, z)$ is bounded above in the upper half-plane. It follows that $E(b, z + i)/E(b, z)$ is bounded in the upper half-plane. So $G(z + i)$ is an element of $\mathcal{H}(E(b))$ such that

$$\limsup_{y \to +\infty} y^{-1} \log |G(iy + i)| \leq h.$$

Such a function $G(z)$ exists also if $E_1(z)$ does not have $r + 3$ zeros. For then

$$E_1(z) = P(z) \exp (uz - ivz)$$

where $P(z)$ is a polynomial of Pólya class, u and v are real numbers, and and $0 \leq v \leq h$. In this case the desired function $G(z)$ is obtained with

$$G(z) = \sin (hz) \exp (uz)/Q(z)$$

where $Q(z)$ is a polynomial of degree $r + 3$ whose zeros are contained in the zeros of $\sin (hz)$.

Let $\mathcal{M}(a)$ be the set of all elements $F(z)$ of $\mathcal{H}(E(b))$ such that the mean types of $F(z)/E(b, z)$ and $F^*(z)/E(b, z)$ in the upper half-plane are at most $h - \tau(b)$. Since the inequality

$$|F(z)|^2 \leq \|F\|^2 K(b, z, z) \exp [2h|y| - 2\tau(b)|y|]$$

holds for every element of $\mathcal{M}(a)$, $\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(E(b))$. It is clear that $F^*(z)$ belongs to $\mathcal{M}(a)$ whenever $F(z)$ belongs to $\mathcal{M}(a)$. If $F(z)$ belongs to $\mathcal{M}(a)$ and has a zero w, then $F(z)/(z - w)$ belongs to $\mathcal{M}(a)$. So $\mathcal{M}(a)$ is a closed subspace of $\mathcal{H}(E(b))$ which satisfies the axioms (H1), (H2), and (H3) in the metric of $\mathcal{H}(E(b))$. We have seen that $\mathcal{M}(a)$ contains a nonzero element. By Theorem 23, $\mathcal{M}(a)$ is equal to a space $\mathcal{H}(E(a))$ in the metric of $\mathcal{H}(E(b))$. Since $F(z)/(z - w)$ belongs to $\mathcal{M}(a)$ whenever $F(z)$ belongs to $\mathcal{M}(a)$ and $F(w) = 0$, $E(a, z)$ has no real zeros. By construction, the maximum mean type of $F(z)/E(b, z)$ in the upper half-plane is $h - \tau(b)$ for elements $F(z)$ of $\mathcal{M}(a)$. It follows that the mean type of $E(a, z)/E(b, z)$ is $h - \tau(b)$ in the upper half-plane.

PROBLEM 300. Let $K(z)$ be an entire function which is of bounded type and of zero mean type in the upper and lower half-planes. Let

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

be an absolutely convergent Fourier transform which vanishes in a neighborhood of the origin and which does not vanish identically. Show that $K(z)$ is a constant if $K(x)$ remains bounded on the support of μ.

66. EXTREME POINTS OF A CONVEX SET

A problem of Fourier analysis is to determine what closed subsets of the real line can support a measure whose Fourier transform vanishes in an interval and does not vanish identically. The problem reduces to one for entire functions.

THEOREM 66. Let $a > 0$ be a given number and let X be a closed subset of the real line. A necessary and sufficient condition that there exist a function $\mu(x)$ of real x which has finite total variation, which is constant in each interval contained in the complement of X, and whose Fourier transform $\int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$ vanishes in the interval $[-a, a]$ without vanishing identically, is that there exist an entire function $S(z)$, which is real for real z and has only real simple zeros, all in X, such that $S(z)$ is of bounded type and of mean type a in the upper half-plane, and such that

$$\sum_{S(t)=0} \frac{1}{|S'(t)|} < \infty.$$

The proof depends on the Kreĭn-Milman convexity theorem and Naĭmark's characterization of extreme points.

LEMMA 15. Let $a > 0$ be a given number and let X be a closed subset of the real line. Let $\mu(x)$ be a real valued function which remains constant in every open interval contained in the complement of X, such that

$$\int_{-\infty}^{+\infty} |d\mu(t)| \leq 1.$$

A necessary and sufficient condition that

$$\int_{-\infty}^{+\infty} e^{ixt} d\mu(t) = 0$$

for $-a \leq x \leq a$ is that

$$\int_{-\infty}^{+\infty} F(t) d\mu(t) = 0$$

for every element $F(z)$ of the Paley-Wiener space of type a. Let $\mathcal{M} = \mathcal{M}(a, X)$ be the convex set of measures corresponding to such functions. If a function $\mu(x)$ determines a nonzero extreme point of $\mathcal{M}$, then the closure of the Paley-Wiener space has deficiency 1 in $L^1(\nu)$,

$$\nu(x) = \int_{-\infty}^{x} |d\mu(t)|.$$

Proof of Lemma 15. If $F(z)$ belongs to the Paley-Wiener space, then it is of bounded type in the upper and lower half-planes and

$$\exp(-a|y|) F(iy)$$

is bounded, $-\infty < y < \infty$, because

$$|F(z)|^2 \leq \frac{\sin(az - a\bar{z})}{\pi(z - \bar{z})} \int_{-\infty}^{+\infty} |F(t)|^2 dt$$

for all complex z by the proof of Theorem 16. By Theorem 61, $F(H)$ is a local operator on Fourier transforms. Since $F(z)$ is bounded on the real axis,

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

is in the domain of $F(H)$. If $f(x)$ vanishes for $-a \leq x \leq a$, then $F(H): f(x) \to g(x)$, where

$$g(0) = \int_{-\infty}^{+\infty} F(t) d\mu(t) = 0.$$

If on the other hand

$$\int_{-\infty}^{+\infty} F(t) d\mu(t) = 0$$

for every element of the Paley-Wiener space of type a, then

$$\int_{-\infty}^{+\infty} \frac{e^{iht} - e^{ihz}}{t - z} d\mu(t) = 0$$

when $-a \leq h \leq a$. By the Lebesgue dominated convergence theorem,

$$\int_{-\infty}^{+\infty} e^{iht} d\mu(t) = \lim_{y \to +\infty} -iy \int_{-\infty}^{+\infty} \frac{e^{iht} d\mu(t)}{t - iy}$$

$$= \lim_{y \to +\infty} -iye^{-hy} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{t - iy} = 0$$

for $0 < h \leq a$. A similar limit process in the lower half-plane will show that the integral vanishes when $-a \leq h < 0$. The integral vanishes by continuity when $h = 0$.

We show that the closure of the Paley-Wiener space of type a has deficiency 1 in $L^1(\nu)$ if μ is an extreme point of $\mathcal{M}$. Let $B(x)$ be a Borel measurable, real valued function of real x, having absolute value 1 everywhere, such that

$$\mu(b) - \mu(a) = \int_a^b B(t) d\nu(t)$$

whenever a and b are points of continuity of $\nu(x)$. Then

$$\int_{-\infty}^{+\infty} F(t)B(t) d\nu(t) = \int_{-\infty}^{+\infty} F(t) d\mu(t) = 0$$

for every element $F(z)$ of the Paley-Wiener space of type a. We must show that any bounded, Borel measurable function $h(x)$ of real x such that

$$\int_{-\infty}^{+\infty} F(t)h(t) d\nu(t) = 0$$

for every element $F(z)$ of the Paley-Wiener space of type a is ν-equivalent to a constant multiple of $B(x)$. Equivalently, we must show that any bounded, Borel measurable function $h(x)$ of real x such that

$$\int_{-\infty}^{+\infty} F(t)h(t)d\mu(t) = 0$$

for every element $F(z)$ of the Paley-Wiener space is ν-equivalent to a constant. Since $F^*(z)$ belongs to the Paley-Wiener space whenever $F(z)$ belongs to the Paley-Wiener space, it is sufficient to obtain this conclusion when $h(x)$ is a real valued function. Since we can add a constant to $h(x)$ without changing these conditions, we can restrict ourselves to the case in which $h(x) \geq 0$ for all real x. By linearity, it is sufficient to consider the case in which

$$\int_{-\infty}^{+\infty} h(t)d\nu(t) = \int_{-\infty}^{+\infty} d\nu(t).$$

If λ is a number, $0 < \lambda < 1$, such that $\lambda h(x)$ is essentially bounded by 1, let

$$\mu_+(x) = \int_{-\infty}^{x} h(t)d\mu(t)$$

and let

$$(1 - \lambda)\mu_-(x) = \mu(x) - \lambda\mu_+(x).$$

Then

$$\mu_+(b) - \mu_+(a) = \int_a^b h(t)d\mu(t),$$

$$\mu_-(b) - \mu_-(a) = \int_a^b \frac{1 - \lambda h(t)}{1 - \lambda}\, d\mu(t)$$

when a and b are points of continuity of $\nu(x)$. It follows that

$$\int_{-\infty}^{+\infty} |d\mu_+(t)| = \int_{-\infty}^{+\infty} h(t)d\nu(t) = \int_{-\infty}^{+\infty} d\nu(t),$$

$$\int_{-\infty}^{+\infty} |d\mu_-(t)| = \int_{-\infty}^{+\infty} \frac{1 - \lambda h(t)}{1 - \lambda}\, d\nu(t) = \int_{-\infty}^{+\infty} d\nu(t).$$

If $F(z)$ belongs to the Paley-Wiener space of type a,

$$\int_{-\infty}^{+\infty} F(t)d\mu_+(t) = \int_{-\infty}^{+\infty} F(t)h(t)d\mu(t) = 0$$

and

$$\int_{-\infty}^{+\infty} F(t)d\mu_-(t)$$

$$= (1 - \lambda)^{-1}\int_{-\infty}^{+\infty} F(t)d\mu(t) - (1 - \lambda)^{-1}\int_{-\infty}^{+\infty} \lambda F(t)h(t)d\mu(t) = 0.$$

So $\mu_+(x)$ and $\mu_-(x)$ determine elements of $\mathcal{M}$. Since

$$\mu(x) = \lambda\mu_+(x) + (1 - \lambda)\mu_-(x)$$

and since we assume that μ is not an extreme point, $\mu(x)$ and $\mu_+(x)$ determine the same measure. It follows that

$$\int_a^b h(t)d\mu(t) = \int_a^b d\mu(t)$$

whenever a and b are points of continuity of $\nu(x)$. So $h(x) = 1$ almost everywhere with respect to ν.

Proof of Theorem 66. For the sufficiency, let $S(z)$ be given as in the statement of the theorem and let $\mu(x)$ be a step function whose points of increase are at the zeros of $S(z)$ and which has a jump of

$$\mu(t+) - \mu(t-) = 1/S'(t)$$

at each zero t. By hypothesis,

$$\int_{-\infty}^{+\infty} |d\mu(t)| = \sum_{S(t)=0} \frac{1}{|S'(t)|} < \infty.$$

By the proof of Theorem 26,

$$L(z) = \int_{-\infty}^{+\infty} \frac{S(t) - S(z)}{t - z}\, d\mu(t) = \sum_{S(t)=0} \frac{S(z)}{S'(t)(z - t)}$$

is an entire function which is real for real z and of bounded type in the upper half-plane. Since the function is equal to 1 at the zeros of $S(z)$, $[L(z) - 1]/S(z)$ is an entire function which is real for real z and of bounded type in the upper half-plane. By Problem 34, $S(z)$ is of Pólya class. Since $S(z)$ has positive mean type in the upper half-plane, it is not a constant. By Problem 39, $S(z)$ is unbounded on the imaginary axis. Since $|S(iy)| = |S(-iy)|$ is a nondecreasing function of $y > 0$,

$$\lim_{y\to+\infty} 1/S(iy) = 0.$$

By the Lebesgue dominated convergence theorem,

$$\lim_{y\to+\infty} \frac{L(iy) - 1}{S(iy)} = \lim_{y\to+\infty} \sum_{S(t)=0} \frac{1}{S'(t)(iy - t)} = 0.$$

By Problem 39, $[L(z) - 1]/S(z)$ vanishes identically. This proves that $L(z) = 1$ identically. We show that

$$\int_{-\infty}^{+\infty} e^{iht} d\mu(t) = 0$$

for $-a \leq h \leq a$.

If h is held fixed, consider the function,

$$G(z) = \int_{-\infty}^{+\infty} \frac{e^{iht} - e^{ihz}}{t - z}\, d\mu(t).$$

By the proof of Theorem 26, $G(z)$ is an entire function which is of bounded type in the upper and lower half-planes. Since

$$1 = \int_{-\infty}^{+\infty} \frac{S(t) - S(z)}{t - z}\, d\mu(t),$$

we have

$$S(z)G(z) - e^{ihz} = \int_{-\infty}^{+\infty} \frac{S(z)e^{iht} - e^{ihz}S(t)}{t - z}\, d\mu(t)$$

for all complex z. By the Lebesgue dominated convergence theorem,

$$0 = \lim_{y\to+\infty} \int_{-\infty}^{+\infty} \frac{e^{iht}d\mu(t)}{t - iy},$$

$$0 = \lim_{y\to+\infty} \int_{-\infty}^{+\infty} \frac{S(t)d\mu(t)}{t - iy}.$$

If $-a < h < a$, then $e^{ihy}/S(iy)$ is bounded, $-\infty < y < \infty$, by the computation of mean type, Theorem 10. In this case it follows that

$$\lim_{y\to+\infty} G(iy) = 0.$$

A similar argument will show that

$$\lim_{y\to-\infty} G(iy) = 0.$$

By Problem 39, $G(z)$ vanishes identically. By the proof of Lemma 15, the Fourier transform of the measure μ vanishes in the open interval $(-a, a)$, and hence by continuity in the closed interval.

The proof of necessity proceeds by an extreme point argument. The set of all measures of total variation at most 1 is a compact Hausdorff space in the weakest topology which makes $\int_{-\infty}^{+\infty} f(t)d\mu(t)$ depend continuously on μ for every continuous function $f(x)$ of real x having limit zero at infinity. Let $\mathcal{M}$ be the convex set of measures defined by Lemma 15. If there exists a function $\mu(x)$ of finite total variation which is constant in each interval contained in the complement of X, and whose Fourier transform vanishes in the interval $[-a, a]$ without vanishing identically, then $\mathcal{M}$ contains a nonzero element. By Lemma 15, $\mathcal{M}$ is a closed subspace of a compact Hausdorff space and so is itself compact. By the Kreĭn-Milman convexity theorem, $\mathcal{M}$ is the closed convex span of its extreme points. So if $\mathcal{M}$ contains a nonzero element, it contains a nonzero extreme point. To prove the necessity we need only show that every nonzero extreme point is obtained from an entire function $S(z)$ as in the proof of sufficiency.

If μ is any element of $\mathcal{M}$ with $\int_{-\infty}^{+\infty} |d\mu(t)| = 1$, let

$$M(z) = \sup |F(z)|$$

where the supremum is taken over the elements of the Paley-Wiener space of type a such that $\int_{-\infty}^{+\infty} |F(t)d\mu(t)| \leq 1$. Since $[F(z) - F(w)]/(z - w)$ belongs to the Paley-Wiener space whenever $F(z)$ belongs to the Paley-Wiener space,

$$\int_{-\infty}^{+\infty} \frac{F(t) - F(z)}{t - z} \, d\mu(t) = 0.$$

When z is not real,

$$\left| F(z) \int_{-\infty}^{+\infty} (t - z)^{-1} d\mu(t) \right| \leq |y|^{-1} \int_{-\infty}^{+\infty} |F(t) d\mu(t)|.$$

By the arbitrariness of $F(z)$,

$$M(z) \left| \int_{-\infty}^{+\infty} (t - z)^{-1} d\mu(t) \right| \leq |y|^{-1}.$$

By the proof of Theorem 62, $M(z) \geq 1$ is finite and locally bounded,

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log M(t) dt < \infty,$$

and

$$\log M(x + iy) \leq a|y| + \frac{|y|}{\pi} \int_{-\infty}^{+\infty} \frac{\log M(t) dt}{(t - x)^2 + y^2}$$

when $y \neq 0$. The majorant $M(z)$ is used to construct a Banach space of entire functions.

We first show that an entire function $G(z)$ vanishes identically if

$$\int_{-\infty}^{+\infty} |G(t) d\mu(t)| = 0$$

and if $G(z)/M(z)$ is bounded in the complex plane. Say that $G(z)/M(z)$ is bounded by 1. From the above estimate of $\log M(z)$ we see that $G(z)$ is of bounded type in the upper and lower half-planes. By the proof of Theorem 26,

$$F(z) = \int_{-\infty}^{+\infty} \frac{G(t) - G(z)}{t - z} \, d\mu(t)$$

is an entire function which is of bounded type in the upper and lower half-planes. Since

$$F(z) = G(z) \int_{-\infty}^{+\infty} \frac{d\mu(t)}{z - t},$$

we obtain the estimate

$$|F(z)| \leq M(z) \left| \int_{-\infty}^{+\infty} \frac{d\mu(t)}{z - t} \right| \leq |y|^{-1}.$$

By Problem 39, $F(z)$, and hence $G(z)$, vanishes identically.

Let $\mathcal{B}$ be the set of entire functions $G(z)$, $G(z)/M(z)$ bounded in the complex plane, for which there exists a sequence $(F_n(z))$ in the Paley-Wiener space of type a such that

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} |G(t) - F_n(t)| \, |d\mu(t)| = 0.$$

Since

$$|F_n(z) - F_k(z)| \leq M(z) \int_{-\infty}^{+\infty} |F_n(t) - F_k(t)| \, |d\mu(t)|$$

for all n and k, there exists an entire function $F(z)$ such that

$$|F(z) - F_n(z)| \leq M(z) \int_{-\infty}^{+\infty} |F(t) - F_n(t)| \, |d\mu(t)|$$

for every n and such that

$$\lim_{n \to \infty} \int_{-\infty}^{+\infty} |F(t) - F_n(t)| \, |d\mu(t)| = 0.$$

Since $[F(z) - G(z)]/M(z)$ is bounded in the complex plane and since

$$\int_{-\infty}^{+\infty} |F(t) - G(t)| \, |d\mu(t)| = 0,$$

$F(z) - G(z)$ vanishes identically. It follows that the inequality

$$|G(z)| \leq M(z) \int_{-\infty}^{+\infty} |G(t) d\mu(t)|$$

holds for every element $G(z)$ of $\mathcal{B}$. It is easily verified that $\mathcal{B}$ is a Banach space in the norm

$$\|G\| = \int_{-\infty}^{+\infty} |G(t) d\mu(t)|.$$

We show that the space contains $[G(z) - G(w)]/(z - w)$ for all complex w if $G(z)$ is any entire function such that

$$\int_{-\infty}^{+\infty} |G(t) d\mu(t)| < \infty$$

and such that $G(z)/M(z)$ is bounded in the complex plane.

By the Hahn-Banach theorem it is sufficient to show that

$$L(z) = \int_{-\infty}^{+\infty} \frac{G(t) - G(z)}{t - z} h(t) d\mu(t)$$

vanishes identically whenever $h(x)$ is a Borel measurable function of real x, which is bounded by 1, such that

$$0 = \int_{-\infty}^{+\infty} F(t)h(t)d\mu(t)$$

for every element $F(z)$ of the Paley-Wiener space of type a. By the proof of Theorem 26, $L(z)$ is an entire function which is of bounded type in the upper and lower half-planes. Since

$$F(z)L(z) = \int_{-\infty}^{+\infty} \frac{F(z)G(t) - G(z)F(t)}{t - z}\, h(t)d\mu(t),$$

we obtain the estimate

$$|F(z)L(z)| \leq \left| F(z) \int_{-\infty}^{+\infty} \frac{G(t)h(t)d\mu(t)}{t - z} \right| + |G(z)|\,|y|^{-1} \int_{-\infty}^{+\infty} |F(t)d\mu(t)|.$$

By the arbitrariness of $F(z)$,

$$|L(z)| \leq \left| \int_{-\infty}^{+\infty} \frac{G(t)h(t)d\mu(t)}{t - z} \right| + \frac{|G(z)/M(z)|}{|y|}.$$

It follows that $L(z)$ has limit zero at both ends of the imaginary axis. By Problem 39, $L(z)$ vanishes identically.

If μ determines an extreme point of $\mathcal{M}$, then by Lemma 15 for every Borel measurable function $f(x)$ of real x such that $\int_{-\infty}^{+\infty} |f(t)d\mu(t)| < \infty$ and $\int_{-\infty}^{+\infty} f(t)d\mu(t) = 0$, there exists an element $F(z)$ of $\mathcal{B}$ such that

$$\int_{-\infty}^{+\infty} |f(t) - F(t)|\,|d\mu(t)| = 0.$$

Since $\int_{-\infty}^{+\infty} |d\mu(t)| = 1$ and $\int_{-\infty}^{+\infty} d\mu(t) = 0$, the measure associated with μ is supported at more than one point. If (a, b) is any finite interval which contains at least two points of support of the measure, there exists a function $f(x)$ which vanishes outside of (a, b) such that $\int_{-\infty}^{+\infty} |f(t)d\mu(t)| = 1$ and $\int_{-\infty}^{+\infty} f(t)d\mu(t) = 0$. If $F(z)$ is the corresponding element of $\mathcal{B}$, then

$$\int_{b}^{\infty} |F(t)d\mu(t)| = 0.$$

It follows that the discontinuities of $\mu(x)$ in the half-line (b, ∞) are contained in the zeros of $F(z)$. Since $F(z)$ does not vanish identically, these discontinuities have no finite limit point. By the arbitrariness of b, $\mu(x)$ is a step function whose discontinuities have no finite limit point.

If t_0 and t_1 are distinct points of discontinuity of $\mu(x)$, consider the function $f(x)$ which vanishes everywhere except these points and whose values at these points are given by

$$f(t_0)[\mu(t_0+) - \mu(t_0-)] = 1/(t_0 - t_1),$$
$$f(t_1)[\mu(t_1+) - \mu(t_1-)] = 1/(t_1 - t_0).$$

Since $\int_{-\infty}^{+\infty} f(t)d\mu(t) = 0$, there exists an element $F(z)$ of $\mathcal{B}$ such that $F(t_0) = f(t_0)$, $F(t_1) = f(t_1)$, and $F(t) = 0$ at all other discontinuities of $\mu(x)$. The function $S(z) = (z - t_0)(z - t_1)F(z)$ is entire and vanishes at the discontinuities of $\mu(x)$. By construction,

$$S'(t_0)[\mu(t_0+) - \mu(t_0-)] = 1,$$
$$S'(t_1)[\mu(t_1+) - \mu(t_1-)] = 1.$$

Let t_k be any other zero of $S(z)$. Since $(z - t_0)^{-1}(z - t_1)^{-1}S(z)$ belongs to $\mathcal{B}$ and since the space contains difference quotients,

$$(z - t_0)^{-1}(z - t_1)^{-1}(z - t_k)^{-1}S(z)$$

belongs to $\mathcal{B}$. Since

$$(t_k - t_1)(z - t_0)^{-1}(z - t_1)^{-1}(z - t_k)^{-1}S(z)$$
$$= (z - t_0)^{-1}(z - t_k)^{-1}S(z) - (z - t_0)^{-1}(z - t_1)^{-1}S(z),$$

the function $(z - t_0)^{-1}(z - t_k)^{-1}S(z)$ belongs to $\mathcal{B}$. The function vanishes at all discontinuities of $\mu(x)$ except t_0, and t_k if t_k is a discontinuity of $\mu(x)$. Since the function does not vanish at t_0 and since it has mean zero with respect to $\mu(x)$, t_k must be a discontinuity of $\mu(x)$. The identity

$$\int_{-\infty}^{+\infty} (t - t_0)^{-1}(t - t_k)^{-1}S(t)d\mu(t) = 0$$

implies that

$$S'(t_0)(t_0 - t_k)^{-1}[\mu(t_0+) - \mu(t_0-)]$$
$$+ S'(t_k)(t_k - t_0)^{-1}[\mu(t_k+) - \mu(t_k-)] = 0$$

and that

$$S'(t_k)[\mu(t_k+) - \mu(t_k-)] = 1.$$

So the entire function

$$P(z) = S(z) \int_{-\infty}^{+\infty} \frac{d\mu(t)}{z - t}$$

has value 1 at the zeros of $S(z)$. Since $S(z)$ is of bounded type in the upper and lower half-planes, $P(z)$ is of bounded type in these half-planes. If K is a bound for $(z - t_0)^{-1}(z - t_1)^{-1}S(z)/M(z)$, then

$$\left|\frac{P(z)}{(z - t_0)(z - t_1)}\right| \leq KM(z)\left|\int_{-\infty}^{+\infty} \frac{d\mu(t)}{z - t}\right| \leq K|y|^{-1}$$

when z is not real. It follows from Problem 39 that $P(z)$ is a linear function. Since $P(z)$ has value 1 at t_0 and t_1, it is identically 1.

From this we see that $S(z)$ is real for real z. Since

$$(z - t_0)^{-1}(z - t_1)^{-1}S(z)/M(z)$$

is bounded in the complex plane, the starting estimate of $\log M(z)$ will show that the mean type of $S(z)$ in the upper half-plane is at most a. Since

$$\int_{-\infty}^{+\infty} \frac{e^{iht} - e^{ihz}}{t - z} \, d\mu(t) = 0$$

when $-a \le h \le a$, the function

$$\frac{e^{ihz}}{S(z)} = \int_{-\infty}^{+\infty} \frac{e^{iht}d\mu(t)}{z - t}$$

is of nonpositive mean type in the upper half-plane. It follows that the mean type of $S(z)$ in the upper half-plane is equal to a.

PROBLEM 301. Let $((a_n, b_n))$ be a sequence of disjoint intervals to the right of $x = 1$ such that

$$\sum_{n=1}^{\infty} \frac{1 + (b_n - a_n)^2}{a_n b_n} = \infty.$$

Show that an absolutely convergent Fourier transform

$$f(x) = \int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$$

vanishes identically if it vanishes in an interval and if $\mu(x)$ is constant in each interval (a_n, b_n).

67. ENTIRE FUNCTIONS WITH ZEROS IN A SET

As an application of the extreme point method, we construct nontrivial entire functions which have zeros in a given set and whose reciprocals have absolutely convergent partial fraction decompositions.

THEOREM 67. Let $\psi(x)$ be a uniformly continuous, increasing function of real x such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1}|\psi(t) - \tau t| dt < \infty$$

for some number $\tau > 0$. For any given number a, $0 < a < \tau$, there exists an entire function $S(z)$ which is real for real z and has only real simple zeros, all at points t such that $\psi(t) \equiv 0$ modulo π, such that $S(z)$ is of bounded type and of mean type a in the upper half-plane, and such that

$$\sum_{S(t)=0} \frac{1}{|S'(t)|} < \infty.$$

By the proof of Theorem 66, these conditions imply that the partial fraction decomposition

$$\frac{1}{S(z)} = \sum_{S(t)=0} \frac{1}{S'(t)(z-t)}$$

is valid.

LEMMA 16. Let $\psi(x)$ be a uniformly continuous, increasing function of real x such that $\psi(x) \equiv 0$ modulo π for at least one value of x. Then there exists a space $\mathcal{H}(E)$ with phase function $\varphi(x)$ which has a bounded derivative and which agrees with $\psi(x)$ whenever $\varphi(x) \equiv 0$ modulo π or $\psi(x) \equiv 0$ modulo π.

Proof of Lemma 16. Since $\psi(x)$ is uniformly continuous,

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1} d\psi(t) < \infty.$$

If (t_n) is an enumeration of the points t such that $\psi(t) \equiv 0$ modulo π, then

$$\sum (1+t_n^2)^{-1} < \infty.$$

Let $B(z)$ be an entire function of Pólya class, which is real for real z, such that

$$\operatorname{Re} \frac{iB'(z)}{B(z)} = \sum \frac{y}{(t_n - x)^2 + y^2}$$

for $y > 0$. Then $A(z) = B'(z)$ is an entire function which is real for real z and $\operatorname{Re} iA(z)/B(z) > 0$ for $y > 0$. It follows that a space $\mathcal{H}(E)$ exists, $E(z) = A(z) - iB(z)$. The phase function $\varphi(x)$ associated with $E(z)$ can be chosen so that $\varphi(x) = \psi(x)$ whenever $\varphi(x) \equiv 0$ modulo π or $\psi(x) \equiv 0$ modulo π. We show that $\varphi'(x)$ is bounded. Since $\tan \varphi(x) = B(x)/A(x)$, $\varphi'(x) = 1$ whenever $\varphi(x) \equiv 0$ modulo π. Since

$$\frac{B(z)\bar{A}(w) - A(z)\bar{B}(w)}{z - \bar{w}} = \sum \frac{B(z)\bar{B}(w)}{(t_n - z)(t_n - \bar{w})},$$

the identity

$$\varphi'(x) = \sum \frac{\sin^2 \varphi(x)}{(t_n - x)^2}$$

holds for all real x. Since we assume that $\psi(x)$ is uniformly continuous, there exists a number $\delta > 0$ such that $|t_n - t_k| > \delta$ whenever $t_n \neq t_k$. It is convenient to choose $\delta < \pi$. If t is a real number such that $\varphi(t) \equiv 0$ modulo π and if $|x - t| \leq \delta$, then

$$\varphi'(x) \leq \sum_{-\infty}^{+\infty} \frac{\sin^2 \varphi(x)}{(x - t + n\delta)^2} = \frac{\pi^2}{\delta^2} \frac{\sin^2 \varphi(x)}{\sin^2 [\pi(x - t)/\delta]}.$$

It follows that

$$\cot \varphi(x) - (\pi/\delta) \cot [\pi(x - t)/\delta]$$

is a nondecreasing function of x in the interval $(t - \delta, t + \delta)$. Since the function is continuous at $x = t$ and vanishes there,

$$\cot^2 \varphi(x) \geq (\pi/\delta)^2 \cot^2 [\pi(x - t)/\delta]$$

for $|x - t| < \delta$. It follows that

$$\sin^2 \varphi(x) \leq \frac{\sin^2 [\pi(x - t)/\delta]}{\sin^2 [\pi(x - t)/\delta] + (\pi/\delta)^2 \cos^2 [\pi(x - t)/\delta]}$$

and that

$$\varphi'(x) \leq \frac{1}{\cos^2 [\pi(x - t)/\delta] + (\delta/\pi)^2 \sin^2 [\pi(x - t)/\delta]}.$$

So $\varphi'(x) \leq (\pi/\delta)^2$ for these values of x. If, on the other hand, x is a number at distance $\frac{1}{2}\delta$ or more from the points t such that $\varphi(t) \equiv 0$ modulo π, then

$$\varphi'(x) \leq 2 \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2(\frac{1}{2}\delta)^2} = \frac{\pi^2}{\delta^2}$$

by Problem 18. So $\varphi'(x) \leq (\pi/\delta)^2$ for all real x.

Proof of Theorem 67. Since the hypotheses imply that $\psi(x)$ is unbounded, there exists a value of x such that $\psi(x) \equiv 0$ modulo π. Let $\mathcal{H}(E) = \mathcal{H}(E(c))$ be a space corresponding to $\psi(x)$ as in Lemma 16. By Theorem 65 there exists a space $\mathcal{H}(E(b))$ contained isometrically in $\mathcal{H}(E(c))$ such that $E(b, z)/E(c, z)$ has no real zeros and such that the mean type of $E(c, z)/E(b, z)$ in the upper half-plane is greater than the given number a. Let $F(z)$ be an element of $\mathcal{H}(E(b))$ such that $F(z)/E(c, z)$ has a nonzero value at every point x such that $\psi(x) \equiv 0$ modulo π. If $-a \leq h \leq a$, then $e^{ihz}F(z)$ belongs to the closure of the domain of multiplication by z in $\mathcal{H}(E(c))$. Since the closed span of these functions does not fill the closure of the domain of multiplication by z in $\mathcal{H}(E(c))$, there exists a nonzero element $G(z)$ of the closure of the domain of multiplication by z in $\mathcal{H}(E(c))$ which is orthogonal to $e^{ihz}F(z)$ for $-a \leq h \leq a$. Let (t_n) be an enumeration of the real numbers t such that $\varphi(t) \equiv 0$ modulo π. By Theorem 22,

$$\pi \sum \frac{\exp(iht_n)\, F(t_n)\bar{G}(t_n)}{\varphi'(t_n)|E(t_n)|^2} = \langle e^{iht}F(t), G(t) \rangle = 0$$

for $-a \leq h \leq a$. The identity can be written

$$\int_{-\infty}^{+\infty} e^{iht} d\mu(t) = 0$$

where $\mu(x)$ is a step function whose only discontinuities are at points t_n and which has a jump of

$$\mu(t_n+) - \mu(t_n-) = \frac{\pi}{\varphi'(t_n)} \frac{F(t_n)\bar{G}(t_n)}{|E(t_n)|^2}$$

at each such point. The integral is absolutely convergent since

$$\int_{-\infty}^{+\infty} |d\mu(t)| \leq \|F\| \, \|G\|$$

by the Schwarz inequality. Since

$$\|G(t)\|^2 = \pi \sum \frac{1}{\varphi'(t_n)} \left| \frac{G(t_n)}{E(t_n)} \right|^2 > 0,$$

there exists some n such that $G(t_n)/E(t_n) \neq 0$. Since $F(t_n)/E(t_n) \neq 0$ for all n, $\mu(t_n+) - \mu(t_n-) \neq 0$ for some n. It follows that $\int_{-\infty}^{+\infty} e^{ixt} d\mu(t)$ does not vanish for all real x. The theorem now follows from Theorem 66.

PROBLEM 302. Let $K(x) \geq 1$ be a continuous function of real x such that $\log K(x)$ is uniformly continuous and

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log K(t)\, dt = \infty.$$

Let $\mu(x)$ be a nondecreasing function of real x such that

$$\int_{-\infty}^{+\infty} \frac{K(t)\, d\mu(t)}{1 + t^2} < \infty.$$

If $\mathcal{H}(E)$ is a given space contained isometrically in $L^2(\mu)$, if $E(z)$ is of bounded type in the upper half-plane, and if $E(z)$ has no real zeros, show that $E(z)$ has zero mean type in the half-plane.

PROBLEM 303. If $K(x)$ is a function of real x with positive values such that $\log K(x)$ is uniformly continuous, construct a differentiable function $K_1(x)$ of real x, which is equal to 1 in a neighborhood of the origin, such that $K_1'(x)/K_1(x)$, $K_1(x)/K(x)$, and $K(x)/K_1(x)$ are bounded functions.

68. NORMS DETERMINED ON A SEQUENCE OF POINTS

A fundamental problem is to determine the spaces $\mathcal{H}(E)$ which are contained isometrically in any given space $L^2(\mu)$. An existence theorem for such spaces can be given when $\mu(x)$ is a step function whose discontinuities are regularly distributed.

THEOREM 68. Let $\psi(x)$ be a uniformly continuous, increasing function of real x such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |\psi(t) - \tau t| \, dt < \infty$$

for some number $\tau > 0$. Let $W(x)$ be a function of real x such that $\log |W(x)|$ is uniformly continuous and

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |\log |W(t)|| \, dt < \infty.$$

For any given number a, $0 < a < \tau$, there exists a space $\mathcal{H}(E)$ such that

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt = \sum_{-\infty}^{+\infty} |F(t_n)/W(t_n)|^2$$

for every $F(z)$ in $\mathcal{H}(E)$ where t_n is for every n the unique point such that $\psi(t_n) = n\pi$. The space can be chosen so that $E(z)$ has no real zeros, so that $E(z)$ is of bounded type in the upper half-plane, and so that its mean type is a.

The proof requires another variant of the Beurling-Malliavin theorem.

LEMMA 17. Let $K(x) > 0$ be a continuous function of real x such that $\log K(x)$ is uniformly continuous and

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |\log K(t)| \, dt < \infty.$$

Let $\psi(x)$ be a uniformly continuous, increasing function of real x such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |\psi(t) - \tau t| \, dt < \infty$$

for some number $\tau > 0$. For any given number a, $0 < a < \tau$, there exists a nonzero entire function $F(z)$ of bounded type and of mean type at most a in the upper and lower half-planes such that $K(x)F(x)$ remains bounded on the set of points x such that $\psi(x) \equiv 0$ modulo π.

Proof of Lemma 17. By Problem 303 it is sufficient to prove the lemma in the case that $K(x)$ is a differentiable function which is equal to 1 in a neighborhood of the origin and $K'(x)/K(x)$ is bounded. If $\log K(x) = -xh(x)$, then $h(x)$ is a differentiable function of x such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |th(t)| \, dt < \infty$$

and such that the derivative of $xh(x)$ is bounded. If $f(z)$ is a function analytic in the upper half-plane such that

$$\operatorname{Re} f(z) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{h(t)\,dt}{(t-x)^2+y^2}$$

for $y > 0$, then $f'(z)$ belongs to $\mathfrak{D}_1$ by Lemma 14. By the proof of Theorem 64 there exists a function $W(z)$ analytic and without zeros in the upper half-plane such that $\operatorname{Re} iW'(z)/W(z) > 0$, such that $W(z)$ is of bounded type and of mean type at most a in the half-plane, and such that

$$K(x)|W(x)| \le |1-ix|^r$$

for all real x, where r is a nonnegative integer and

$$|W(x)| = \lim_{y\searrow 0} |W(x+iy)|.$$

By the proof of Theorem 15 there exists an entire function $E_1(z)$ of Pólya class such that

$$\operatorname{Re}\,[E_1(z)/W(z)] \ge 0$$

for $y > 0$. The function is of bounded type and of mean type at most a in the upper half-plane. If $\sqrt{E_1(z)/W(z)}$ is defined continuously so that its real part is nonnegative in the half-plane, then the real parts of

$$\exp\,(\tfrac{1}{4}i\pi)\sqrt{E_1(z)/W(z)}$$

and

$$\exp\,(-\tfrac{1}{4}i\pi)\sqrt{E_1(z)/W(z)}$$

are nonnegative in the half-plane. By the Poisson representation,

$$\int_{-\infty}^{+\infty} (1+t^2)^{-1}|E_1(t)/W(t)|^{\frac{1}{2}}dt < \infty.$$

It follows that

$$\int_{-\infty}^{+\infty} |(1-it)^{-r-4}K(t)E_1(t)|^{\frac{1}{2}}dt < \infty.$$

For every integer n, let t_n be the unique real number such that $\varphi(t_n) = n\pi$. Since $\varphi(x)$ is uniformly continuous, there exists a number $\delta > 0$ such that $|t_n - t_k| \ge 2\delta$ whenever $n \ne k$. Since

$$\int_{-\delta}^{\delta}\sum_{-\infty}^{+\infty}|(1-it_n-it)^{-r-4}K(t_n-t)E_1(t_n-t)|^{\frac{1}{2}}dt$$
$$= \sum_{-\infty}^{+\infty}\int_{-\delta}^{\delta}|(1-it)^{-r-4}K(t_n-t)E_1(t_n-t)|^{\frac{1}{2}}dt < \infty,$$

there exists at least one number t, $-\delta < t < \delta$, such that the function

$$(1-ix+it)^{-r-4}K(x-t)E_1(x-t)$$

remains bounded on the set (t_n). Since $\log K(x)$ is uniformly continuous, the function

$$(1 - ix + it)^{-r-4} K(x) E_1(x - t)$$

remains bounded on the set. If $E_1(z - t)$ has $r + 4$ zeros $w_1, \cdots, w_{r+4}$, the desired function $F(z)$ is given by

$$E_1(z - t) = F(z)(z - w_1) \cdots (z - w_{r+4}).$$

If $E_1(z - t)$ has fewer than $r + 4$ zeros, then

$$E_1(z - t) = P(z) e^{ihz}$$

where $P(z)$ is a polynomial of Pólya class and $0 \leq h \leq a$. In this case $F(z)$ is given by

$$F(z) = \sin (az)/Q(z)$$

where $Q(z)$ is a polynomial whose zeros are contained in the zeros of $\sin (az)$ and whose degree is $r + 4$ minus the degree of $P(z)$.

Proof of Theorem 68. Let b be a number, $a < b < \tau$. By Theorem 67, there exists an entire function $S(z)$ which is real for real z and has only real simple zeros, all at points t such that $\varphi(t) \equiv 0$ modulo π, such that $S(z)$ is of bounded type and of mean type b in the upper half-plane, such that

$$\sum_{S(t)=0} \frac{1}{|S'(t)|} < 1.$$

By Lemma 17 there exists a nonzero entire function $L(z)$ of bounded type and of mean type less than $b - a$ in the upper and lower half-planes, such that $L(x) W(x)$ is bounded by 1 on the set of points x such that $\varphi(x) \equiv 0$ modulo π.

Let $\mathcal{H}$ be the set of all entire functions $F(z)$ of bounded type and of mean type at most a in the upper and lower half-planes such that

$$\|F\|^2 = \sum_{-\infty}^{+\infty} |F(t_n)/W(t_n)|^2 < \infty.$$

We show that $\mathcal{H}$ contains a nonzero element. By Lemma 17 there exists a nonzero entire function $G(z)$ of bounded type and of mean type at most a in the upper and lower half-planes, such that $G(x)/W(x)$ remains bounded on the set of points x such that $\varphi(x) \equiv 0$ modulo π. If $G(z)$ has a zero w, then $G(z) = (z - w)F(z)$ where $F(z)$ is in $\mathcal{H}$. If $G(z)$ has no zeros, then $G(z) = G(0)e^{ihz}$ for some number h, $0 \leq h \leq a$, and the function $1/W(x)$ remains bounded on the set of points x such that $\varphi(x) \equiv 0$ modulo π. In this case $\sin (az) = zF(z)$ where $F(z)$ is in $\mathcal{H}$. We show that $\mathcal{H}$ is a Hilbert space.

By the proof of Theorem 66,

$$\frac{1}{S(z)} = \sum_{-\infty}^{+\infty} \frac{1}{S'(t_n)(z - t_n)}$$

and

$$0 = \sum_{-\infty}^{+\infty} \frac{e^{iht_n}}{S'(t_n)}$$

when $-b \leq h \leq b$. If $F(z)$ is in $\mathcal{H}$, then $F(z)L(z)$ is an entire function of bounded type and of mean type less than b in the upper and lower half-planes. It follows that

$$\exp[-b|y|] F(iy)L(iy)$$

remains bounded, $-\infty < y < \infty$. Since $F(x)L(x)$ remains bounded on the set of points x such that $\varphi(x) \equiv 0$ modulo π, the identity

$$0 = \sum_{-\infty}^{+\infty} \frac{F(t_n)L(t_n) - F(z)L(z)}{t_n - z} \frac{1}{S'(t_n)}$$

holds for all complex z by the proof of Theorem 61. By the partial fraction decomposition of $1/S(z)$, the identity becomes an interpolation formula

$$F(z)L(z) = \sum_{-\infty}^{+\infty} \frac{F(t_n)L(t_n)}{S'(t_n)} \frac{S(z)}{z - t_n}.$$

By the Schwarz inequality,

$$|F(z)L(z)|^2 \leq \|F\|^2 \sum_{-\infty}^{+\infty} \left| \frac{S(z)}{z - t_n} \right|^2 \leq \|F\|^2 \frac{S(z)\bar{S}'(z) - S'(z)\bar{S}(z)}{z - \bar{z}}$$

for all complex z. If $(F_n(z))$ is a Cauchy sequence in $\mathcal{H}$, $\lim F_n(z) = F(z)$ exists uniformly on bounded sets. The limit is an entire function such that

$$\sum_{-\infty}^{+\infty} |F(t_n)/W(t_n)|^2 < \infty$$

and such that the same interpolation formula holds. By Problem 65, $F(z)L(z)/S(z)$ is of bounded type and of nonpositive mean type in the upper half-plane. It follows that $F(z)$ is of bounded type in the upper half-plane and that its mean type does not exceed the difference between τ and the mean type of $L(z)$. Since $L(z)$ can be chosen with any mean type less than $b - a$, the mean type of $F(z)$ is at most a. A similar argument will show that $F(z)$ is of bounded type and of mean type at most a in the lower half-plane. So $F(z)$ belongs to $\mathcal{H}$. Since

$$\|F - F_n\| \leq \lim_{k \to \infty} \|F_k - F_n\|$$

and since the sequence $(F_n(z))$ is Cauchy, $F(z) = \lim F_n(z)$ in the metric of $\mathcal{H}$. This completes the proof that $\mathcal{H}$ is a Hilbert space.

The axioms (H1), (H2), and (H3) are easily verified. Since $\mathcal{H}$ contains a nonzero element, it is equal isometrically to a space $\mathcal{H}(E)$ by Theorem 23. Since $F(z)/(z - w)$ belongs to $\mathcal{H}(E)$ whenever $F(z)$ belongs to $\mathcal{H}(E)$ and has a zero w, $E(z)$ has no real zeros. It is clear from the definition of $\mathcal{H}$ that $E(z)$ is of bounded type and of mean type a in the upper half-plane.

PROBLEM 304. Let $W(x) \geq 1$ be a continuous function of real x such that $\log W(x)$ is uniformly continuous and such that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} \log W(t) dt = \infty.$$

Show that there exists a nonconstant entire function $F(z)$ of bounded type and of zero mean type in the upper and lower half-planes such that $F(x)/W(x)$ is bounded on the real axis.

LAGUERRE CLASSES

A generalization of the Pólya class theory is given by the Laguerre classes of entire functions.

PROBLEM 305. Let ρ be a nonnegative integer. An entire function $F(z)$ is said to belong to the ρth Laguerre class if it is real for real z, has only real zeros, and has value one at the origin, and if

$$\operatorname{Re} iz^{-2\rho} F'(z)/F(z) \geq 0$$

for $y > 0$. Show that the function

$$(1 - hz) \exp\left(hz + \tfrac{1}{2}h^2z^2 + \cdots + \frac{1}{2\rho + 1} h^{2\rho+1} z^{2\rho+1}\right)$$

belongs to the ρth Laguerre class if h is real. Show that a finite product of functions which belong to the ρth Laguerre class is a function which belongs to the ρth Laguerre class. Show that a limit of functions which belong to the ρth Laguerre class is a function which belongs to the ρth Laguerre class if convergence is uniform on bounded sets. If (h_n) is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} h_n^{2\rho+2} < \infty,$$

show that the product

$$P(z) = \prod_{n=1}^{\infty} (1 - h_n z) \exp\left(h_n z + \tfrac{1}{2}h_n^2 z^2 + \cdots + \frac{1}{2\rho + 1} h_n^{2\rho+1} z^{2\rho+1}\right)$$

converges uniformly on bounded sets and represents an entire function which belongs to the ρth Laguerre class. Show that it satisfies the estimate

$$\log (1 + |P(z) - 1|) \leq \sum_{n=1}^{\infty} h_n^{2\rho+2} |z|^{2\rho+2}$$

for all complex z. Show that the function $\exp(-az^{2\rho+2})$ belongs to the ρth Laguerre class if $a \geq 0$.

PROBLEM 306. If an entire function $F(z)$ belongs to the ρth Laguerre class and has a zero w, show that

$$F(z) = G(z)(1 - hz) \exp\left(hz + \tfrac{1}{2}h^2 z^2 + \cdots + \frac{1}{2\rho + 1} h^{2\rho+1} z^{2\rho+1}\right)$$

where $G(z)$ is an entire function which belongs to the ρth Laguerre class and $h = 1/w$.

PROBLEM 307. If an entire function $F(z)$ belongs to the ρth Laguerre class and has no zeros, show that

$$F(z) = \exp(-az^{2\rho+2})$$

where $a \geq 0$.

PROBLEM 308. If an entire function $F(z)$ belongs to the ρth Laguerre class, show that $F(z)$ is equal to

$$\exp(-az^{2\rho+2}) \prod (1 - h_n z) \exp\left(h_n z + \tfrac{1}{2}h_n^2 z^2 + \cdots + \frac{1}{2\rho + 1} h_n^{2\rho+1} z^{2\rho+1}\right)$$

where $a \geq 0$ and (h_n) is a sequence of real numbers such that $\sum h_n^{2\rho+2} < \infty$.

PROBLEM 309. If an entire function $F(z)$ belongs to the ρth Laguerre class, show that

$$\lim_{z \to 0} F'(z)/z^{2\rho+1} = -(2\rho + 2)\delta$$

where $\delta \geq 0$ and

$$\log (1 + |F(z) - 1|) \leq \delta |z|^{2\rho+2}$$

for all complex z.

PROBLEM 310. Let $F(z)$ be a function which is real for real z, has only real zeros, and has value one at the origin. Define $\log F(z)$ continuously in the upper half-plane so as to have limit zero at the origin. Show that $F(z)$ belongs to the ρth Laguerre class if, and only if,

$$\text{Re } i[\log F(z)]/z^{2\rho+1} \geq 0$$

for $y > 0$.

PROBLEM 311. Let $F(z)$ and $G(z)$ be entire functions which are real for real z, have only real zeros, and have value one at the origin. Assume that $G(z)/F(z)$ is of bounded type in the upper half-plane. Let $P(z)$ and $Q(z)$ be the unique polynomials of degree at most $2\rho + 1$, which are real for real z and have value zero at the origin, such that the derivatives of $F(z) \exp P(z)$ and $G(z) \exp Q(z)$ have zeros of order at least $2\rho + 1$ at the origin. Show that $G(z) \exp Q(z)$ belongs to the ρth Laguerre class if $F(z) \exp P(z)$ belongs to the ρth Laguerre class.

PROBLEM 312. Let $F(z)$ be an entire function of Pólya class which is real for real z, has only real zeros, and has value one at the origin. Let $P(z)$ be the unique polynomial of degree at most $2\rho + 1$, which is real for real z and has value zero at the origin, such that the derivative of $F(z) \exp P(z)$ has a zero of order at least $2\rho + 1$ at the origin. Show that $F(z) \exp P(z)$ belongs to the ρth Laguerre class.

PROBLEM 313. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Show that the transformation $F(z) \to [F(z) - A(z)F(0)]/z$ is self-adjoint in the space. Show that the space admits an orthogonal basis consisting of eigenfunctions of the transformation. Show that the nonzero eigenvalues of the transformation are the numbers $(1/t_n)$ where (t_n) are the zeros of $A(z)$.

PROBLEM 314. A bounded transformation T of a Hilbert space into itself is said to be of Schmidt class if

$$\sigma(T)^2 = \sum \|Tf_n\|^2 < \infty$$

for some orthonormal basis (f_n) of the space. Show that the sum does not depend on the choice of orthonormal basis.

PROBLEM 315. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Let T be the transformation

$$F(z) \to [F(z) - A(z)F(0)]/z$$

in the space. Show that $T^{1+\rho}$ is of Schmidt class if $A(z)$ belongs to the ρth Laguerre class. Show that $\sigma(T^{1+\rho})^2 \leq \delta$ where

$$-(2\rho + 2)\delta = \lim_{z\to 0} A'(z)/z^{2\rho+1}.$$

Show that equality holds if

$$\lim_{y\to+\infty} \frac{\log |A(iy)|}{y^{2\rho+2}} = 0.$$

PROBLEM 316. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Let T be the transformation

$$F(z) \to [F(z) - A(z)F(0)]/z$$

in the space. If $T^{1+\rho}$ is of Schmidt class, show that $E(z) = S(z)E_0(z)$ where $S(z)$ is an entire function which is real for real z and has no zeros and $\mathcal{H}(E_0)$ is a space such that $A_0(z)$ belongs to the ρth Laguerre class. Show that $A_0(z)$ can be chosen so that

$$\lim_{y\to+\infty} \frac{\log |A_0(iy)|}{y^{2\rho+2}} = 0.$$

PROBLEM 317. Let $\{\mathcal{H}(E(t))\}$ be a family of spaces, $t > 0$, associated with a nondecreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

so that $E(t, z)$ is a continuous function of t for each fixed z and

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

when $0 < a < b < \infty$. Assume that $E(a, z)$ has no real zeros and has value one at the origin for every index a and that

$$\lim_{a\searrow 0} K(a, z, z) = 0$$

for all complex z. If for some index a there exists a polynomial $P(a, z)$ of degree at most $2\rho + 1$, which is real for real z and has value zero at the origin, such that $A(a, z) \exp P(a, z)$ belongs to the ρth Laguerre class, show that such a polynomial exists for every index a. If

$$\lim_{y\to+\infty} \frac{\log |A(a, iy)|}{y^{2\rho+2}} = 0$$

for some index a, show that the same is true for every index a. If

$$-(2\rho + 2)\delta(a) = \lim_{z \to 0} \frac{1}{z^{2\rho+1}} \frac{d}{dz} [A(a, z) \exp P(a, z)],$$

show that $\delta(t)$ is a continuous, nondecreasing function of t such that $\lim \delta(t) = 0$ as $t \searrow 0$. Show that

$$1 + |A(a, z) \exp P(a, z) - 1| \leq \exp [\delta(a)|z|^{2\rho+2}]$$

for all complex z.

PROBLEM 318. Let c be a point which is regular with respect to $m(t)$ in Problem 317. If $(f_2(t), g_2(t))$ belongs to $L^2(m)$ and vanishes in (c, ∞), show that there exists an element $(f_1(t), g_1(t))$ of $L^2(m)$, which vanishes in (c, ∞), such that

$$g_1(a) = \int_0^a [f_2(t)d\alpha(t) + g_2(t)d\beta(t)],$$

$$f_1(a) = \int_a^c [f_2(t)d\beta(t) + g_2(t)d\gamma(t)]$$

whenever $0 < a < c$. If $F_1(z)$ and $F_2(z)$ are the elements of $\mathcal{H}(E(c))$ such that

$$\pi F_k(w) = \int_0^a (f_k(t), g_k(t))dm(t)(A(t, \bar{w}), B(t, \bar{w}))^-$$

for all complex w, show that

$$F_1(z) = [F_2(z) - A(z)F_2(0)]/z.$$

LAGUERRE SPACES

Some interesting examples of Hilbert spaces of entire functions appear in the theory of Laguerre polynomials. The spaces satisfy an axiom which depends on the choice of a real index ν. The axiom is equivalent to a recurrence relation for the defining functions $A(z)$ and $B(z)$ of the space.

PROBLEM 319. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has a nonzero value at the origin, and let ν be a given real number. Assume that $zF(z + 1)$ belongs to the space whenever $F(z)$ belongs to the space, that $G(z - 1)/(z - 1)$ belongs to the space whenever $G(z)$ belongs to the space and vanishes at the origin, and that the identity

$$\langle (t - \nu)F(t + 1), G(t) \rangle = \langle F(t), G(t - 1)/(t - 1) \rangle$$

holds for all such elements $F(z)$ and $G(z)$. Show that $K(0, z)/E(z)$ has a nonzero value at ν, and that there exists a matrix

$$P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

having real entries and determinant one such that

$$(z - \nu)(A(z + 1), B(z + 1))P$$
$$= (A(z), B(z)) - (A(\nu), B(\nu))K(0, z)/K(0, \nu).$$

PROBLEM 320. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 319 for some index ν, show that there exists a space $\mathcal{H}(E_0)$ which satisfies the hypotheses of the problem for the same index ν and that there exists an entire function $S(z)$, which is real for real z, such that

$$E(z) = S(z)E_0(z),$$

$E_0(z)$ is of Pólya class and has no real zeros, and $S(z + 1) = \pm S(z)$.

PROBLEM 321. Let $\mathcal{H}(E_0)$ be a space which satisfies the hypotheses of Problem 319 such that $E_0(z)$ has value one at the origin. If multiplication by z is not densely defined in $\mathcal{H}(E_0)$, show that the closure of the domain of multiplication by z in $\mathcal{H}(E_0)$ is a space $\mathcal{H}(E_1)$ which satisfies the hypotheses of Problem 319 in the metric of $\mathcal{H}(E_0)$. Show that $E_1(z)$ can be chosen so that the identity

$$(A_n(z), B_n(z)) = (A_{n+1}(z), B_{n+1}(z))\begin{pmatrix} 1 - u_n v_n z & u_n^2 z \\ -v_n^2 z & 1 + u_n v_n z \end{pmatrix}$$

holds with $n = 0$ for some real numbers u_0 and v_0. Assume that multiplication by z is not densely defined in $\mathcal{H}(E_1)$. Show that there exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, with these properties: Multiplication by z is not densely defined in $\mathcal{H}(E_n)$. The space $\mathcal{H}(E_{n+1})$ is contained isometrically in $\mathcal{H}(E_n)$ and coincides with the closure of the domain of multiplication by z in $\mathcal{H}(E_n)$. There exist real numbers u_n and v_n such that the previous identity holds for every n. Show that $\mathcal{H}(E_n)$ satisfies the hypotheses of Problem 319. If P_n is defined for $E_n(z)$ as in Problem 319, show that there exists a nonzero number λ, $-1 < \lambda < 1$, such that

$$P_n\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lambda\begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \text{and} \quad P_{n+1}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \lambda^{-1}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

for every n. Show that there exists a real number κ, not a negative integer, such that

$$(u_{n+1}/u_n)^2 = \lambda^2(\kappa + \nu + n + 1)/(\kappa + n + 1)$$

and

$$u_{n+1}^2(v_n/u_n - v_{n+1}/u_{n+1}) = (1 - \lambda^2)/(\kappa + n + 1)$$

for every $n = 0, 1, 2, \cdots$. Show that the functions

$$\Phi_n(z) = \lambda^n[A_n(z) + B_n(z)v_n/u_n]$$

satisfy the identities

$$\begin{aligned}(1 - \lambda^2)z\Phi_n(z) \\ = \lambda(\kappa + n)\Phi_{n-1}(z) - [(\kappa + n) + \lambda^2(\kappa + \nu + n + 1)]\Phi_n(z) \\ + \lambda(\kappa + \nu + n + 1)\Phi_{n+1}(z)\end{aligned}$$

for $n > 0$ and

$$\lambda\Phi_n(z) = (z - \nu)\Phi_{n+1}(z + 1) + (\kappa + \nu + n + 1)[\lambda\Phi_n(z) - \Phi_{n+1}(z)]/z$$

for $n \geq 0$. Show that the intersection of the spaces $\mathcal{H}(E_n)$ contains no nonzero element. Show that λ is positive if $E_0(z)$ has no real zeros.

Examples of Laguerre spaces can be constructed from the space of square summable power series and its generalizations. A Hilbert space $\mathcal{C}(a, b; c; z)$ is associated with the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{1!\,c}\,z + \frac{a(a + 1)b(b + 1)}{2!\,c(c + 1)}\,z^2 + \cdots$$

if the parameters are such that all coefficients of the series are positive. The space is the set of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = |a_0|^2 + \frac{1!\,c}{ab}\,|a_1|^2 + \frac{2!\,c(c + 1)}{a(a + 1)b(b + 1)}\,|a_2|^2 + \cdots < \infty.$$

The series which belong to $\mathcal{C}(a, b; c; z)$ converge and represent analytic functions in the unit disk. The series $F(a, b; c; \bar{w}z)$ belongs to the space when $|w| < 1$, and the identity

$$f(w) = \langle f(z), F(a, b; c; \bar{w}z)\rangle$$

holds for every element $f(z)$ of the space. When $a = b = c = 1$, the space reduces to the usual space of square summable power series.

PROBLEM 322. Let κ and ν be real numbers such that the coefficients of $F(\kappa + \nu + 1, 1; \kappa + 1; z)$ are all positive, and let λ be a given number,

$0 < \lambda < 1$. Define

$$\Phi_n(z) = \lambda^{n+z} \frac{\Gamma(\kappa + n + 1)}{\Gamma(\kappa + z + n + 1)} F(\kappa + \nu + n + 1, z; \kappa + z + n + 1; \lambda^2)$$

for every $n = 0, 1, 2, \cdots$. Show that $\Phi_n(z)$ is an entire function which is real for real z. Show that there exist spaces $\mathcal{H}(E_n)$, $n = 0, 1, 2, \cdots$, satisfying the hypotheses of Problem 319, such that $\mathcal{H}(E_{n+1})$ is contained isometrically in $\mathcal{H}(E_n)$ and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathcal{H}(E_{n+1})$ in $\mathcal{H}(E_n)$. Show that the function $E_0(z)$ can be chosen so that

$$\int_{-\infty}^{+\infty} |\Phi_n(t)/E_0(t)|^2 dt = \frac{\pi(\kappa + 1)(\kappa + 2)\cdots(\kappa + n)}{(\kappa + \nu + 1)(\kappa + \nu + 2)\cdots(\kappa + \nu + n)},$$

where the products on the right are taken to be 1 when $n = 0$.

PROBLEM 323. Let $\mathcal{H}(E)$ be a given space which satisfies the hypotheses of Problem 321 for some index ν. Show that there exist real numbers κ and λ as in Problem 322 and that there exists an entire function $S(z)$ which is real for real z such that $S(z + 1) = \pm S(z)$ and such that the transformation $F(z) \to S(z)F(z)$ is an isometry of the space $\mathcal{H}(E_0)$ of Problem 322 onto the given space $\mathcal{H}(E)$.

PROBLEM 324. Let $E_n(z)$ and $\Phi_n(z)$ be defined as in Problem 322 for some real numbers κ, ν, and λ. Show that

$$\Phi(w, z) = \sum_{n=0}^{\infty} \Phi_n(\bar{w}) z^n$$

belongs to $\mathcal{C}(\kappa + \nu + 1, 1; \kappa + 1; z)$ for every complex number w. If $f(z)$ belongs to $\mathcal{C}(\kappa + \nu + 1, 1; \kappa + 1; z)$ show that its eigentransform $F(z)$, defined by

$$F(w) = \langle f(z), \Phi(w, z)\rangle$$

for all complex numbers w, belongs to $\mathcal{H}(E_0)$, and that

$$\int_{-\infty}^{+\infty} |F(t)/E_0(t)|^2 dt = \pi \|f(z)\|^2.$$

Show that every element of $\mathcal{H}(E_0)$ is of this form. Show that a necessary and sufficient condition that $F(z)$ belong to $\mathcal{H}(E_n)$ is that the coefficient of z^r in $f(z)$ be zero for all indices $r < n$. If the coefficient of z^r in $f(z)$ is zero for all indices $r < -\kappa$, show that the formula

$$F(w) = \frac{1}{\lambda^\kappa (1 - \lambda^2)^{1+\nu} \Gamma(w)} \int_0^\lambda \frac{(\lambda - t)^{w-1}}{(1 - \lambda t)^{w-\nu}} t^\kappa f(t) dt$$

is valid for $\operatorname{Re} w > 0$.

PROBLEM 325. In Problem 324 let $f(z)$ and $g(z)$ be elements of $\mathcal{C}(\kappa + \nu + 1, 1; \kappa + 1; z)$ and let $F(z)$ and $G(z)$ be their eigentransforms. Show that $G(z) = zF(z)$ is a necessary and sufficient condition that $f(0) = 0$ and that

$$(1 - \lambda^2)g(z) = (\lambda - z)(1 - \lambda z)f'(z) - \lambda(1 + \nu)(\lambda - z)f(z) \\ + \kappa(\lambda - z)(1 - \lambda z)f(z)/z.$$

PROBLEM 326. In Problem 324 let $f(z)$ and $g(z)$ be elements of $\mathcal{C}(\kappa + \nu + 1, 1; \kappa + 1; z)$ and let $F(z)$ and $G(z)$ be their eigentransforms. Show that $G(z) = zF(z + 1)$ is a necessary and sufficient condition that

$$g(z) = f(z)(\lambda - z)/(1 - \lambda z).$$

MEIXNER AND POLLACZEK SPACES

Some finite dimensional examples of Hilbert spaces of entire functions are associated with Pollaczek's orthogonal polynomials. The spaces are characterized by an axiom which involves two parameters, h and ω. The axiom implies a recurrence relation for the defining functions $A(z)$ and $B(z)$.

PROBLEM 327. Let $\mathcal{H}(E)$ be a given space, and let h and ω be numbers, $h > 0$, $\omega \neq \bar{\omega}$, and $|\omega| = 1$. Assume that the functions

$$(h - iz)[F(z + i) - F(z)] \quad \text{and} \quad (h + iz)[F(z - i) - F(z)]$$

belong to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle \bar{\omega}(h - it)F(t + i), G(t)\rangle + \langle \omega(h + it)F(t), G(t + i)\rangle = 0$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Show that there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \quad \text{and} \quad S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the recurrence relations

$$\bar{\omega}(h - iz)S_+(z + i) + i(\omega + \bar{\omega})zS_+(z) - \omega(h + iz)S_+(z - i) \\ = \lambda_+(\bar{\omega} - \omega)S_+(z),$$

$$\bar{\omega}(h - iz)S_-(z + i) + i(\omega + \bar{\omega})zS_-(z) - \omega(h + iz)S_-(z - i) \\ = \lambda_-(\bar{\omega} - \omega)S_-(z),$$

$$\bar{\omega}^2(h - iz)S_-(z + i) + 2izS_-(z) - \omega^2(h + iz)S_-(z - i) \\ = (\lambda_- + h)(\bar{\omega} - \omega)S_+(z),$$

$$(h - iz)S_+(z + i) + 2izS_+(z) - (h + iz)S_+(z - i) \\ = (\lambda_+ - h)(\bar{\omega} - \omega)S_-(z)$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

PROBLEM 328. If $\mathcal{H}(E)$ satisfies the hypotheses of Problem 327 for some indices h and ω and if $E(z) = S(z)E_0(z)$ as in Problem 221, show that $\mathcal{H}(E_0)$ satisfies these hypotheses for the same h and ω. If $S(z)$ is not periodic of period i, show that the numbers λ_+ and λ_- which appear in the recurrence relations for $A_0(z)$ and $B_0(z)$ are the negatives of the corresponding numbers in the recurrence relations for $A(z)$ and $B(z)$.

PROBLEM 329. Show that a space $\mathcal{H}(E)$ which satisfies the hypotheses of Problem 327 is finite dimensional. If λ_+ is positive, show that $S_-(z)$ belongs to the space and that $\lambda_+ = r + h$ where r is the dimension of the space.

PROBLEM 330. Let h and ω be given numbers, $h > 0$, $\omega \neq \bar{\omega}$, and $|\omega| = 1$. Show that the polynomials $\Phi_n(z)$ defined by

$$\Phi_n(z) = \bar{\omega}^n F(-n, h + iz; 2h; 1 - \omega^2)$$

are real for real z and satisfy the identities

$$\bar{\omega}(h - iz)\Phi_n(z + i) + i(\omega + \bar{\omega})z\Phi_n(z) - \omega(h + iz)\Phi_n(z - i) = (h + n)(\bar{\omega} - \omega)\Phi_n(z),$$

$$iz(\bar{\omega} - \omega)\Phi_n(z) = -n\Phi_{n-1}(z) + (h + n)(\omega + \bar{\omega})\Phi_n(z) - (2h + n)\Phi_{n+1}(z),$$

$$(h - iz)\Phi_n(z + i) + 2iz\Phi_n(z) - (h + iz)\Phi_n(z - i) = n(\bar{\omega} - \omega)\Phi_{n-1}(z),$$

$$\bar{\omega}^2(h - iz)\Phi_n(z + i) + 2iz\Phi_n(z) - \omega^2(h + iz)\Phi_n(z - i) = (2h + n)(\bar{\omega} - \omega)\Phi_{n+1}(z).$$

Show that there exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, satisfying the hypotheses of Problem 327, such that $\mathcal{H}(E_n)$ is contained isometrically in $\mathcal{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathcal{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathcal{H}(E_n)$ in $\mathcal{H}(E_{n+1})$ when $n > 0$. Show that the spaces can be chosen so that

$$\|\Phi_n\|^2 = \Gamma(1 + n)\Gamma(2h)/\Gamma(2h + n)$$

for every n.

PROBLEM 331. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 327, show that there exists an index r in Problem 330 and an entire function $S(z)$ which is real for real z and periodic of period $2i$ such that the transformation $F(z) \to S(z)F(z)$ takes $\mathcal{H}(E_r)$ isometrically onto $\mathcal{H}(E)$.

PROBLEM 332. In Problem 330 if $f(z) = \sum a_n z^n$ is a polynomial of degree at most r, show that its eigentransform $F(z)$, defined by $F(z) = \sum a_n \Phi_n(z)$, belongs to $\mathcal{H}(E_r)$ and that

$$\int_{-\infty}^{+\infty} |F(t)/E_r(t)|^2 dt = \|f(z)\|^2$$

where the norm of $f(z)$ is taken in $\mathcal{C}(2h, 1; 1; z)$. Show that every element of $\mathcal{H}(E_r)$ is of this form. Show that the identity

$$\Gamma(h - iz)\Gamma(h + iz)F(z)$$
$$= 2^{1-2h}\Gamma(2h) \int_{-1}^{1} (1 - t)^{h+iz-1}(1 + t)^{h-iz-1} f(\tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega} - \tfrac{1}{2}t\omega + \tfrac{1}{2}t\bar{\omega})dt$$

holds for $-h < y < h$. Let $f(z)$ and $g(z)$ be polynomials and let $F(z)$ and $G(z)$ be their eigentransforms. Show that

$$(\bar{\omega} - \omega)G(z) = \bar{\omega}(h - iz)F(z + i) + i(\omega + \bar{\omega})zF(z) - \omega(h + iz)F(z - i)$$

is a necessary and sufficient condition that

$$g(z) = hf(z) + zf'(z).$$

Show that

$$(\bar{\omega} - \omega)G(z) = \bar{\omega}^2(h - iz)F(z + i) + 2izF(z) - \omega^2(h + iz)F(z - i)$$

is a necessary and sufficient condition that

$$g(z) = 2hzf(z) + z^2f'(z).$$

Show that

$$(\bar{\omega} - \omega)G(z) = (h - iz)F(z + i) + 2izF(z) - (h + iz)F(z - i)$$

is a necessary and sufficient condition that

$$g(z) = f'(z).$$

Show that $G(z) = i(\omega - \bar{\omega})zF(z)$ is a necessary and sufficient condition that

$$g(z) = h(2z - \omega - \bar{\omega})f(z) + (z^2 - \omega z - \bar{\omega}z + 1)f'(z).$$

Similar finite dimensional spaces are associated with Meixner's polynomials.

PROBLEM 333. Let $\mathcal{H}(E)$ be a given space, and let h and ω be positive numbers, $\omega \neq \omega^{-1}$. Assume that $E(z)$ has nonzero values at h and $-h$. Assume that the functions $(h + z)[F(z + 1) - F(z)]$ and $(h - z)[F(z - 1) - F(z)]$ belong to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle \omega^{-1}(h - t)F(t - 1), G(t)\rangle + \langle \omega(h + t)F(t), G(t + 1)\rangle = 0$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Show that there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \quad \text{and} \quad S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the recurrence relations

$$\omega(h + z)S_+(z + 1) - (\omega + \omega^{-1})zS_+(z) - \omega^{-1}(h - z)S_+(z - 1) = \lambda_+(\omega - \omega^{-1})S_+(z),$$

$$\omega(h + z)S_-(z + 1) - (\omega + \omega^{-1})zS_-(z) - \omega^{-1}(h - z)S_-(z - 1) = \lambda_-(\omega - \omega^{-1})S_-(z),$$

$$\omega^2(h + z)S_-(z + 1) - 2zS_-(z) - \omega^{-2}(h - z)S_-(z - 1) = (\lambda_- + h)(\omega - \omega^{-1})S_+(z),$$

$$(h + z)S_+(z + 1) - 2zS_+(z) - (h - z)S_+(z - 1) = (\lambda_+ - h)(\omega - \omega^{-1})S_-(z)$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$. Show that such a space is finite dimensional and that either $S_+(z)$ or $S_-(z)$ belongs to the space.

PROBLEM 334. Let h and ω be given positive numbers, $\omega \neq \omega^{-1}$. Show that the polynomials $\Phi_n(z)$ defined by

$$\Phi_n(z) = \omega^{-n}F(-n, h + z; 2h; 1 - \omega^2)$$

are real for real z and satisfy the identities

$$\omega(h + z)\Phi_n(z + 1) - (\omega + \omega^{-1})z\Phi_n(z) - \omega^{-1}(h - z)\Phi_n(z - 1) = (h + n)(\omega - \omega^{-1})\Phi_n(z),$$

$$z(\omega - \omega^{-1})\Phi_n(z) = n\Phi_{n-1}(z) - (h + n)(\omega + \omega^{-1})\Phi_n(z) + (2h + n)\Phi_{n+1}(z),$$

$$\omega^2(h + z)\Phi_n(z + 1) - 2z\Phi_n(z) - \omega^{-2}(h - z)\Phi_n(z - 1) = (2h + n)(\omega - \omega^{-1})\Phi_{n+1}(z),$$

$$(h + z)\Phi_n(z + 1) - 2z\Phi_n(z) - (h - z)\Phi_n(z - 1) = n(\omega - \omega^{-1})\Phi_{n-1}(z).$$

Show that there exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, satisfying the hypotheses of Problem 333, such that $\mathcal{H}(E_n)$ is contained isometrically in $\mathcal{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathcal{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathcal{H}(E_n)$ in $\mathcal{H}(E_{n+1})$ when $n > 0$. Show that the spaces can be chosen so that

$$\|\Phi_n\|^2 = \Gamma(1 + n)\Gamma(2h)/\Gamma(2h + n)$$

for every n.

PROBLEM 335. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 333, show that there exists an index r in Problem 334 and an entire function $S(z)$ which is real for real z and periodic of period 2 such that the transformation $F(z) \to S(z)F(z)$ takes $\mathcal{H}(E_r)$ isometrically onto $\mathcal{H}(E)$.

PROBLEM 336. In Problem 334 if $f(z) = \sum a_n z^n$ is a polynomial of degree at most r, show that its eigentransform $F(z)$, defined by $F(z) = \sum a_n \Phi_n(z)$, belongs to $\mathcal{H}(E_r)$ and that

$$\int_{-\infty}^{+\infty} |F(t)/E_r(t)|^2 dt = \|f(z)\|^2 .$$

where the norm of $f(z)$ is taken in $\mathcal{C}(2h, 1; 1; z)$. Show that every element of $\mathcal{H}(E_r)$ is of this form. Show that the identity

$$\begin{aligned}\Gamma(h-z)\Gamma(h+z)F(z) &\\ = 2^{1-2h}\Gamma(2h)\int_{-1}^{1} &(1-t)^{h+z-1}(1+t)^{h-z-1}f(\tfrac{1}{2}\omega + \tfrac{1}{2}\omega^{-1} - \tfrac{1}{2}t\omega + \tfrac{1}{2}t\omega^{-1})dt\end{aligned}$$

holds for $-h < x < h$. Let $f(z)$ and $g(z)$ be polynomials and let $F(z)$ and $G(z)$ be their eigentransforms. Show that

$$\begin{aligned}(\omega - \omega^{-1})G(z) &\\ = \omega(h+z)F(z+1) &- (\omega + \omega^{-1})zF(z) - \omega^{-1}(h-z)F(z-1)\end{aligned}$$

is a necessary and sufficient condition that

$$g(z) = hf(z) + zf'(z).$$

Show that

$$(\omega - \omega^{-1})G(z) = \omega^2(h+z)F(z+1) - 2zF(z) - \omega^{-2}(h-z)F(z-1)$$

is a necessary and sufficient condition that

$$g(z) = 2hzf(z) + z^2 f'(z).$$

Show that

$$(\omega - \omega^{-1})G(z) = (h+z)F(z+1) - 2zF(z) - (h-z)F(z-1)$$

is a necessary and sufficient condition that

$$g(z) = f'(z).$$

Show that $G(z) = (\omega - \omega^{-1})zF(z)$ is a necessary and sufficient condition that

$$g(z) = h(2z - \omega - \omega^{-1})f(z) + (z^2 - \omega z - \omega^{-1}z + 1)f'(z).$$

SONINE SPACES

Some other Hilbert spaces of entire functions which are of known structure occur in the theory of self-reciprocal functions for the Hankel transformation. In 1880 N. Sonine constructed a nontrivial example of a function which belongs to $L^2(0, \infty)$, which vanishes in an interval $(0, a)$, and whose Hankel transform of order ν vanishes in the same interval. A fundamental problem is to determine all such functions.

PROBLEM 337. Let $\nu > 0$, and let λ and σ be numbers such that $\lambda > -\frac{1}{2}$, $\sigma > -\frac{1}{2}$, and $\lambda + \sigma > \nu - 1$. Define

$$f(x) = a^{-\lambda}(x^2 - a^2)^{\lambda/2} J_\lambda(a\sqrt{x^2 - a^2}) x^{\frac{1}{2}-\nu},$$
$$g(x) = a^{-\sigma}(x^2 - a^2)^{\sigma/2} J_\sigma(a\sqrt{x^2 - a^2}) x^{\frac{1}{2}-\nu}$$

for $x > a$, and let $f(x)$ and $g(x)$ vanish for $0 < x < a$. Show that $f(x)$ and $g(x)$ belong to $L^2(0, \infty)$ and are Hankel transforms of order ν.

PROBLEM 338. Let $f(x)$ and $g(x)$ be functions which belong to $L^2(0, \infty)$, which are Hankel transforms of order ν, and which vanish in an interval $(0, a)$. Let $\nu = 2h - 1$ where $h > 0$ and let

$$F(z) = 2^{h-iz}\Gamma(h - iz) \int_0^\infty f(t) t^{-\frac{1}{2}+iz} dt,$$
$$G(z) = 2^{h-iz}\Gamma(h - iz) \int_0^\infty g(t) t^{-\frac{1}{2}+iz} dt$$

for $y > 0$. Show that $F(z)$ and $G(z)$ have entire extensions such that $G(z) = F(-z)$. Show that

$$\int_{-\infty}^{+\infty} |F(t)/\Gamma(h - it)|^2 dt = \pi 4^h \int_0^\infty |f(t)|^2 dt,$$

that $F(z)/\Gamma(h - iz)$ and $F^*(z)/\Gamma(h - iz)$ are of bounded type in the upper half-plane, and that these ratios are of mean type at most $-\log(\frac{1}{2}a^2)$ in the half-plane.

PROBLEM 339. Let $F(z)$ be an entire function such that

$$\int_{-\infty}^{+\infty} |F(t)/\Gamma(h - it)|^2 dt < \infty$$

for some $h > 0$, and such that the ratios $F(z)/\Gamma(h - iz)$ and $F^*(z)/\Gamma(h - iz)$ are of bounded type in the upper half-plane. If these ratios are of mean type at most $-\log(\frac{1}{2}a^2)$ in the half-plane, show that

$$F(z) = 2^{h-iz}\Gamma(h - iz) \int_0^\infty f(t) t^{-\frac{1}{2}+iz} dt,$$
$$F(-z) = 2^{h-iz}\Gamma(h - iz) \int_0^\infty g(t) t^{-\frac{1}{2}+iz} dt$$

for $y > 0$, where $f(x)$ and $g(x)$ belong to $L^2(0, \infty)$, are Hankel transforms of order $\nu = 2h - 1$, and vanish in $(0, a)$.

PROBLEM 340. Let $\nu = 2h - 1$ where $h > 0$, and let $a > 0$. Consider the set of entire functions $F(z)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/\Gamma(h - it)|^2 dt < \infty,$$

such that $F(z)/\Gamma(h - iz)$ and $F^*(z)/\Gamma(h - iz)$ are of bounded type in the upper half-plane, and such that these ratios are of mean type at most $-\log a$ in the half-plane. Show that this set is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element. Show that the space is equal isometrically to a space $\mathcal{H}(E)$ for a function $E(z)$ such that $E^*(z) = E(-z)$. Show that $F(z + i)/(h - iz)$ belongs to the space whenever $F(z)$ belongs to the space and vanishes at $i - ih$. Show that the identity

$$\langle F(t + i)/(h - it), G(t)\rangle = \langle F(t), G(t + i)/(h - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space which vanish at $i - ih$. Show that $E(z)$ is of Pólya class.

The Sonine spaces are Kummer spaces when $h = \frac{1}{2}$ or $h = 1$. In what follows we assume that $h \neq \frac{1}{2}$.

PROBLEM 341. Let $\mathcal{H}(E)$ be a given space of dimension greater than one such that $E^*(z) = E(-z)$. Let h be a given positive number, $h \neq \frac{1}{2}$. Assume that $F(z + i)/(h - iz)$ belongs to the space whenever $F(z)$ belongs to the space and vanishes at $i - ih$, and that the identity

$$\langle F(t + i)/(h - it), G(t)\rangle = \langle F(t), G(t + i)/(h - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space which vanish at $i - ih$. Show that there exist numbers u and v, u real and v imaginary, such that $L(z) = A(z)u + B(z)v$ has value one at $-ih$, such that

$$[F(z + i) - L(z)F(i - ih)]/(h - iz)$$

belongs to the space whenever $F(z)$ belongs to the space, and such that the identity

$$\langle [F(t + i) - L(t)F(i - ih)]/(h - it), G(t)\rangle$$
$$= \langle F(t), [G(t + i) - L(t)G(i - ih)]/(h - it)\rangle$$

holds for all elements $F(z)$ and $G(z)$ of the space. Show that there exist real numbers p, r, and s such that $pr = s^2$ and such that the recurrence relations

$$[A(z+i) - L(z)A(i-ih)]/(h-iz) + [B(z)\bar{A}(i-ih) - A(z)\bar{B}(i-ih)]v/(1-h-iz) = A(z)s - iB(z)r,$$

$$[B(z+i) - L(z)B(i-ih)]/(h-iz) + [B(z)\bar{A}(i-ih) - A(z)\bar{B}(i-ih)]u/(1-h-iz) = iA(z)p + B(z)s$$

hold. Show that

$$1 - L(ih-i)^2 = (2h-1)[A(ih-i)s - iB(ih-i)r]u - i(2h-1)[A(ih-i)p - iB(ih-i)s]v.$$

Show that $s = 2/a$ if $E(z)/\Gamma(h-iz)$ is of bounded type and of mean type $\log(4/a)$ in the upper half-plane.

PROBLEM 342. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 341 for some index h, show that $E(z) = S(z)E_0(z)$ where $\mathcal{H}(E_0)$ is a space which satisfies the hypotheses of the problem for the same index, $E_0(z)$ is of Pólya class, and $S(z)$ is an even entire function which is real for real z and periodic of period i.

PROBLEM 343. Assume that $s \neq 0$ in Problem 341. Show that there exist numbers U and V, $U\bar{V} - V\bar{U} \neq 0$, such that

$$sU + ipV = 2sU \quad \text{and} \quad -irU + sV = 2sV.$$

Show that the function $P(z) = A(z)U + B(z)V$ satisfies the identities

$$[P(z+i) - L(z)P(i-ih)]/(h-iz) + [B(z)\bar{A}(i-ih) - A(z)\bar{B}(i-ih)](vU+uV)/(1-h-iz) = 2sP(z),$$

$$[P^*(z+i) - L(z)P^*(i-ih)]/(h-iz) + [B(z)\bar{A}(i-ih) - A(z)\bar{B}(i-ih)](v\bar{U}+u\bar{V})/(1-h-iz) = 0.$$

Show that $i(V\bar{U} - U\bar{V}) > 0$. Show that

$$\lim_{y\to+\infty} y^{-1}P(iy+i)/P(iy) = 2s.$$

Show that p, r, and s are positive.

PROBLEM 344. Let $\mathcal{H}(E(a))$ be a given space which satisfies the hypotheses of Problem 341 for some index h. Assume that $s(a) > 0$. Let $\mathcal{H}(E(b))$ be a space contained isometrically in $\mathcal{H}(E(a))$ such that $E(b,z)/E(a,z)$ has no real zeros and such that $E^*(b, z) = E(b, -z)$. Show that $\mathcal{H}(E(b))$ satisfies the hypotheses of Problem 341 for the same index h. Show that $s(b) \neq 0$ and that the mean type of $E(a, z)/E(b, z)$ in the upper half-plane is equal to $\log [s(a)/s(b)]$.

PROBLEM 345. Show that the domain of multiplication by z is dense in any space which satisfies the hypotheses of Problem 341 with $s > 0$.

PROBLEM 346. Let $\mathcal{H}(E)$ be a given space which satisfies the hypotheses of Problem 341 for some index h. Assume that $s > 0$. For each number a, $2/s < a < \infty$, show that there exists a space $\mathcal{H}(E(a))$ contained isometrically in $\mathcal{H}(E)$ such that $E(a, z)/E(z)$ has no real zeros, such that $E^*(a, z) = E(a, -z)$, and such that the recurrence relations of Problem 341 hold with $s(a) = 2/a$. If $E(a, z)$ is chosen so that $E(a, 0) = E(0)$, show that $E(t, z)$ is a continuous function of t for every z and that the integral equation

$$(A(b, z), B(b, z))I - (A(a, z), B(a, z))I = z \int_a^b (A(t, z), B(t, z))dm(t)$$

holds for some continuous, nonincreasing, matrix valued function

$$m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$$

such that $\beta(t) = 0$. Show that the intersection of the spaces $\mathcal{H}(E(a))$ contains no nonzero element. Show that $\alpha(t)$ and $\gamma(t)$ are differentiable functions of t such that

$$-t\alpha'(t) = p(t)/s(t) \quad \text{and} \quad -t\gamma'(t) = r(t)/s(t).$$

Show that $u(t)$ and $v(t)$ are differentiable functions of t such that

$$u'(t) = ihv(t)\alpha'(t) \quad \text{and} \quad v'(t) = -ihu(t)\gamma'(t).$$

Show that $t\alpha'(t)$ and $t\gamma'(t)$ are differentiable functions of t such that

$$[t\alpha'(t)]' = -L(t, ih - i)t\alpha'(t),$$
$$[t\gamma'(t)]' = L(t, ih - i)t\gamma'(t).$$

PROBLEM 347. In Problem 346 let $A_1(a, z)$ and $B_1(a, z)$ be the unique entire functions, which are real for real z, such that

$$(1 + iz/h)[A(a, z)u(a) + B(a, z)v(a)] = A_1(a, z)u(a) + B_1(a, z)v(a).$$

Show that a space $\mathcal{H}(E_1(a))$ exists and that the transformation $F(z) \to (1 + iz/h)F(z)$ takes $\mathcal{H}(E(a))$ isometrically onto the subspace of those elements of $\mathcal{H}(E(a))$ which vanish at ih. Show that $\mathcal{H}(E_1(a))$ satisfies the hypotheses of Problem 341 with h replaced by $h + 1$. Show that

$$A_1(a, -ih)u(a) = 1, \qquad B_1(a, -ih)v(a) = 1,$$
$$K_1(a, -ih, z) = L(a, z)K_1(a, -ih, -ih).$$

Show that $E_1(t, z)$ is a continuous function of t for every z and that the integral equation

$$(A_1(b, z), B_1(b, z))I - (A_1(a, z), B_1(a, z))I = z\int_a^b (A_1(t, z), B_1(t, z))\,dm_1(t)$$

holds with

$$m_1(t) = \begin{pmatrix} \alpha_1(t) & \beta_1(t) \\ \beta_1(t) & \gamma_1(t) \end{pmatrix}$$

a matrix valued function whose entries are differentiable functions of t such that

$$\alpha_1'(t) = [iv(t)/u(t)]^{-2}\gamma'(t),$$
$$\beta_1'(t) = 0,$$
$$\gamma_1'(t) = [iv(t)/u(t)]^2\alpha'(t).$$

LAGUERRE POLYNOMIAL SPACES

There also exist more elementary spaces associated with Laguerre polynomials.

PROBLEM 348. Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has a nonzero value at the origin. Let ν be a given number, $\nu = 2h - 1$ where $h > 0$. Assume that the function $zF'(z)$ belongs to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle tF'(t), G(t)\rangle + \langle F(t), tG'(t)\rangle = \langle (t - 2h)F(t), G(t)\rangle$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Show that there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \quad \text{and} \quad S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the differential equations

$$hS_+(z) + (z - 2h)S_+'(z) - zS_+''(z) = \lambda_+ S_+(z),$$
$$hS_-(z) + (z - 2h)S_-'(z) - zS_-''(z) = \lambda_- S_-(z),$$
$$-2hS_+'(z) - zS_+''(z) = (\lambda_+ - h)S_-(z),$$
$$-(z - 2h)S_-(z) + 2(z - h)S_-'(z) - zS_-''(z) = (\lambda_- + h)S_+(z)$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

PROBLEM 349. If h is a given positive number, show that the Laguerre polynomials $\Phi_n(z)$, defined by

$$\Phi_n(z) = F(-n; 2h; z)$$

for $n = 0, 1, 2, \cdots$, are real for real z and satisfy the identities

$$h\Phi_n(z) + (z - 2h)\Phi_n'(z) - z\Phi_n''(z) = (h + n)\Phi_n(z),$$
$$-2h\Phi_n'(z) - z\Phi_n''(z) = n\Phi_{n-1}(z),$$
$$-(z - 2h)\Phi_n(z) + 2(z - h)\Phi_n'(z) - z\Phi_n''(z) = (2h + n)\Phi_{n+1}(z),$$
$$z\Phi_n(z) = -(2h + n)\Phi_{n+1}(z) + 2(h + n)\Phi_n(z) - n\Phi_{n-1}(z).$$

Show that there exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, satisfying the hypotheses of Problem 348, such that $\mathcal{H}(E_n)$ is contained isometrically in $\mathcal{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathcal{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathcal{H}(E_n)$ in $\mathcal{H}(E_{n+1})$ for $n > 0$. Show that the spaces can be chosen so that

$$\|\Phi_n(t)\|^2 = \Gamma(1 + n)\Gamma(2h)/\Gamma(2h + n)$$

for every n. Show that the identity

$$\Gamma(2h)\langle F(t), G(t)\rangle = \int_0^\infty F(t)\bar{G}(t)e^{-t}t^{2h-1}dt$$

holds for all polynomials $F(z)$ and $G(z)$.

PROBLEM 350. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 348, show that there exists an index r in Problem 349 and a positive constant S such that the transformation $F(z) \to SF(z)$ is an isometry of $\mathcal{H}(E)$ onto $\mathcal{H}(E_r)$.

PROBLEM 351. In Problem 349 if $f(z) = \Sigma\, a_n z^n$ is a polynomial of degree less than r, show that its eigentransform $F(z)$, defined by $F(z) = \Sigma\, a_n\Phi_n(z)$, belongs to $\mathcal{H}(E_r)$ and that

$$\int_{-\infty}^{+\infty} |F(t)/E_r(t)|^2 dt = \|f(z)\|^2$$

where the norm of $f(z)$ is taken in $\mathcal{C}(2h, 1; 1; z)$. Show that every element of $\mathcal{H}(E_r)$ is of this form. Show that the identity

$$\Gamma(2h)f(w) = (1 - w)^{-2h}\int_0^\infty F(t)\exp\left[-t/(1 - w)\right] t^{2h-1}dt$$

holds for all complex numbers w whenever $f(z)$ is a polynomial and $F(z)$ is its eigentransform. Let $f(z)$ and $g(z)$ be polynomials, and let $F(z)$ and $G(z)$

be their eigentransforms. Show that the condition

$$G(z) = hF(z) + (z - 2h)F'(z) - zF''(z)$$

is necessary and sufficient that

$$g(z) = hf(z) + zf'(z).$$

Show that the condition

$$G(z) = -2hF'(z) - zF''(z)$$

is necessary and sufficient that

$$g(z) = f'(z).$$

Show that the condition

$$G(z) = -(z - 2h)F(z) + 2(z - h)F'(z) - zF''(z)$$

is necessary and sufficient that

$$g(z) = 2hzf(z) + z^2f'(z).$$

Show that the condition

$$G(z) = zF(z)$$

is necessary and sufficient that

$$g(z) = 2h(1 - z)f(z) - (1 - z)^2f'(z).$$

Show that $F(0) = f(1)$ for every polynomial $f(z)$ and that $f(z)$ has a zero of order r at the point one if, and only if, $F(z)$ has a zero of order r at the origin.

STIELTJES SPACES

Some remarkable examples of Hilbert spaces of entire functions originate in work of Stieltjes on continued fractions.

PROBLEM 352. Let κ and q be given numbers, $\kappa < 1$ and $0 < q < 1$, and let $\mathcal{H}(E)$ be a given space such that $E(z)$ has a nonzero value at the origin. Assume that the functions $F(qz)$ and $F(q^{-1}z)$ belong to the space whenever $F(z)$ belongs to the space and that the identity

$$\langle F(t), G(t) \rangle = \langle (\kappa + t)F(qt), G(qt) \rangle$$

holds for every element $G(z)$ of the space when $F(z)$ belongs to the domain of multiplication by z in the space. Show that there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \quad \text{and} \quad S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the identities

$$\lambda_+ S_+(z) = \frac{S_+(z) - S_+(qz)}{1-q} + \frac{1-\kappa}{1-q^2} S_+(qz)$$
$$+ \kappa \frac{\kappa+q}{1+q} \frac{S_+(z) - S_+(qz)}{(1-q)z} - q \frac{\kappa+q}{1+q} \frac{S_+(q^{-1}z) - S_+(z)}{(1-q)z},$$

$$\lambda_- S_-(z) = \frac{S_-(z) - S_-(qz)}{1-q} + \frac{1-\kappa}{1-q^2} S_-(qz)$$
$$+ \kappa \frac{\kappa+q}{1+q} \frac{S_-(z) - S_-(qz)}{(1-q)z} - q \frac{\kappa+q}{1+q} \frac{S_-(q^{-1}z) - S_-(z)}{(1-q)z},$$

$$\left(\frac{\kappa-q^2}{1-q}\frac{\lambda_+}{\kappa+q} + \frac{1-\kappa}{1-q}\frac{q\lambda_-}{\kappa+q}\right) S_-(z)$$
$$= q\kappa \frac{S_+(q^{-1}z) - S_+(z)}{(1-q)z} - q^2 \frac{S_+(q^{-2}z) - S_+(q^{-1}z)}{(1-q)z},$$

$$\left(\frac{\kappa-q^2}{1-q}\frac{\lambda_-}{\kappa+q} + \frac{1-\kappa}{1-q}\frac{q\lambda_+}{\kappa+q}\right) S_+(z)$$
$$= -\frac{z}{1-q} S_-(q^2 z) + \frac{\kappa+q}{q(1-q)} S_-(qz) - \frac{\kappa}{q}\frac{1+q}{1-q} S_-(q^2 z)$$
$$- \kappa \frac{S_-(z) - S_-(qz)}{(1-q)z} + \frac{\kappa^2}{q} \frac{S_-(qz) - S_-(q^2 z)}{(1-q)z}$$

hold for some real numbers λ_+ and λ_- such that $\lambda_+ - q\lambda_- = 1$.

The construction of such spaces is made from Heine's generalization of the hypergeometric series. The series is similar to Gauss's series except that it satisfies a difference equation rather than a differential equation. A unit of discreteness, or quantum, q is used in forming these differences. In what follows we take the quantum to be a fixed number, $0 < q < 1$. The notation

$$\varphi(a, b; c; z) = 1 + \frac{(1-a)(1-b)}{(1-q)(1-c)} z$$
$$+ \frac{(1-a)(1-qa)(1-b)(1-qb)}{(1-q)(1-q^2)(1-c)(1-qc)} z^2 + \cdots$$

is used for Heine's series, $c \neq q^{-n}$ for every $n = 0, 1, 2, \cdots$. The corresponding confluent series are

$$\varphi(a; c; z) = \lim_{b \to \infty} \varphi(a, b; c; -z/b)$$
$$= 1 + \frac{(1-a)}{(1-q)(1-c)} z + \frac{(1-a)(1-qa)q}{(1-q)(1-q^2)(1-c)(1-qc)} z^2 + \cdots,$$

where $q^{n(n-1)/2}$ appears as a factor in the coefficient of z^n, and

$$\varphi(c; z) = \lim_{a\to\infty} \varphi(a; c; -z/a)$$

$$= 1 + \frac{1}{(1-q)(1-c)} z + \frac{q^2}{(1-q)(1-q^2)(1-c)(1-qc)} z^2 + \cdots,$$

where $q^{n(n-1)}$ appears as a factor in the coefficient of z^n.

PROBLEM 353. If κ is a given number, $\kappa < 1$, show that the polynomials $\Phi_n(z)$, defined by

$$\Phi_n(z) = \varphi(q^{-n}; \kappa; q^n z)$$

for $n = 0, 1, 2, \cdots$, are real for real z and satisfy the identities

$$\left(\frac{1-q^n}{1-q} + \frac{1-\kappa}{1-q^2} q^n\right)\Phi_n(z) = \frac{\Phi_n(z) - \Phi_n(qz)}{1-q} + \frac{1-\kappa}{1-q^2}\Phi_n(qz)$$

$$+ \kappa \frac{\kappa+q}{1+q} \frac{\Phi_n(z) - \Phi_n(qz)}{(1-q)z} - q\frac{\kappa+q}{1+q} \frac{\Phi_n(q^{-1}z) - \Phi_n(z)}{(1-q)z},$$

$$\frac{1-q^n}{1-q}\Phi_{n-1}(z) = q\kappa \frac{\Phi_n(q^{-1}z) - \Phi_n(z)}{(1-q)z} - q^2 \frac{\Phi_n(q^{-2}z) - \Phi_n(q^{-1}z)}{(1-q)z},$$

$$\frac{1-q^n\kappa}{1-q}\Phi_{n+1}(z) = -\frac{z}{1-q}\Phi_n(q^2z) + \frac{\kappa+q}{q(1-q)}\Phi_n(qz) - \frac{\kappa}{q}\frac{1+q}{1-q}\Phi_n(q^2z)$$

$$- \kappa \frac{\Phi_n(z) - \Phi_n(qz)}{(1-q)z} + \frac{\kappa^2}{q} \frac{\Phi_n(qz) - \Phi_n(q^2z)}{(1-q)z},$$

$$z\Phi_n(z) = q^{-2n}(1 - q^n\kappa)[\Phi_n(z) - \Phi_{n+1}(z)]$$

$$+ q^{1-2n}(1-q^n)[\Phi_n(z) - \Phi_{n-1}(z)].$$

Show that there exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, satisfying the hypotheses of Problem 352, such that $\mathcal{H}(E_n)$ is contained isometrically in $\mathcal{H}(E_{n+1})$ for every n, such that $\Phi_0(z)$ spans $\mathcal{H}(E_1)$, and such that $\Phi_n(z)$ spans the orthogonal complement of $\mathcal{H}(E_n)$ in $\mathcal{H}(E_{n+1})$ when $n > 0$. Show that the spaces can be chosen so that $\|\Phi_0(t)\| = 1$ and so that

$$q^n \|\Phi_n(t)\|^2 = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-\kappa)(1-q\kappa)\cdots(1-q^{n-1}\kappa)}$$

for $n > 0$. Show that there exists a space $\mathcal{H}(E_\infty)$ containing the union of the spaces $\mathcal{H}(E_n)$ isometrically as a dense set. If $0 < \kappa < 1$, show that the identity

$$\frac{\varphi(0;0;-q)\varphi(0;0;qs^{-1})\varphi(0;0;s)}{\varphi(0;0;-\kappa)\varphi(0;0;q\kappa s^{-1})\varphi(0;0;s\kappa^{-1})}\langle F(t), G(t)\rangle = \sum_{n=-\infty}^{+\infty} F(sq^n)\bar{G}(sq^n)\kappa^n/\varphi(0;0;q^n s/\kappa)$$

holds for all elements $F(z)$ and $G(z)$ of $\mathcal{H}(E_\infty)$ when $s > 0$.

PROBLEM 354. If $\mathcal{H}(E)$ is a given space which satisfies the hypotheses of Problem 352, show that there exists a positive constant S and a finite index r in Problem 353 such that the transformation $F(z) \to SF(z)$ is an isometry of $\mathcal{H}(E_r)$ onto $\mathcal{H}(E)$.

These spaces are related to a generalized space of square summable power series. Let a, b, and c be numbers such that the coefficients of $\varphi(a;b;c;z)$ are all positive. By $\Omega(a,b;c;qz)$ we mean the Hilbert space of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = |a_0|^2 + \frac{(1-q)(1-c)}{(1-a)(1-b)}\frac{|a_1|^2}{q} + \frac{(1-q)(1-q^2)(1-c)(1-qc)}{(1-a)(1-qa)(1-b)(1-qb)}\frac{|a_2|^2}{q^2} + \cdots < \infty.$$

The elements of the space are convergent power series in the disk $q|w|^2 < 1$. The series $\varphi(a,b;c;q\bar{w}z)$ belongs to the space when $q|w|^2 < 1$, and the identity

$$f(w) = \langle f(z), \varphi(a,b;c;q\bar{w}z)\rangle$$

holds for every element $f(z)$ of the space.

PROBLEM 355. In Problem 353 if $f(z) = \sum a_n z^n$ belongs to $\Omega(\kappa,0;0;qz)$, show that its eigentransform $F(z) = \sum a_n \Phi_n(z)$ belongs to $\mathcal{H}(E_\infty)$ and that

$$\int_{-\infty}^{+\infty} |F(t)/E_\infty(t)|^2 dt = \|f(z)\|^2$$

where the norm of $f(z)$ is taken in $\Omega(\kappa,0;0;qz)$. Show that every element of $\mathcal{H}(E_\infty)$ is of this form. Show that the function $\varphi(q\kappa\bar{w};-q\bar{w}z)$ belongs to $\mathcal{H}(E_\infty)$ when $q|w|^2 < 1$ and that the identity

$$f(w) = \varphi(\kappa,0;0;qw)\langle F(t), \varphi(q\kappa\bar{w};-q\bar{w}t)\rangle$$

holds for the eigentransform $F(z)$ of any element $f(z)$ of $\mathfrak{Q}(\kappa, 0; 0; qz)$. Let $f(z)$ and $g(z)$ be elements of $\mathfrak{Q}(\kappa, 0; 0; qz)$, and let $F(z)$ and $G(z)$ be their eigentransforms. Show that the condition

$$G(z) = \frac{F(z) - F(qz)}{1 - q} + \frac{1 - \kappa}{1 - q^2} F(qz)$$
$$+ \kappa \frac{\kappa + q}{1 + q} \frac{F(z) - F(qz)}{(1 - q)z} - q \frac{\kappa + q}{1 + q} \frac{F(q^{-1}z) - F(z)}{(1 - q)z}$$

is necessary and sufficient that

$$g(z) = \frac{f(z) - f(qz)}{1 - q} + \frac{1 - \kappa}{1 - q^2} f(qz).$$

Show that the condition

$$G(z) = q\kappa \frac{F(q^{-1}z) - F(z)}{(1 - q)z} - q^2 \frac{F(q^{-2}z) - F(q^{-1}z)}{(1 - q)z}$$

is necessary and sufficient that

$$g(z) = \frac{f(z) - f(qz)}{(1 - q)z}.$$

Show that the condition

$$G(z) = -\frac{z}{1 - q} F(q^2z) + \frac{\kappa + q}{q(1 - q)} F(qz) - \frac{\kappa}{q} \frac{1 + q}{1 - q} F(q^2z)$$
$$- \kappa \frac{F(z) - F(qz)}{(1 - q)z} + \frac{\kappa^2}{q} \frac{F(qz) - F(q^2z)}{(1 - q)z}$$

is necessary and sufficient that

$$g(z) = z \frac{f(z) - \kappa f(qz)}{1 - q}.$$

Show that the condition $G(z) = zF(z)$ is necessary and sufficient that

$$g(z) = -z[f(q^{-2}z) - \kappa f(q^{-1}z)] + (1 + q)f(q^{-2}z) - (\kappa + q)f(q^{-1}z)$$
$$- q[f(q^{-2}z) - f(q^{-1}z)]/z.$$

Show that $F(0) = f(1)$ for every element $F(z)$ of $\mathcal{H}(E_\infty)$. Show that $f(z)$ vanishes at the points $1, q, \cdots, q^r$ if, and only if, $F(z)$ has a zero of order r at the origin.

Notes on the Theorems

CHAPTER 1

THEOREM 1. Applications of the Phragmén-Lindelöf principle are given by Boas [2] and Levin [56].

THEOREMS 2, 3, and 4. These are classical results of analytic function theory. See SSPS [29] for analogous results in the unit disk.

THEOREMS 5 and 6. The theory of $\mathfrak{L}(\varphi)$ spaces [17] is used in the perturbation theory of self-adjoint transformations, de Branges and Shulman [32].

THEOREMS 7–11. The background of the Pólya class theory is presented by Boas [2] and Levin [56]. They also treat asymptotic behavior outside of the bounded type theory. See Whittaker and Watson [75] for a solution of Problem 19.

THEOREMS 12 and 13. Cauchy's formula in a half-plane and the factorization of positive functions are classical applications of the bounded type theory.

THEOREM 14. These conditions for Pólya class appear in previous work [18].

THEOREM 15. See Boas [2] and Levin [56] for relations between growth and zeros of entire functions.

CHAPTER 2

THEOREMS 16, 17, and 18. The L^2-theory of finite Fourier transforms is due to Paley and Wiener [59].

THEOREMS 19, 20, and 21. The original construction [10] of the space $\mathcal{H}(E)$ did not use the bounded type theory.

THEOREM 22. The formula for mean squares of entire functions was originally stated [9] without the $\mathcal{H}(E)$ theory.

THEOREM 23. The axioms (H1), (H2), and (H3) state that multiplication by z is a closed, symmetric transformation of deficiency index (1, 1) which is real with respect to a conjugation [10]. Symmetric transformations having arbitrary deficiency indices can be studied in the same way using the vector theory of Hilbert spaces of entire functions [26], [30].

THEOREM 24. The uniqueness of spaces with given phase functions was originally obtained [18] without the bounded type theory. See Levin [56] for the theory of pairs of entire functions which are real for real z and which have alternating real zeros.

THEOREM 25. The theory of functions $S(z)$ associated with $\mathcal{H}(E)$ was originally developed [11] in the case $S(z) = 1$. The general case is given in Trutt's thesis [72].

THEOREM 26. The original theorem [12] concludes that a function belongs to $\mathcal{H}(E)$ from an estimate on the imaginary axis.

THEOREM 27. The characterization of functions associated with $\mathcal{H}(E)$ was originally given [11] in the case $S(z) = 1$. The general case is taken from Trutt's thesis [72].

THEOREM 28. The construction of the space $\mathcal{H}_S(M)$ was originally made in the case $S(z) = 1$. See [26] for a vector generalization of the theory.

THEOREM 29. Several variants of the density theorem for the domain of multiplication by z are known [11], [12].

THEOREM 30. The use of $\mathcal{L}(\varphi)$ spaces to study measures associated with $\mathcal{H}(E)$ is a new device to obtain an old result originating in Stieltjes' theory of the moment problem [69].

THEOREM 31. See [26] for a vector generalization of the theory of $\mathcal{L}(\varphi)$ spaces associated with $\mathcal{H}(E)$.

THEOREM 32. The original results [11] on measures associated with $\mathcal{H}(E)$ are stated for special choices of $S(z)$. The general case is taken from Trutt's thesis [72].

THEOREMS 33 and 34. The original proofs [11] of the theorems on isometric inclusions did not use the space $\mathcal{H}_S(M)$.

THEOREM 35. The ordering theorem was originally stated [14] in special cases in which the bounded type hypothesis is automatically satisfied. The ordering theorem generalizes a theorem of M. G. Kreĭn [53]. His result is equivalent to the special case of the theorem in which $E^*(z) = E(-z)$. The Carleman method [35] is not needed to prove the theorem in this case.

THEOREM 36. The original proof of existence of subspaces [14] proceeded through the Pólya class theory.

THEOREMS 37 and 38. The integral representation for $M(z)$ is a special case of the product representation of characteristic operator functions of nonself-adjoint transformations T such that $T - T^*$ is of trace class. See [20], [23], and [24] for an expository account of the theory and for generalizations. A proof independent of the trace class theory is also known [26].

THEOREM 39. The original proof [14] of the formula for mean type proceeded without Lemma 9.

THEOREM 40. The original method [13] of obtaining the integral equation for $E(z)$ went through the Pólya class theory. A vector generalization of the integral representation is known [26] when sufficiently many invariant subspaces exist and a complete continuity condition is satisfied.

THEOREM 41. The original proof [12] of existence of solutions of the integral equation for $E(z)$ did not use the results of Problem 170. Generalizations of the theorem can be obtained from the results of Problems 305–318 on Laguerre classes.

THEOREM 42. See [12] for the original proof of existence and uniqueness of measures associated with the integral equation. The theorem originates in Stieltjes' theory of continued fractions [69].

THEOREM 43. Completeness of $L^2(m)$ is stated without proof in previous work [13]. The present definition of a regular point is different since an interval (a, b) can now be regular when $m(t)$ is constant in (a, b).

THEOREMS 44 and 45. See [26] for a different conception of the expansion theorem for Hilbert spaces of entire functions.

THEOREM 46. The comparison theorem for Hilbert spaces of entire functions [19] has a generalization to the vector theory.

CHAPTER 3

THEOREM 47. The theory of symmetry is [16].

THEOREMS 48 and 49. The original treatment [15] of periodic spaces depends on the theorem that a periodic entire function of exponential type is a trigonometric polynomial. The proof that periodicity is hereditary in subspaces follows an argument of Eidswick's thesis [36]. The results of Problem 221 were supplied by R. Bolstein.

THEOREM 50. An apparently more general definition of homogeneous space was originally given [15]. A space $\mathcal{H}(E)$ was said to be homogeneous if for every number a, $0 < a < 1$, there exists a number $k(a) > 0$ such that $k(a)F(az)$ belongs to the space whenever $F(z)$ belongs to the space and such that $k(a)F(az)$ always has the same norm as $F(z)$. It is shown there that $k(a) = a^{1+\nu}$ for some real constant ν.

THEOREM 51. The theory of analytic weight functions is easily generalized to the vector theory of Hilbert spaces of entire functions [26] using the trace class concept.

THEOREMS 52, 53, and 54. A definitive treatment of the hypergeometric function theory is given by Gauss [42]. The hypergeometric series is, however, due to Euler [40] who discovered all the essential features of the theory in a lifetime of research along lines which seem to have been suggested by his teacher, Johann Bernoulli. The present approach to hypergeometric functions through the theory of entire functions is new.

THEOREMS 55, 56, and 57. The confluent form of the hypergeometric function is due to Kummer [55]. I use the notation $F(a; c; z)$ for Kummer's series $\lim F(a, b; c; z/b)$ as $b \to \infty$, and the notation $F(c; z)$ for the series $\lim F(a; c; z/a)$ as $a \to \infty$. The confluent hypergeometric expansion is used in M. Rosenblum's theory of the Hilbert matrix [64]. I am indebted to him for discussions of the Hilbert matrix theory (Institute for Advanced Study, 1959) which stimulated the present work on Gauss and Kummer spaces. The Hardy space $\mathfrak{H}_\nu$ is taken from the theory of self-reciprocal functions for the Hankel transformation, Hardy and Titchmarsh [47]. See Whittaker and Watson [75] for the theory of Whittaker functions.

THEOREMS 58, 59, and 60. The use of the hypergeometric function in connection with Jacobi polynomials is due to Jacobi [50].

THEOREMS 61, 62, and 63. The theory of local operators on Fourier transforms is taken from my thesis [5] and the companion paper [6]. The construction of local operators was originally made outside of the bounded type theory. The theorem on nonvanishing Fourier transforms is a variant of a theorem of Levinson [57], who obtains a weaker result by a different method. The results of Problems 301, 302, and 304 are also related to Levinson's work. See [18] for hints to solutions.

THEOREM 64. I first learned of the Beurling-Malliavin theorem [1] at the 1961 Summer Institute on Functional Analysis at Stanford, when I was working on a conjecture now stated as Problem 295. See [18] for a proof of the conjecture from their theorem.

THEOREM 65. This variant of the Beurling-Malliavin theorem was originally stated [18] on the hypothesis that

$$\int_{-\infty}^{+\infty} (1 + t^2)^{-1} |\varphi(b, t) - \tau(b)t|^2 dt < \infty,$$

in which case it follows from Problem 295. The present statement was later announced without proof [21].

THEOREMS 66 and 67. The extreme point method was originally applied to the Bernstein approximation problem [8]. The density property of extremal measures, which is due to Naĭmark [58], gives a short proof of the Stone-Weierstrass theorem [7]. A fundamental problem is to find the extreme points of the convex set of measures which determine the norm of any given space $\mathcal{H}(E)$. See Trutt [73] for the current status of the problem.

THEOREM 68. This construction of spaces $\mathcal{H}(E)$ contained isometrically in $L^2(\mu)$ is a new application of the methods of [18].

LAGUERRE CLASSES. See Boas [2] and Levin [56] for Laguerre's contribution to the theory of entire functions with real zeros. The theory of Laguerre classes can be used to generalize the results of Theorem 41.

LAGUERRE SPACES. Previously published work on Laguerre spaces [25] is restricted to spaces in which multiplication by z is densely defined. These new results are taken from Klopfenstein's thesis [51].

MEIXNER AND POLLACZEK SPACES. Solutions of these problems and literature references are given by de Branges and Trutt [33].

SONINE SPACES. The results of Problem 337 are due to Sonine [68]. The Sonine spaces [22] have been extensively studied by J. and V. Rovnyak [65], and by V. Rovnyak [66]. See Bolstein and de Branges [3], [4] for related hypergeometric spaces.

LAGUERRE POLYNOMIAL SPACES. Solutions of these problems can be given using the methods of de Branges and Trutt [33], [34]. The theory of Laguerre polynomials, as we know it, owes more to Sonine [68] than it does to Laguerre.

STIELTJES SPACES. Stieltjes [69] gives a continued fraction expansion which is equivalent to knowledge of these orthogonal polynomials when $\kappa = 0$. See Szegö [70] for a discussion of the polynomials in this case. The general case is due to Hahn [45]. Solutions of these problems can be given by the same methods as are used for the Laguerre, Meixner, and Pollaczek spaces.

References

1. A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Math. 107 (1962), 291–302.

2. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.

3. R. Bolstein and L. de Branges, *Gauss spaces of entire functions*, unpublished manuscript, 1967.

4. ———, *Jacobi spaces of entire functions*, unpublished manuscript, 1967.

5. L. de Branges, *Local operators on Fourier transforms*, Duke Math. J. 25 (1958), 143–153.

6. ———, *The a-local operator problem*, Canadian J. Math. 11 (1959), 583–592.

7. ———, *The Stone-Weierstrass theorem*, Proc. Amer. Math. Soc. 10 (1959), 822–824.

8. ———, *The Bernstein problem*, Proc. Amer. Math. Soc. 10 (1959), 825–832.

9. ———, *Some mean squares of entire functions*, Proc. Amer. Math. Soc. 10 (1959), 833–839.

10. ———, *Some Hilbert spaces of entire functions*, Proc. Amer. Math. Soc. 10 (1959), 840–846.

11. ———, *Some Hilbert spaces of entire functions*, Trans. Amer. Math. Soc. 96 (1960), 259–295.

12. ———, *Some Hilbert spaces of entire functions II*, Trans. Amer. Math. Soc. 99 (1961), 118–152.

13. ———, *Some Hilbert spaces of entire functions III*, Trans. Amer. Math. Soc. 100 (1961), 73–115.

14. ——, *Some Hilbert spaces of entire functions IV*, Trans. Amer. Math. Soc. 105 (1962), 43–83.

15. ——, *Homogeneous and periodic spaces of entire functions*, Duke Math. J. 29 (1962), 203–224.

16. ——, *Symmetry in spaces of entire functions*, Duke Math. J. 29 (1962), 383–392.

17. ——, *Perturbations of self-adjoint transformations*, Amer. J. Math. 84 (1962), 543–560.

18. ——, *Some applications of spaces of entire functions*, Canadian J. Math. 15 (1963), 563–583.

19. ——, *A comparison theorem for spaces of entire functions*, Proc. Amer. Math. Soc. 14 (1963), 490–496.

20. ——, *Some Hilbert spaces of analytic functions I*, Trans. Amer. Math. Soc. 106 (1963), 445–468.

21. ——, *New and old problems for entire functions*, Bull. Amer. Math. Soc. 70 (1964), 214–223.

22. ——, *Self-reciprocal functions*, J. Math. Anal. Appl. 9 (1964), 433–457.

23. ——, *Some Hilbert spaces of analytic functions II*, J. Math. Anal. Appl. 11 (1965), 44–72.

24. ——, *Some Hilbert spaces of analytic functions III*, J. Math. Anal. Appl. 12 (1965), 149–186.

25. ——, *Some examples of spaces of entire functions*, J. Reine Angew. Math. 222 (1966), 20–54.

26. ——, *The expansion theorem for Hilbert spaces of entire functions*, Entire Functions and Related Parts of Analysis, American Mathematical Society, Providence, 1968.

27. ——, *Kummer spaces of entire functions*, unpublished manuscript, 1967.

28. L. de Branges and K. F. Klopfenstein, *Laguerre spaces of entire functions*, unpublished manuscript, 1967.

29. L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart, and Winston, New York, 1966.

30. ——, *Canonical models in quantum scattering theory*, Perturbation Theory and its Applications in Quantum Mechanics, Wiley, New York, 1966, pp. 295–392.

31. L. de Branges and V. Rovnyak, *Self-reciprocal functions*, unpublished manuscript, 1967.

32. L. de Branges and L. Shulman, *Perturbations of unitary transformations*, J. Math. Anal. Appl., to appear in 1968.

33. L. de Branges and D. Trutt, *Meixner and Pollaczek spaces of entire functions*, J. Math. Anal. Appl., to appear in 1968.

34. ——, *Charlier spaces of entire functions*, Proc. Amer. Math. Soc., to appear in 1968.

35. T. Carleman, *Sur une inégalité différentielle dans la théorie des fonctions analytiques*, C.R. Acad. Sci. Paris 196 (1933), 995–997.

36. J. A. Eidswick, *A hereditary class of Hilbert spaces of entire functions*, Dissertation, Purdue University, 1964.

37. L. Euler, *Variae observationes circa series infinitas*, Opera Omnia (I) 14, Teubner, Berlin, 1925, pp. 216–244.

38. ——, *Introductio in analysin infinitorum, Caput XVI, De partitione numerorum*, Opera Omnia (I) 8, Teubner, Berlin, 1922, pp. 313–338.

39. ——, *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques*, Opera Omnia (I) 15, Teubner, Berlin, 1927, pp. 70–80.

40. ——, *Specimen transformationis singularis serierum*, Opera Omnia (I) 16, sectio altera, Teubner, Berlin, pp. 41–55.

41. C. F. Gauss, *Summatio quarumdam serierum singularium*, Werke, volume 2, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1863, pp. 9–45.

42. ——, *Disquisitiones generales circa seriem infinitam . . .* , Werke, volume 3, Königlichen Gesellschaft der Wissenschaften, Göttingen, 1876, pp. 125–162 and 207–229.

43. ——, *Zur Theorie der transscendenten Functionen gehörig*, Werke, volume 3, pp. 433–480.

44. I. C. Gohberg and M. G. Kreĭn, *Theory of Volterra operators in Hilbert space and its applications*, Izdatelstvo Nauka, Moscow, 1967. (Russian)

45. W. Hahn, *Über Orthogonalpolynome, die q-Differenzengleichungen genügen*, Math. Nach. 2 (1949), 4–34.

46. ——, *Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören*, Math. Nach. 2 (1949), 263–278.

47. G. H. Hardy and E. C. Titchmarsh, *Self-reciprocal functions*, Quart. J. Math. (Oxford) 1 (1930), 196–231.

48. E. Heine, *Handbuch der Kugelfunktionen*, G. Reimer, Berlin, 1878.

49. C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticarum*, Gesammelte Werke, volume 1, G. Reimer, Berlin, 1881, pp. 49–239 and 497–538.

50. ———, *Untersuchungen über die Differentialgleichung der hypergeometrischen Reihe*, J. Reine Angew. Math. 56 (1859), 149–165.

51. K. F. Klopfenstein, *Laguerre spaces of entire functions*, Duke Math. J., to appear in 1968.

52. H. L. Krall and O. Frink, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc. 65 (1949), 100–115.

53. M. G. Kreĭn, *On some cases of effective determination of the density of an inhomogeneous string from its spectral function*, Dokl. Akad. Nauk SSSR 93 (1953), 617–620. (Russian)

54. E. E. Kummer, *Über die hypergeometrische Reihe . . .* , J. Reine Angew. Math. 15 (1836), 39–83.

55. ———, *De integralibus quibusdam definitis et seriebus infinitis*, J. Reine Angew. Math. 17 (1837), 228–242.

56. B. Ya. Levin, *Distribution of zeros of entire functions*, American Mathematical Society, Providence, 1964.

57. N. Levinson, *Gap and density theorems*, American Mathematical Society, Providence, 1940.

58. M. A. Naĭmark, *Extreme spectral functions of a symmetric operator*, Izv. Akad. Nauk SSSR 11 (1947), 327–344. (Russian)

59. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, American Mathematical Society, Providence, 1934.

60. O. Perron, *Die Lehre von den Kettenbrüchen*, third edition, Teubner, Stuttgart, 1929.

61. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebene Grösse*, Gesammelte Mathematische Werke, Teubner, Leipzig, 1876, pp. 145–153.

62. L. J. Rogers, *On a three-fold symmetry in the elements of Heine's series*, Proc. London Math. Soc. 24 (1893), 171–179.

63. ———, *On the expansion of some infinite products*, Proc. London Math. Soc. 24 (1893), 337–352; 25 (1894), 318–343; 26 (1895), 15–32.

64. M. Rosenblum, *On the Hilbert matrix*, Proc. Amer. Math. Soc. 9 (1958), 137–140 and 581–585.

65. J. Rovnyak and V. Rovnyak, *Self-reciprocal functions for the Hankel transformation of integer order*, Duke Math. J., to appear in 1968.

66. V. Rovnyak, *Self-reciprocal functions*, Duke Math. J. 33 (1966), 363–378.

67. I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, Akademie der Wissenschaften, Berlin, Sitzungsberichte, 1917, pp. 302–321.

68. N. Sonine, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Math. Annalen 16 (1880), 1–80.

69. T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse 8 (1894), 1–122; 9 (1895), 1–47.

70. G. Szegö, *Ein Beitrag zur Theorie der Thetafunktionen*, Akademie der Wissenschaften, Berlin, Sitzungsberichte, 1926, pp. 242–252.

71. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations, part I*, second edition, Oxford University Press, London, 1962.

72. D. Trutt, *Some mean squares of entire functions*, Dissertation, Purdue University, 1964.

73. ———, *Extremal norm-determining measures for Hilbert spaces of entire functions*, J. Math. Anal. Appl. 20 (1967), 74–89.

74. G. N. Watson, *A treatise on the theory of Bessel functions*, second edition, Cambridge University Press, London, 1944.

75. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, fourth edition, Cambridge University Press, London, 1927.

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