The Measure Problem*

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A problem of Banach is to determine the structure of a nonnegative (countably additive) measure which is defined on all subsets of a set. The problem is trivial for countable sets. Banach and Kuratowski [1] show the existence of a countable set whose complement has zero measure when cardinality hypotheses are satisfied. These hypotheses are not verified generally for the classical continuum, which has the cardinality of the class of all subsets of a countably infinite set. Ulam [5] obtains the same conclusion for the classical continuum when the measure has no nonzero value other than one. The same conclusion is now obtained for the classical continuum for all nonnegative measures. According to a theorem of Solovay [4] the existence of a countable set whose complement has zero measure follows for all infinite sets. These results are valid in Zermelo–Fraenkel set theory with the axiom of choice.

The integral representation of continuous linear functionals on spaces of bounded measurable functions is a fundamental problem. Measures are assumed to be bounded and countably additive with domain the measurable subsets of a measure space. A bounded measure determines by integration a linear functional which is continuous for the topology of uniform convergence. Every continuous linear functional is represented by a unique bounded measure when the function space is suitably chosen. The construction of the space is the principal new contribution.

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An example is given to motivate the construction. Consider the set Z of nonnegative integers with all subsets accepted as measurable. A bounded measure μ on the subsets of Z is absolutely continuous with respect to counting measure and has derivative which is a summable function $\mu'(n)$ of elements n of Z. The value

$$\mu(E) = \sum \mu'(n)$$

of the measure on a set E is an absolutely convergent sum taken over the elements n of Z. The total variation

$$\|\mu\| = \sum |\mu'(n)|$$

is a convergent sum taken over the elements n of Z. The set of bounded measures is a Banach space in the total variation norm. The space is the dual space of a Banach algebra $\mathcal{C}(Z)$ of functions defined on Z.

By definition $\mathcal{C}(Z)$ is the set of functions $f(n)$ of n in Z with complex values which converge to zero in the limit of large n: For every positive number ϵ all but a finite number of elements n of Z satisfy the inequality

$$|f(n)| < \epsilon.$$

The set $\mathcal{C}(Z)$ is a commutative Banach algebra with norm

$$\|f\|_\infty = \max |f(n)|$$

defined as a maximum taken over the elements n of Z.

The Banach algebra $\mathcal{B}(Z)$ is defined as the set of all bounded functions $f(n)$ of n in Z with norm

$$\|f\|_\infty = \sup |f(n)|$$

a least upper bound taken over the elements n of Z.

A bounded measure μ on the subsets of Z defines a continuous linear functional L on $\mathcal{B}(Z)$ by the integral

$$Lf = \int f(n) d\mu(n)$$

which reduces to an absolutely convergent sum

$$Lf = \sum f(n)\mu'(n)$$

taken over the elements n of Z. The inequality

$$|Lf| \leq \|f\|_\infty \|\mu\|$$

is satisfied.

Every continuous linear functional L on $\mathcal{C}(Z)$ is represented as an integral with respect to a unique bounded measure μ. When a subset E of Z has only a finite number of elements, the value

$$\mu(E) = Lf$$

of the measure on E is the action of the linear functional on the element f of $\mathcal{C}(Z)$ which has value one on E and which vanishes elsewhere. The measure is determined on all subsets of Z since its derivative is determined on every element of Z.

An example of a continuous linear functional L on $\mathcal{C}(Z)$ is a linear functional which is contractive. The inequality

$$|Lf| \leq 1$$

holds for an element f of $\mathcal{C}(Z)$ which satisfies the inequality

$$|f(n)| \leq 1$$

for every element n of Z.

A linear functional L on $\mathcal{C}(Z)$ is said to be nonnegative if Lf is a nonnegative number whenever f is an element of $\mathcal{C}(Z)$ with nonnegative values:

$$f(n) \geq 0$$

for every element n of Z. A nonnegative linear functional on $\mathcal{C}(Z)$ is continuous. A continuous linear functional on $\mathcal{C}(Z)$ is nonnegative if, and only if, it is represented by a measure with nonnegative values.

A linear functional on $\mathcal{C}(Z)$ is nonnegative and contractive if, and only if, it is represented by a nonnegative measure whose value on Z is at most one. The value of the measure on Z is then equal to its total variation.

A measure space is a set $\mathcal{S}$ with a σ–complete field of subsets called measurable. A countable union of measurable sets is assumed to be measurable. The complement of a measurable set is assumed to be measurable.

A mapping π of $\mathcal{S}$ onto Z is said to be measurable if for every subset E of Z a measurable subset of $\mathcal{S}$ is defined as the set of elements s of $\mathcal{S}$ such that πs belongs to E.

A Banach algebra $\mathcal{B}(\mathcal{S})$ is defined as the set of all bounded measurable functions $f(s)$ of s in $\mathcal{S}$ with norm

$$\|f\|_\infty = \sup |f(s)|$$

a least upper bound taken over the elements s of $\mathcal{S}$.

A function $f(s)$ of s in $\mathcal{S}$ belongs to $\mathcal{B}(\mathcal{S})$ if it can be factored

$$f(s) = f_\pi(\pi s)$$

by a measurable mapping π of $\mathcal{S}$ onto Z with a function $f_\pi(n)$ of n in Z which belongs to $\mathcal{B}(Z)$. A dense set of elements of $\mathcal{B}(\mathcal{S})$ are factorable.

A property of measurable mappings of $\mathcal{S}$ onto Z is required for the verification that sums and products of factorable elements of $\mathcal{B}(\mathcal{S})$ are factorable. The conjugate function

$$f^-(s) = f(s)^-$$

of s in $\mathcal{S}$ belongs to $\mathcal{B}(\mathcal{S})$ whenever the function $f(s)$ of s in $\mathcal{S}$ belongs to $\mathcal{B}(\mathcal{S})$.

If α and β are measurable mappings of $\mathcal{S}$ onto Z, then a measurable mapping γ of $\mathcal{S}$ onto Z exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of $\mathcal{S}$ is equivalent to the equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z exist such that

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

for every element s of $\mathcal{S}$.

Functions $f(s)$ and $g(s)$ of s in $\mathcal{S}$ which can be factored

$$f(s) = f_\alpha(\alpha s)$$

and

$$g(s) = g_\beta(\beta s)$$

with f_α and g_β in $\mathcal{B}(Z)$ can be factored

$$f(s) = f_\gamma(\gamma s)$$

and

$$g(s) = g_\gamma(\gamma s)$$

with f_γ and g_γ in $\mathcal{B}(Z)$ with

$$f_\gamma(n) = f_\alpha(\alpha' n)$$

and

$$g_\gamma(n) = g_\beta(\beta' n)$$

for every element n of Z.

The functions

$$f(s) + g(s) = f_\gamma(\gamma s) + g_\gamma(\gamma s)$$

and

$$f(s)g(s) = f_\gamma(\gamma s)g_\gamma(\gamma s)$$

of s in $\mathcal{S}$ can be factored with

$$f_\gamma(n) + g_\gamma(n)$$

and

$$f_\gamma(n)g_\gamma(n)$$

functions of n in Z which belong to $\mathcal{B}(Z)$.

The total variation of a bounded measure μ on the measurable subsets of $\mathcal{S}$ is the least upper bound

$$\|\mu\| = \sup(|\mu(E_1)| + \ldots + |\mu(E_r)|)$$

taken over all finite choices of disjoint measurable subsets $E_1, \ldots, E_r$ of $\mathcal{S}$. The set of all bounded measures is a Banach space $\mathcal{M}(\mathcal{S})$ with total variation as norm.

If π is a measurable mapping of $\mathcal{S}$ onto Z, then for every bounded measure μ on the measurable subsets of $\mathcal{S}$ a bounded measure μ_π on the subsets of Z is defined whose value $\mu_\pi(E)$ on a set E is the value of μ on the set of elements s of $\mathcal{S}$ such that πs belongs to E. The linear transformation which takes μ into μ_π is contractive in the total variation norm: The inequality

$$\|\mu_\pi\| \leq \|\mu\|$$

holds for every element μ of $\mathcal{M}(\mathcal{S})$.

A vector subspace $\mathcal{C}_\infty(\mathcal{S})$ of $\mathcal{B}(\mathcal{S})$ is defined as the topological completion of the vector span of functions

$$f(s) = f_\pi(\pi s)$$

of s in $\mathcal{S}$ which can be factored by a measurable mapping π of $\mathcal{S}$ into Z with a function $f_\pi(n)$ of n in Z which belongs to $\mathcal{C}(Z)$.

The conjugate

$$f^-(s) = f(s)^-$$

of a function $f(s)$ of s in $\mathcal{S}$ which can be so factored is a function which can be so factored.

If α and β are measurable mappings of $\mathcal{S}$ onto Z, a measurable mapping γ of $\mathcal{S}$ onto Z exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of $\mathcal{S}$ is equivalent to the simultaneous equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z exist such that the functions

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

of s in $\mathcal{S}$ are factored by the function $\gamma(s)$ of s in $\mathcal{S}$.

Factorizations

$$f(s) = f_\gamma(\gamma s)$$

and

$$g(s) = g_\gamma(\gamma s)$$

are obtained with

$$f_\gamma(n) = f_\alpha(\alpha' n)$$

and

$$g_\gamma(n) = g_\beta(\beta' n)$$

functions of n in Z which belong to $\mathcal{B}(Z)$.

A vector subspace of $\mathcal{B}(\mathcal{S})$ which is closed under conjugation is defined as the set of finite linear combinations of functions which can be factored by measurable mappings of $\mathcal{S}$ onto Z to produce a function which belongs to $\mathcal{C}(Z)$.

A locally convex topology is defined on the vector space whose open sets are the unions of nonempty convex open sets: A nonempty convex subset of the vector space is defined as open if it is disjoint from the closure in $\mathcal{B}(\mathcal{S})$ of every disjoint convex subset of the vector space.

An alternative description of the topology of the vector space results from the Stone formulation of the Hahn–Banach theorem: If A is a nonempty open convex subset of the vector space and if B is a disjoint nonempty convex subset of the vector space, then by the Kuratowski–Zorn lemma a maximal convex subset of the vector space exists which contains B and is disjoint from A. The maximal convex set is the intersection with the vector space of its closure in $\mathcal{B}(\mathcal{S})$. The complement of the maximal convex set in the vector subspace is convex.

A continuous linear functional on $\mathcal{B}(\mathcal{S})$ exists by the Hahn–Banach theorem which maps the convex sets A and B into disjoint convex subsets of the complex plane. The linear functional is constructed from a convex open subset A' of $\mathcal{B}(\mathcal{S})$ which is disjoint from B and has a nonempty intersection with A. It can be assumed that B is a maximal convex subset of the vector space which is disjoint from A. A continuous linear functional on $B(\mathcal{S})$ exists which maps A' and B into disjoint convex subsets of the complex plane. The linear functional maps A and B into disjoint convex subsets of the complex plane.

The inclusion of the vector space in $\mathcal{B}(\mathcal{S})$ is continuous since every open subset of $\mathcal{B}(\mathcal{S})$ is a union of convex open subsets of $\mathcal{B}(\mathcal{S})$ and since the intersection with the vector space of a convex open subset of $\mathcal{B}(\mathcal{S})$ is a convex open subset of the vector space.

Since the vector space is contained continuously in $\mathcal{B}(\mathcal{S})$, the completion $\mathcal{C}_\infty(\mathcal{S})$ of the vector space in its uniform topology is mapped continuously into $\mathcal{B}(\mathcal{S})$. The continuous extension of the inclusion is injective since an element of $\mathcal{C}_\infty(\mathcal{S})$ whose image in $\mathcal{B}(\mathcal{S})$ is annihilated by every continuous linear functional on $\mathcal{B}(\mathcal{S})$ is annihilated by every continuous linear functional on the vector space. The space $\mathcal{C}_\infty(\mathcal{S})$ is contained continuously in $\mathcal{B}(\mathcal{S})$. A nonempty convex subset of $\mathcal{C}_\infty(\mathcal{S})$ is open if, and only if, it is disjoint from the closure in $\mathcal{B}(\mathcal{S})$ of every disjoint convex subset of $\mathcal{C}_\infty(\mathcal{S})$.

A good relationship between measure and integration follows for every measure space $\mathcal{S}$.

Theorem 1. *Every continuous linear functional L on $\mathcal{C}_\infty(\mathcal{S})$ is represented by a unique bounded measure μ on the measurable subsets of $\mathcal{S}$. The action*

$$Lf = \int f(s) d\mu(s)$$

of the linear functional on a function $f(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{C}_\infty(\mathcal{S})$ is equal to the integral of the function with respect to the measure.

Proof of Theorem 1. If π is a measurable mapping of $\mathcal{S}$ into Z, a continuous linear functional L_π on $\mathcal{C}_\infty(Z)$ is defined by

$$L_\pi f = L f_\pi$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}_\infty(Z)$ with

$$f_\pi(s) = f(\pi s)$$

the composed function of s in $\mathcal{S}$ which belongs to $\mathcal{C}_\infty(\mathcal{S})$.

A unique measure μ_π on the subsets of Z exists such that

$$L_\pi f = \int f(n) d\mu_\pi(n)$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}(Z)$. The measure μ_π has finite total variation since L_π is continuous. If a function

$$f(s) = f_\pi(\pi s)$$

of s in $\mathcal{S}$ can be factored with $f_\pi(n)$ a function of n in Z which belongs to $\mathcal{C}(Z)$, then

$$Lf = \int f_\pi(n) d\mu_\pi(n).$$

A subset E of $\mathcal{S}$ is said to be saturated with respect to π if an image subset E_π of Z exists such that an element s of $\mathcal{S}$ belongs to E if, and only if, if image πs in Z belongs to E_π. The measure μ is defined by

$$\mu(E) = \mu_\pi(E_\pi)$$

for every subset E of $\mathcal{S}$ which is saturated with respect to π.

The definition of $\mu(E)$ applies to every measurable subset E of $\mathcal{S}$ since the set is saturated with respect to some measurable mapping of $\mathcal{S}$ onto Z, but it needs to be shown that the definition is independent of the measurable mapping π with respect to which E is saturated.

If α and β are measurable mappings of $\mathcal{S}$ onto Z with respect to which E is saturated, then a measurable mapping γ of $\mathcal{S}$ onto Z with respect to which E is saturated exists such that equality

$$\gamma(s_1) = \gamma(s_2)$$

for elements s_1 and s_2 of $\mathcal{S}$ is equivalent to the equalities

$$\alpha(s_1) = \alpha(s_2)$$

and

$$\beta(s_1) = \beta(s_2).$$

Mappings α' and β' of Z onto Z then exist such that

$$\alpha(s) = \alpha'(\gamma s)$$

and

$$\beta(s) = \beta'(\gamma s)$$

for every element s of $\mathcal{S}$.

Continuous linear functionals L_α, L_β, and L_γ on $\mathcal{C}_\infty(Z)$ are defined by

$$L_\alpha f = L f_\alpha$$

and

$$L_\beta f = L f_\beta$$

and

$$L_\gamma f = L f_\gamma$$

for every element f of $\mathcal{C}_\infty(Z)$ with

$$f_\alpha(s) = f(\alpha s)$$

and

$$f_\beta(s) = f(\beta s)$$

and

$$f_\gamma(s) = f(\gamma s)$$

functions of s in $\mathcal{S}$ which belong to $\mathcal{C}_\infty(\mathcal{S})$. It follows that

$$L_\alpha f = L_\gamma f'_\alpha$$

and

$$L_\beta f = L_\gamma f'_\beta$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}_\infty(Z)$ with

$$f'_\alpha(n) = f(\alpha' n)$$

and

$$f'_\beta(n) = f(\beta' n)$$

functions of n in Z which belong to $\mathcal{B}(Z)$.

Measures μ_α, μ_β, and μ_γ on the subsets of Z are defined by

$$L_\alpha f = \int f(n) d\mu_\alpha(n) = \sum f(n)\mu'_\alpha(n)$$

and

$$L_\beta f = \int f(n) d\mu_\beta(n) = \sum f(n)\mu'_\beta(n)$$

and

$$L_\gamma f = \int f(n) d\mu_\gamma(n) = \sum f(n)\mu'_\gamma(n)$$

for every function $f(n)$ of n in Z which belongs to $\mathcal{C}(Z)$.

If functions $f(n)$ and $g(n)$ of n in Z belong to $\mathcal{C}(Z)$, then the function

$$h(n) = f(\alpha' n)g(\beta' n)$$

of n in Z belongs to $\mathcal{C}(Z)$. The action

$$L_\gamma h = \sum f(\alpha' n)g(\beta' n)\mu'_\gamma(n)$$

on the product function is a sum over the elements n of Z.

When the functions $f(n)$ and $g(n)$ of n in Z have no nonzero values other than one and the function $f(\alpha' n)$ of n in Z vanishes whenever the function $g(\beta' n)$ of n in Z vanishes, the function

$$h(n) = f(\alpha' n)$$

of n in Z has no nonzero value other than one. Since

$$L_\alpha f = L_\gamma h,$$

the identity

$$\sum f(n)\mu'_\alpha(n) = \sum f(\alpha' n)\mu'_\gamma(n)$$

holds with summation over the elements n of Z. If $g(\beta' n)$ does not vanish for some element n of Z, the choice of f can be made so that the identity reads

$$\mu'_\alpha(n) = \sum \mu'_\gamma(k)$$

with summation over the elements k of Z such that

$$n = \alpha' k.$$

The identity holds for every element n of Z by the arbitrariness of g.

A subset E of $\mathcal{S}$ is saturated with respect to γ if it is saturated with respect to α and β. The identity

$$\mu_\alpha(E_\alpha) = \mu_\gamma(E_\gamma)$$

follows since E_α is the set of elements n of Z such that

$$n = \alpha' k$$

for an element k of E_γ. The identity

$$\mu_\beta(E_\beta) = \mu_\gamma(E_\gamma)$$

is obtained by a similar argument. It follows that

$$\mu_\alpha(E_\alpha) = \mu_\beta(E_\beta).$$

The identity

$$Lf = \int f_\pi(n) d\mu_\pi(n) = \int f(s) d\mu(s)$$

holds for every function

$$f(s) = f_\pi(\pi s)$$

of s in $\mathcal{S}$ which can be factored by a measurable mapping π of $\mathcal{S}$ onto Z with $f_\pi(n)$ a function of n in Z which belongs to $\mathcal{C}(Z)$. Since a dense set of elements f of $\mathcal{C}(\mathcal{S})$ can be so factored and since L is continuous, the total variation

$$\|\mu\| = \sup \|\mu_\pi\| < \infty$$

of the measure μ is finite. The identity

$$Lf = \int f(s) d\mu(s)$$

follows for every function $f(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{C}_\infty(\mathcal{S})$.

This completes the proof of the theorem.

The Stone–Weierstrass theorem states than an algebra of continuous functions on a compact Hausdorff space is uniformly dense in the algebra of all continuous functions on the space if the algebra contains constants, if the algebra contains the complex conjugate of every function which it contains, and if distinct elements of the space are mapped into distinct complex numbers by some element of the algebra.

The Stone–Weierstrass theorem applies also to a locally compact Hausdorff space since it is a subspace of a compact Hausdorff space which contains only one other element, said to be at infinity. A continuous function on the locally compact space is said to vanish at infinity if it admits a continuous extension to the compact space which vanishes at infinity: For every positive number ϵ the function is bounded by ϵ outside of some compact set. An algebra of continuous functions which vanish at infinity is uniformly dense in the algebra of all continuous functions which vanish at infinity if it contains the complex conjugate of every function it contains and if distinct elements of the space are mapped into distinct complex numbers by some element of the algebra.

A proof of the Stone–Weierstrass theorem [2] is given from the Krein–Milman theorem [4].

The convex set of all nonnegative elements μ of $\mathcal{M}(\mathcal{S})$ of total variation

$$\mu(\mathcal{S}) \leq 1$$

at most one is a compact Hausdorff space in the weakest topology with respect to which the integral

$$\int f(s) d\mu(s)$$

is a continuous function of μ for every element f of $\mathcal{C}(\mathcal{S})$. The convex set is the closed convex span of its extreme points by the Krein–Milman theorem.

If μ is an element of the convex set, a convex decomposition

$$\mu = \tfrac{1}{2}\mu_+ + \tfrac{1}{2}\mu_-$$

into elements μ_+ and μ_- of the convex set is defined by a function $h(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{B}(\mathcal{S})$ and satisfies the inequalities

$$-1 \leq h(s) \leq 1$$

for every element s of $\mathcal{S}$ if the integral

$$\int h(s) d\mu(s) = 0$$

vanishes. The measures μ_+ and μ_- are defined on every measurable subset E of $\mathcal{S}$ by integration

$$\mu_+(E) = \int_E [1 + h(s)] d\mu(s)$$

and

$$\mu_-(E) = \int_E [1 - h(s)] d\mu(s).$$

The measures

$$\mu_+ = \mu = \mu_-$$

are equal when μ is an extreme point of the convex set. The function $h(s)$ of s in $\mathcal{S}$ vanishes almost everywhere with respect to μ. An element of $\mathcal{C}(\mathcal{S})$ is equal to a constant almost everywhere with respect to μ. The measure μ has no nonzero value other than one.

The Stone representation of the σ–algebra of measurable subsets of $\mathcal{S}$ is an isomorphism into the Boolean algebra of regular open subsets of a compact Hausdorff space. A computation of the Stone representation is made from the set $\mathcal{S}^\wedge$ of nonzero extreme points of the convex set of nonnegative elements of $\mathcal{M}(\mathcal{S})$ of total variation at most one. The set of all extreme points is a compact Hausdorff space in the weakest topology with respect to which the integral

$$\int f(s) d\mu(s)$$

is a continuous function of μ for every function $f(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{C}(\mathcal{S})$. The set $\mathcal{S}^\wedge$ is a locally compact Hausdorff space with the measure which is identically zero at infinity.

When E is a measurable subset of $\mathcal{S}$ whose complement contains infinitely many measurable subsets, a function $f(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{C}(\mathcal{S})$ is defined to have value one on E and value zero on the complement of E. The function of $f^\wedge(\mu)$ of μ in $\mathcal{S}^\wedge$ has no nonzero value other than one. The $E^\wedge$ on which $f^\wedge$ has the value one is compact and open.

A classical application of measure theory is made to assertions which are true almost everywhere with respect to a nonnegative measure. A modern application is made to assertions concerning subsets of Z which, as Paul Cohen has shown, are not determined by the Zermelo–Fraenkel axioms of set theory with axiom of choice. A probabilistic construction of subsets of Z is stimulated by the Cohen construction of sets.

In probabilistic logic assertions have truth values which are measurable subsets of a measure space $\mathcal{S}$. A probability measure μ on the measurable subsets of $\mathcal{S}$ assigns a number in the interval $[0, 1]$ to the truth value.

In a given model of set theory the question whether A belongs to B is answered by yes or no when it is meaningful. A probabilistic model of set theory is constructed from the given model in which the same question is answered for B by a function whose values are measurable subsets of $\mathcal{S}$ and whose domain is the set of candidates A for belonging to B.

The construction of the probabilistic model is made so that an isomorphic image of the given model of set theory is obtained for every probability measure on the measurable subsets of $\mathcal{S}$. An isomorphism C into $C^\wedge$ is determined of the given model of set theory into the probabilistic model of set theory. The truth value of sets in the image of the isomorphism is the full space $\mathcal{S}$.

A probabilistic set B is a function whose domain is the class of all probabilistic sets A such that the image of A belongs to the image of B under some such homomorphism. Every function which has the same domain as B and whose values are measurable subsets of $\mathcal{S}$ is assumed to be a probabilistic set. The probabilistic model of set theory is constructed inductively from those probabilistic sets which are identified with sets of the given model.

Classical applications of probabilistic logic create the expectation that probabilistic sets can be successfully treated by the same methods. The Zermelo–Fraenkel axioms of set theory and the axiom of choice are satisfied in the probabilistic model. If a set C has cardinality γ, then the image probabilistic set $C^\wedge$ has almost surely cardinality γ.

An application of probabilistic logic is a proof of the Cohen theorem that the cardinality of the class of all subsets of Z is not determined by the axioms of set theory.

A measure space $\mathcal{S}$ is defined whose elements are the subsets of a given infinite set E. The measurable subsets of $\mathcal{S}$ are generated under countable unions and countable intersections by sets determined by elements of E. If c is an element of E, the subsets of $\mathcal{S}$ which are determined by c are the class $\iota_+ c$ of subsets of E which contain c and the class $\iota_- c$ of subsets of E which do not contain c. The canonical measure of $\mathcal{S}$ is the probability measure μ on measurable subsets of $\mathcal{S}$ whose value is one–half on $\iota_+ c$ and $\iota_- c$ for every element c of E.

A probabilistic subset of $Z^\wedge$ is a function defined on $Z^\wedge$ whose values are measurable subsets of $\mathcal{S}$. Since an element of $Z^\wedge$ is almost surely equal to $n^\wedge$ for an element n of Z, a probabilistic subset is identified with a function defined on Z whose values are measurable subsets of $\mathcal{S}$.

An example of such a function is defined by every mapping γ of Z into E and every subset C of Z. The value of the function at an element n of Z is $\iota_+\gamma(n)$ when n belongs to C and $\iota_-\gamma(n)$ when n does not belong to C.

If α and β are mappings of Z into E and if A and B are subsets of Z, then the probability that the probabilistic subset of $Z^\wedge$ determined by α and A is equal to the probabilistic subset of $Z^\wedge$ determined by β and B is

$$2^{-k}$$

where k is the number of positive integers n such that $\alpha(n)$ is unequal to $\beta(n)$ when n belongs to the intersection of A and B or the intersection of the complements of A and B or such that n does not belong to the intersection of A and B or to the intersection of the complements of A and B.

If a function $F(n)$ of n in Z has measurable subsets of $\mathcal{S}$ as values, then the probabilistic subset of $Z^\wedge$ determined by F is almost surely equal to some probabilistic subset of $Z^\wedge$ determined by a mapping γ of Z into E and a subset C of Z.

In the probabilistic model of set theory the cardinality of the class of probabilistic subsets of $Z^\wedge$ is equal to the cardinality of the class of mappings of Z into E.

The measure problem concerns the structure of a nonnegative measure which is defined on all subsets of a set. The structure of such a measure has been described when the set is Z. Banach and Kuratowski [1] treat the problem when the cardinality of the set does not exceed the cardinality of the class of all subsets of Z. When the cardinality of the set is sufficiently small, they show that if a nonnegative measure is defined on all subsets of a set E, then a countable subset of E exists whose complement has zero measure.

The measure problem is treated by Ulam [5] for sets of unrestricted cardinality. The first theorem applies to a set E whose cardinality is greater than the cardinality of the class of all subsets of Z: If a nonnegative measure exists which is defined on all subsets of E, which has value zero in all countable subsets of E, and does not vanish identically, then such a measure exists whose only nonzero value is one.

The second theorem applies to a set E whose cardinality is less than or equal to the cardinality of the class of all subsets of Z: If a nonnegative measure is defined on all subsets of E and has no nonzero value other than one, then a subset of E containing at most one element exists whose complement has zero measure.

The theorem has an elementary proof. Since the class of finite subsets of Z is countable, the desired conclusion is immediate when this class is a subset of $\mathcal{S}$ of positive measure. Otherwise an element k of Z defines a measurable mapping π of $\mathcal{S}$ onto Z which takes an infinite subset into its k–th element in the inherited ordering and which is defined arbitrarily on finite subsets. Since π maps the measure μ into a nontrivial measure on the subsets of Z whose only nonzero value is one, an element n_k of Z exists such that the measure μ has value one on the class of infinite subsets of Z whose k–th element is n_k and has value zero on the class of infinite subsets of Z which do not have n_k as k–th element. The infinite subset of Z whose k–th element is n_k for every element k of Z is then an element of $\mathcal{S}$ which determines μ.

An application of probabilistic logic which is due to Solovay [4] reduces the measure problem to sets whose cardinality is less than or equal to the cardinality of the class of all subsets of Z.

Assume that a probability measure σ, which is defined on all subsets of a set E, has no nonzero value other than one and has value zero on all countable subsets of E. A probabilistic model of set theory is applied whose measure space $\mathcal{S}$ is the class of all subsets of E and whose measurable sets are generated under countable unions and intersections by the sets $\iota_+ c$ and $\iota_- c$ for every element c of E. The measure space $\mathcal{S}$ is applied with its canonical measure μ. A nonnegative measure is defined by Solovay on the class of all probabilistic subsets of the class of all probabilistic subsets of $Z^\wedge$.

The Solovay construction is elementary in nature, but notationally difficult. A logically satisfactory argument requires expertize which a majority of interested readers do not have.

A probabilistic subset of the class of probabilistic subsets of $Z^\wedge$ is determined by a function F which is defined on the subsets of Z and whose values are measurable subsets of $\mathcal{S}$. Such a function is almost surely equal to a function which is determined by a mapping γ of the class of all subsets of Z into E and by a class $\mathcal{C}$ of subsets of Z. The value of the function F on a subset C of Z is $\iota_+\gamma(C)$ when C belongs to $\mathcal{C}$ and $\iota_-\gamma(C)$ when C does not belong to $\mathcal{C}$.

A related function μF with values in the interval $[0, 1]$ is obtained by the action of the canonical measure on the measurable subsets of $\mathcal{S}$. The value of μF on a subset C of Z is $\mu\iota_+\gamma(C)$ when C belongs to $\mathcal{C}$ and $\mu\iota_-\gamma(C)$ when C does not belong to $\mathcal{C}$.

The function μF is implicitly defined on the Cartesian product of E with itself taken as many times as there are subsets of Z. The Cartesian product measure of σ with itself is taken as many times as there are subsets of Z.

The function μF is integrated with respect to the Cartesian product measure to produce a number, which defines the action of a probability measure ν on the class of all probabilistic subsets F of the class of all probabilistic subsets of Z. The measure ν extends the canonical measure on the class of all probabilistic subsets of the class of all probabilistic subsets of Z. Countable subsets have zero ν-measure.

A hypothesis in the first Ulam theorem is removed by a corollary to Theorem 1.

Theorem 2. *If there exists a nonnegative measure which is defined on all subsets of a set $\mathcal{S}$, which has value zero on all countable subsets of $\mathcal{S}$, and which does not vanish identically, then there exists a nonnegative measure which is defined on all subsets of $\mathcal{S}$ and whose only nonzero value is one, which has value zero on all countable subsets of $\mathcal{S}$, and which does not vanish identically.*

Proof of Theorem 2. The set of all nonnegative measures μ which are defined on all subsets of $\mathcal{S}$, whose values are taken in the interval $[0, 1]$, and which have value zero on all countable subsets of $\mathcal{S}$, is a convex set which is compact in the weakest topology with respect to which

$$\int f(s) d\mu(s)$$

is a continuous function of μ for every function $f(s)$ of s in $\mathcal{S}$ which belongs to $\mathcal{C}_\infty(\mathcal{S})$: If a subset C of $\mathcal{S}$ is countable, a function $f(s)$ of s in $\mathcal{S}$ exists which belongs to $\mathcal{C}_\infty(\mathcal{S})$, has positive values on C, and vanishes outside of C. The product of such a function with a function which belongs to $\mathcal{B}(\mathcal{S})$ is a function which belongs to $\mathcal{C}_\infty(\mathcal{S})$.

The convex set of measures is the closed convex span of its extreme points by the Krein–Milman theorem. An extreme point is a measure whose only nonzero value is one.

This completes the proof of the theorem.

A solution of the measure problem results from the strengthening of the first Ulam theorem.

Theorem 3. *If a nonnegative measure μ is defined on all subsets of a set $\mathcal{S}$, then a countable subset C of $\mathcal{S}$ exists whose complement has zero μ-measure.*

Proof of Theorem 3. The set C is defined as the set of elements of $\mathcal{S}$ which belong to every countable subset of $\mathcal{S}$ of positive measure. Since an element s of C defines a set of positive measure whose only element is s and since the sum of these measures is the measure of C, the set C is countable. The complement of C will be shown to have zero measure.

It remains to be shown that a nonnegative measure which is defined on all subsets of a set $\mathcal{S}$ vanishes identically if it vanishes on all countable subsets of $\mathcal{S}$. By Theorem 2 it is sufficient to treat a measure whose only nonzero value is one. By the Solovay theorem it can be assumed that the set $\mathcal{S}$ is the class of all subsets of Z. The measure vanishes identically by the second Ulam theorem.

This completes the proof of the theorem.

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