RESEARCH PROPOSAL
RIEMANN HYPOTHESIS

The Riemann hypothesis is the conjecture made by Riemann that the Euler zeta function remains without zeros in a larger half–plane than the half–plane of convergence of the Euler product. The zeta function was introduced by Euler as an analogue of his previously discovered gamma function, which is a product determined by zeros. The Riemann hypothesis is treated as the conjecture that the Euler zeta function has properties similar to those of the gamma function.

The gamma function is the simplest analytic function giving information about the set of positive integers. The function $\Gamma(s)$ of s is analytic in the complex plane with the exception of singularities at the nonpositive integers and satisfies the recurrence relation

$$s\Gamma(s) = \Gamma(s+1).$$

An interpolation of the factorial follows from the identity

$$\Gamma(n+1) = n!$$

for every nonnegative integer n. The interpolation is interesting because of the associated hypergeometric function theory introduced by Euler. The Euler zeta function is linked to the gamma function in a relation discovered by Euler.

A generalization of the gamma function is obtained in which the factor of s in the functional identity is replaced by an arbitrary analytic function of s in the right half–plane whose real part is positive. Such a function admits a function theory which generalizes the hypergeometric function theory of the gamma function.

The function theory originates in the Stieltjes representation of positive linear functionals on polynomials. A nontrivial linear functional on polynomials is said to be positive if it has nonnegative values on polynomials whose values on the real axis are nonnegative. A positive linear functional defines a scalar product on the space of polynomials of degree less than r for every positive integer r. The scalar self–product of a polynomial $F(z)$ is defined as the action of the positive linear functional on the polynomial

$$F^*(z)F(z)$$

with the conjugate polynomial

$$F^*(z) = F(z^-)^-$$

defined by conjugate values on the real axis.

Stieltjes determined the structure of the space of polynomials of degree less than r when the scalar product is nondegenerate. A polynomial $E(z)$ of degree r, which has no zeros on or above the real axis, exists such that the scalar self–product of a polynomial $F(z)$ is equal to the integral

$$\int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt.$$

A Stieltjes space of entire functions is defined as a Hilbert space whose elements are entire functions and which has these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same norm as $F(z)$.

(H2) For every nonreal number w a continuous linear functional is defined on the space by taking an entire function $F(z)$ of z into its value $F(w)$ at w.

(H3) The entire function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the function $F(z)$ of z belongs to the space, and it always has the same norm as $F(z)$.

An example of a Stieltjes space is constructed from an entire function $E(z)$ of z which satisfies the inequality

$$|E(z^-)| < |E(z)|$$

when z is in the upper half–plane. The space $\mathcal{H}(E)$ is defined as the set of entire functions $F(z)$ of z such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |E(t)/E(t)|^2 dt$$

converges and such that the inequality

$$|F(z)|^2 \leq \|F\|^2 [|E(z)|^2 - |E(z^-)|^2]/[2\pi i(z^- - z)]$$

holds for all complex numbers z. Every nontrivial Stieltjes space of entire functions is isometrically equal to a space $\mathcal{H}(E)$.

Methods of functional analysis were introduced in the complex analysis of the Riemann hypothesis independently by Hilbert and Hardy. The Hardy space for the upper half–plane is the Hilbert space of analytic functions $F(z)$ of z in the upper half–plane such that the integral

$$\int_{-\infty}^{+\infty} |F(x + iy)|^2 dx$$

converges when y is positive and is a bounded function of y. The scalar self–product of the function is the least upper bound taken over all positive y, which is obtained in the limit as y decreases to zero.

A generalization of the Hardy space for the upper half–plane is defined for application to Stieltjes spaces of entire functions. An analytic weight function is defined as an analytic function $W(z)$ of z in the upper half–plane which has no zeros in the upper half–plane.

A trivial example of an analytic weight function is the function which is identically one. The weighted Hardy space $\mathcal{F}(1)$ is the Hardy space for the upper half–plane. If $W(z)$ is an analytic weight function, the weighted Hardy space $\mathcal{F}(W)$ is the isometric image of the space $\mathcal{F}(1)$ under multiplication by $W(z)$.

The defining function $E(z)$ of a Stieltjes space $\mathcal{H}(E)$ is an analytic weight function $W(z)$. The space $\mathcal{H}(E)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ and contains every entire function $F(z)$ such that the entire functions $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(W)$.

If $W(z)$ is an analytic weight function, the set of entire functions $F(z)$ such that the entire functions $F(z)$ and $F^*(z)$ of z belong to the weighted Hardy space $\mathcal{F}(W)$ is a Stieltjes space of entire functions which is contained isometrically in the weighted Hardy space. The space is a space $\mathcal{H}(E)$ if it contains a nonzero element.

A motivating example of an analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

is defined by the gamma function. Hypergeometric function theory constructs Stieltjes spaces of entire functions which are contained isometrically in the weighted Hardy space $\mathcal{F}(W)$. The spaces appear in Fourier analysis on the complex plane.

In 1880 Nikolai Sonine constructed nontrivial examples of functions which are square integrable with respect to plane measure, which vanish in a disk $|z| < a$ about the origin, and whose Fourier transform vanishes in the disk. These are functions of distance from the origin whose Fourier transforms are computed by the Hankel transformation of order zero.

A Stieltjes space of entire functions which is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ is constructed from the Sonine functions for every positive number a. The spaces have a maximality property: An entire function belongs to the space whenever its product with a nonconstant polynomial belongs to the space. A Stieltjes space of entire functions which is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$, which has the maximality property, and which contains a nonzero element, is a space obtained from the Sonine examples for some positive number a.

These Stieltjes spaces do not have finite dimension and do not contain nontrivial polynomials. The recurrence relations for orthogonal polynomials applied by Stieltjes are replaced by ordinary differential equations of first order. The defining functions of the Stieltjes spaces are solutions of the differential equations in confluent hypergeometric series. The analytic weight function $W(z)$ is recovered asymptotically from the defining functions of the Stieltjes spaces as the scattering function of a one–dimensional dynamical system.

Analogous constructions are made from generalizations of the gamma function. An Euler weight function is defined as an analytic weight function $W(z)$ such that for every h in the interval $[0, 1]$ the function

$$W(z + \tfrac{1}{2}ih)/W(z - \tfrac{1}{2}ih)$$

of z is analytic and has nonnegative real part in the upper half–plane.

It is not sufficient to impose the condition only when h is one since the analytic weight function can then be multiplied by an arbitrary entire function which is periodic of period one and has no zeros. But if an analytic weight function satisfies the condition when h is one, it can be multiplied by an entire function which is periodic of period one and has no zeros to produce an analytic weight function which satisfies the condition for all h in the interval $[0, 1]$.

An equivalent condition for an analytic weight function to be an Euler weight function follows from the Poisson representation of a function $\phi(z)$ of z which is analytic and has positive real part in the upper half–plane: A Hilbert space exists whose elements are functions analytic in the upper half–plane and which contains the function

$$[\phi(z) + \phi(w)^-]/[\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane: A function which belongs to the space has a value at w which is a scalar product with the reproducing kernel function.

An Euler weight function is an analytic weight function $W(z)$ such that for every h in the interval $[0, 1]$ a Hilbert space of functions analytic in the upper half–plane exists which contains the function

$$[W(z + \tfrac{1}{2}ih)W(w - \tfrac{1}{2}ih)^- + W(z - \tfrac{1}{2}ih)W(w + \tfrac{1}{2}ih)^-]/[\pi i(w^- - z)]$$

of z as reproducing kernel function for function values at w when w is in the upper half–plane.

An equivalent condition for an Euler weight function is formulated by the concept of a maximal accretive transformation. A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$\langle a, b\rangle + \langle b, a\rangle \geq 0$$

of scalar products in the space is nonnegative for all elements (a, b) of the graph of the relation. An accretive relation is said to be maximal accretive if it is not the proper restriction of an accretive relation with domain and range in the same Hilbert space. A transformation is said to be maximal accretive if it is maximal accretive as a relation.

An analytic weight function $W(z)$ is an Euler weight function if, and only if, a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ for all h in the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the space.

The analytic weight function defined from the gamma function is an Euler weight function. For an arbitrary Euler weight function there are associated Stieltjes spaces of entire functions.

A Stieltjes space of entire functions is associated with an Euler weight function $W(z)$ when it is contained contractively in the weighted Hardy space $\mathcal{F}(W)$ and when the inclusion is isometric on functions $F(z)$ such that the functions $F(z)$ and $zF(z)$ of z belong to the Stieltjes space. The orthogonal complement in the Stieltjes space of the set of functions for which the inclusion is isometric has dimension zero or one. The associated Stieltjes space is required to contain an entire function $F(z)$ which belongs to the weighted Hardy space whenever the product of $F(z)$ with a nonconstant polynomial belongs to the Stieltjes space. A maximal accretive transformation is then defined in the Stieltjes space when h belongs to the interval $[0,1]$ by taking $F(z)$ into $F(z+ih)$ whenever the functions of z belong to the Stieltjes space.

The Stieltjes spaces of entire functions which are associated with an Euler weight function are totally ordered: For any two spaces one is contained in the other.

The intersection of the Stieltjes spaces of entire functions which are associated with an Euler weight function contains no nonzero element.

The union of the Stieltjes spaces of entire functions which are associated with an Euler weight function is dense in the weighted Hardy space for the Euler weight function.

The defining functions of the Stieltjes spaces of entire functions associated with an Euler weight function are parametrized by positive numbers so that the space with parameter b is contained in the space with parameter a when a is less than b. The parametrized functions satisfy an ordinary differential equation of first order as in the case of the gamma function.

The hypergeometric function theory of the gamma function applies to Fourier analysis on the complex plane. The maximal accretive property of the shift originates in the properties of a Radon transformation in the Hilbert space of functions $f(z)$ of z which are square integrable with respect to plane measure and which satisfy the identity

$$f(\omega z) = f(z)$$

for every element ω of the unit circle. The transformation takes $f(z)$ into $g(z)$ when these functions of z belong to the space and satisfy the identity

$$g(\omega z) = \int_{-\infty}^{+\infty} f(\omega z + \omega i t) dt$$

for every element ω of the unit circle. The integral is interpreted as a limit in the metric topology of the Hilbert space of integrals over bounded subsets of the real line. The adjoint of the Radon transformation is an inverse transformation of the differential operator in the Fourier model of heat flow. The maximal accretive property of the Radon transformation generates a flow of heat in which energy can never be gained but can only be lost.

The maximal accretive property is verified using the Laplace transformation, which permits a spectral analysis of the Radon transformation. The Radon transformation is unitarily equivalent to multiplication by i/z in a Hilbert space of analytic functions of z in the upper half–plane. The maximal accretive property of the Radon transformation is a consequence of the positivity of the real part of the multiplier in the half–plane.

A generalization of the Radon transformation applies in a locally compact skew–field. Every skew–field contains the algebra of quaternions

$$\xi = t + ix + jy + kz$$

whose coordinates are rational numbers. The skew–field is assumed to be an algebra of quaternions whose coordinates are in a field and to contain

$$\xi^- = t - ix + jy - kz$$

whenever it contains ξ. The coordinates are elements of the field of self–conjugate elements of the skew–field. The skew–field is assumed to be a vector space of finite dimension over the smallest skew–field. There is a double vector space structure since multiplication by an element of the smallest skew–field can be in the left or right of a vector. The same dimension is obtained in each case. The skew–field is given the discrete topology.

The discrete skew–field and its completions are treated as a model for the orbital electrons in the atoms of a molecule. An analogy is seen between the electronic structure of molecules and the analytic structure of locally compact skew–fields. The analogy gives a purpose to the harmonic analysis of skew–fields which goes beyond applications to number theory.

Harmonic analysis on a skew–field applies its relationship to a maximal commutative subalgebra. The subalgebra is a field which is mapped into itself by conjugation, which acts as an automorphism of the field.

The complementary space to the field in the skew–field is the set of elements ξ of the skew–field which satisfy the identity

$$\xi\eta = \eta^-\xi$$

for every element η of the field. An element of the skew–field is the unique sum of an element of the field and an element of its complementary space. Multiplication by an element of the field maps the field into itself and the complementary space into itself. Multiplication by an element of the complementary space maps the field into the complementary space and the complementary space into the field. Elements of the complementary space are skew–conjugate.

When the skew–field is locally compact, the field, the skew–field, and the complementary space have canonical measures which are unique within constant factors. A canonical measure is a nonnegative measure on Baire subsets which is finite on compact sets and positive on open sets such that a measure preserving transformation is defined by taking ξ into $\xi + \eta$ for every element η of the space. Canonical measures are normalized so that the canonical measure for the skew–field is the Cartesian product measure of the canonical measure for the field and the canonical measure for its complementary space. The modulus $\lambda(s)$ of a nonzero element of the field is the positive number such that multiplication by ξ multiplies the canonical measure of the field by a factor of $\lambda(\xi)^2$. Multiplication by ξ multiplies the canonical measure for the skew–field by a factor of $\lambda(\xi)^4$. The modulus of a nonzero element ξ of the skew–field is the positive number $\lambda(\xi)$ such that multiplication by ξ multiplies the canonical measure of the skew–field by a factor of $\lambda(\xi)^4$. Canonical measures are normalized so that multiplication by a nonzero element ξ of the complementary space takes the canonical measure for the field into $\lambda(\xi)^2$ times the canonical measure for the complementary space. The origin of the skew–field is given zero modulus.

The identity

$$\lambda(\xi) = \lambda(\xi^-)$$

holds for every element ξ of the skew–field. The identity

$$\lambda(\xi\eta) = \lambda(\xi)\lambda(\eta)$$

holds for all elements ξ and η of the skew–field.

A locally compact skew–field is said to be complex if its field of self–conjugate elements is isomorphic to the real line. A locally compact skew–field is said to be p–adic for a prime p if its field of self–conjugate elements contains an isomorphic image of the field of p–adic numbers. A locally compact skew–field which is not discrete is either complex or p–adic for a prime p.

The canonical measure for a discrete skew–field is counting measure. The modulus of a self–conjugate element of the skew–field is a rational number. An element of the skew–field is said to be integral if it is a quaternion whose coordinates are all integral or all nonintegral with product by two integral. Sums and products of integral elements are integral. A rational number is a self–conjugate element of the skew–field which is integral if, and only if, it is an integer.

An ideal of the ring of integral elements of a discrete skew–field is said to be conjugated if it contains ξ^- whenever it contains ξ. A conjugated ideal which contains an element but which does not contain every element has a finite quotient ring which inherits a conjugation. The discrete skew–field has p–adic completions constructed from the topologies of quotient rings.

A completion of a discrete skew–field is a locally compact skew–field which contains an isomorphic image of the discrete skew–field as a dense subset. A discrete skew–field has a finite number of complex completions and a finite number of p–adic completions for every prime p.

Radon transformations for a locally compact skew–field apply to irreducible representations of the group of elements ξ of the skew–field with conjugate as inverse: $\xi^-\xi = 1$. The representations are defined in the Hilbert space of functions which are square integrable with respect to the canonical measure for the skew–field. The transformation defined by an element ω with conjugate as inverse takes a function $f(\xi)$ of ξ in the skew–field into the function $f(\omega\xi)$ of ξ in the skew–field.

An irreducible representation for the complex skew–field is determined by a harmonic polynomial in the coordinates. An analogue of a harmonic polynomial exists for a p–adic skew–field.

The Radon transformation defined by a harmonic ϕ is a maximal accretive transformation whose domain and range are contained in the Hilbert space of square integrable functions with respect to the canonical measure for the skew–field. The functions $f(\xi)$ of ξ in the skew–field which belong to domain and range satisfy the identity

$$\phi(\xi)f(\omega\xi) = \varphi(\omega\xi)f(\xi)$$

for every element ω of the skew–field with conjugate as inverse.

The Radon transformation takes $f(\xi)$ into $g(\xi)$ when the functions of ξ in the skew–field satisfy the identity

$$g(\omega\xi)/\phi(\omega\xi) = \int f(\omega\xi + \omega\eta)/\phi(\omega\xi + \omega\eta)d\eta$$

for every element ω of the skew–field with conjugate as inverse with integration with respect to the canonical measure for the complementary space. The integral is taken as a limit in the metric topology of the Hilbert space of square integrable functions with respect to the canonical measure for the skew–field. The limit is a limit of integrals over compact subsets of the complementary space.

The maximal accretive property of the Radon transformation is verified by a generalization of the Laplace transformation. The adjoint of the Radon transformation is unitarily equivalent to multiplication by a function whose values have positive real part in a Hilbert space of functions square integrable with respect to a nonnegative measure.

The functions are analytic in the upper half–plane when the skew–field is complex. The classical Laplace transformation applies in that case. The Radon transformation is self–adjoint and nonnegative when the skew–field is p–adic.

Zeta functions for a discrete skew–field are generated in harmonic analysis on Cartesian products of completions. In the Cartesian product are taken all complex completions and all p–adic completions for an arbitrary finite number of primes p. The Cartesian product is a locally compact ring which has a conjugation. The canonical measure is the Cartesian product measure of the canonical measures for factor skew–fields.

The product ring contains a maximal commutation subring whose elements are the elements of the product whose components belong to maximal commutative subfields. The canonical measure for the commutative subring is the Cartesian product measure of the canonical measures for commutative subfields.

The complementary space to the maximal commutative subring is the set whose elements are the elements of the product whose components belong to complementary spaces to commutative subfields. The canonical measure for the complementary space to the subring is the Cartesian product measure of the canonical measures for complementary spaces to subfields.

An element of the product ring which has conjugate as inverse is an element whose components in skew–fields have conjugate as inverse. The group of elements of the product ring with conjugate as inverse is compact. The elements of the group act as isometric transformations of the Hilbert space of square integrable functions with respect to the canonical measure for the product ring into itself. The transformation defined by an element ω with conjugate as inverse takes a function $f(\xi)$ of ξ in the product ring into the function $f(\omega\xi)$ of ξ in the product ring.

The Hilbert space of square integrable functions with respect to the canonical measure for the product ring decomposes into irreducible invariant subspaces of finite dimension under the action of elements of the group. A harmonic function is a function $\phi(\xi)$ of elements ξ of the ring with conjugate as inverse which is a harmonic function of every component of ξ in a skew–field. Harmonic functions belong to a Hilbert space constructed from Hilbert spaces of harmonic functions for every component skew–field. An isometric transformation of the Hilbert space into itself is defined by taking a function $f(\xi)$ of ξ into the function $f(\xi\omega)$ of ξ for every nonzero element ω of the discrete skew–field.

Hecke operators are commuting self–adjoint transformations in the Hilbert space of harmonic functions for the product ring. A Hecke operator $\Delta(n)$ is defined for every positive integer n which is the modulus of the inverse of a self-conjugate integral element of the discrete skew–field. The Hecke operator $\Delta(n)$ takes a harmonic function $f(\xi)$ of ξ into the harmonic function $g(\xi)$ of ξ defined by summation

$$g(\xi)\sum 1 = \sum f(\xi\omega)$$

on the left over the integral elements ω of the discrete skew–field which represent

$$1 = \lambda(\omega^{-}\omega)^{-1}$$

and on the right over the elements ω of the discrete skew–field which represent

$$n = \lambda(\omega^{-}\omega)^{-1}.$$

The Hilbert space of harmonic functions for the product ring is the orthogonal sum of invariant subspaces whose elements are characterized as eigenfunctions of Hecke operators for given eigenvalues. The Hecke operator $\Delta(1)$ is the orthogonal projection onto the subspace of functions $f(\xi)$ of ξ which satisfy the identity

$$f(\xi\omega) = f(\xi)$$

for every element ω of the product ring with conjugate as inverse. The kernel of $\Delta(1)$ is contained in the kernel of $\Delta(n)$ and the range of $\Delta(n)$ is contained in the range of $\Delta(1)$ for every positive integer n for which $\Delta(n)$ is defined. The range of $\Delta(1)$ is the orthogonal sum of eigenfunctions of the Hecke operator $\Delta(n)$ for a real eigenvalue $\tau(n)$ for every positive integer n such that $\Delta(n)$ is defined.

The Radon transformation of harmonic ϕ for the product ring applies a harmonic function ϕ which is an eigenfunction of $\Delta(n)$ for the eigenvalue $\tau(n)$ for every n for which $\Delta(n)$ is defined. The definition of the Radon transformation is analogous for the definition for a component skew–field and produces a maximal accretive transformation. The proof applies a generalization of the Laplace transformation constructed from the Laplace transformations for component skew–fields.

The Laplace transformation for a p–adic skew–field is a variant of the Fourier transformation for a p–adic subfield. In the p–adic case the Radon transformation and its adjoint are self–adjoint and nonnegative. The adjoint is shown to be unitarily equivalent to multiplication by $\lambda(\xi)^{-1}$ on functions of ξ in the field which are square integrable with respect to the canonical measure. The multiplier is undefined at the origin since the modulus then vanishes. No difficulty occurs when the canonical measure for the field is used since the set whose only element is the origin has zero measure. But in constructions which follow topologies appear in which the set has positive measure. The adjoint of the Radon transformation is then not densely defined. This causes the appearance of accretive transformations which are not maximal.

The Hilbert space of square integrable functions with respect to the canonical measure for the product ring is acted upon by a group of isometric transformations defined by nonzero elements ω of the discrete skew–field. The transformation defined by ω takes a function $f(\xi)$ of ξ in the product ring into the function $f(\xi\omega)$ of ξ in the product ring. The Hilbert space decomposes into irreducible invariant subspaces under the action of the group. The invariant subspaces are however not contained in the given Hilbert space. The given Hilbert space decomposes as an integral of invariant subspaces, not an orthogonal sum.

Only one of the invariant subspaces is applied in harmonic analysis on the product skew–plane. The value of other invariant subspaces is unknown. The invariant subspace of known value contains functions which are left fixed by all isometric transformations of the group. The subspace originates in the application of Poisson summation by Jacobi to construct theta functions. An analogue of Jacobian theta functions is applied in harmonic analysis on the product ring.

The invariant subspace is a Hilbert space of functions $f(\xi)$ of ξ in the product ring which satisfy the identity

$$f(\xi\omega) = f(\xi)$$

for every nonzero element ω of the discrete skew–field and which are square integrable with respect to the canonical measure for the product ring over the set of elements whose components in p–adic skew–fields are integral and have integral inverse.

The invariant subspace inherits a Radon transformation from the product ring as well as a Laplace transformation showing that the adjoint of the Radon transformation is unitarily equivalent to multiplication by a function in a Hilbert space whose elements are square integrable functions with respect to a measure. The accretive property of the adjoint of the Radon transformation follows from the positivity of the real part of values of the multiplying function. The adjoint of the Radon transformation fails to be maximal accretive when the multiplier is undefined on a set of positive measure and when the Hilbert space contains functions which do not vanish on the set.

A zeta function coupled with a gamma function factor appears on application of the Mellin transformation. The product is an analytic weight function which is the Mellin transform of a theta function. The zeta function is a Dirichlet series whose coefficients are eigenvalues of Hecke operators. An Euler weight function is obtained when the adjoint of the Radon transformation is maximal accretive.

When the maximal accretive property fails, the weighted Hardy space decomposes into the orthogonal sum of a subspace whose elements are symmetric about the imaginary axis and a subspace of functions which are anti–symmetric about the imaginary axis. A maximal accretive transformation in the subspace of anti–symmetric functions is defined by taking $F(z)$ into $F(z + i)$ whenever the functions of z belong to the space.

The maximal accretive property is initially verified for an analytic weight function defined by a finite number of primes. The zeta function for a finite number of primes is a partial sum of the zeta function for an infinite number of primes. The partial sums converge to the full sum in the limit which accepts all primes. The maximal accretive property is preserved in the limit. The limiting zeta function has the positioning of zeros which is conjectured by the Riemann hypothesis.

The Riemann hypothesis originates as a conjecture for the Euler zeta function, which is not produced in harmonic analysis on skew–fields. A zeta function which is so produced is obtained by the duplication formula for the gamma function and its counterpart for zeta functions. A proof of the Riemann hypothesis results for the Euler zeta function.

The formulation of the Riemann hypothesis in harmonic analysis on skew–fields surpasses the original context of Euler and Riemann in number theory. The significance of the generalization is an issue of importance in the determination of future directions of research.

A striking feature of the proof of the Riemann hypothesis is its resemblance to the quantum mechanical theory of electrons in atoms and molecules. The complex skew–field is an algebra of quaternions with real coordinates which contains the physical space in which electrons are seen as moving objects. Quaternions parametrize rotations of a three dimensional space about a chosen origin. Quaternions apply to a solar system in which electrons orbit about a nucleus.

The harmonic analysis created by a discrete skew–field is the quantum mechanical theory of an assembly of electrons. Each complex completion of the discrete skew–field describes the motion of an electron. The resulting quantum mechanical model has a feature which is not found elsewhere. Namely the electrons are related to each other under the action of a group. This is the group of automorphisms of the discrete skew–field.

There is a physical context in which a group of automorphisms can be expected. It occurs when the electrons are held toether in a molecule. A conjecture is implied by these observations. This is that the structure of molecules is an aspect of the structure of discrete skew–fields. The key to electronic structure in molecules is symmetry. And symmetry is formulated in automorphisms.

The decomposition of a molecule into atoms suggests a decomposition of a discrete skew–field into elementary skew fields. In terms of symmetries this suggests the decomposition of a group into elementary groups. Commutative subgroups are a natural candidate for decomposition.

A group of automorphisms is always noncommutative since it contains the inner automorphisms defined by integral elements of the Gauss skew–plane with conjugate as inverse. The subgroup is normal and has a quotient group of outer automorphisms which can be commutative.

Although discrete skew–fields are mysterious, discrete skew–fields having a commutative group of outer automorphisms are accessible for research. Such skew–fields can be constructed for example from cyclotomic fields. A relationship is expected between these skew–fields and the periodic table of the elements.

The advantage of a discrete skew–field in treating atomic structure is that electrons are coordinated to create a single oscillator. The symmetry of the resulting structure imposes a constraint on its function theory. It is however not true that symmetry considerations alone determine atomic structure.

The need for additional considerations is seen in the case of a one electron atom. The discrete skew–field is then the algebra of quaternions which have rational numbers as coordinates. This is the skew–field applied in the proof of the Riemann hypothesis for the Euler zeta function, which has another function theory that that of the Dirac electron. It is a simpler function theory in which the electron moves through free space without the action of an electromagnetic field. The Dirac electron has the symmetry properties of the skew–field of quaternions with rational coordinates, but it has a more complicated function theory.

The desired application to quantum mechanics indicates a weakness in the harmonic analysis of skew–fields which is sufficient for a proof of the Riemann hypothesis. There is a more general function theory which allows the gamma function factor to be replaced by an arbitrary Euler weight function. The general theory is conjectured to exhibit the same zeta functions. The difference lies in the treatment of the complex completions of the discrete skew–field.

The generalization is proposed as objective for coming research. Funding is requested for that purpose.

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electromagnetic theory." The germs of the present molecular theory of electrons are to be found there.

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PROJECT SUMMARY

Overview: The aim of the project is a proof of a conjecture made by Riemann in the nineteenth century which consolidates mathematics of the eighteenth century principally due to Euler. The original conjecture is treated in a generalization made in the twentieth century. A proof of the conjecture restores mathematics to its classical position as queen of the sciences.

Intellectual Merit: A unification of mathematics is made which consolidates apparently unrelated disciplines such as complex analysis, Fourier analysis, and number theory so as to create a coherent whole.

Broader Impact: A successful treatment of the Riemann hypothesis reveals a natural application of mathematics to the structure of chemical molecules.