RESEARCH PROPOSAL
RIEMANN HYPOTHESIS

A proof of the Riemann hypothesis is proposed for zeta functions constructed from an algebra of finite dimension over the discrete skew-field of quaternions with rational numbers as coordinates. The algebra is assumed to have a unit and a conjugation which acts as an anti-automorphism.

The set of elements of the algebra with conjugate as inverse is a finite subgroup of the multiplicative infinite group of invertible elements. An element of the algebra is said to be integral if it is a linear combination with integer coefficients of elements of the finite group. A zeta function for the algebra is a Dirichlet series whose coefficients are eigenfunctions of Hecke operators defined when the elements of the infinite group act as isometric transformations on a Hilbert space and the elements of the finite group act trivially. A Hecke operator Δn is defined for a positive integer n as a sum over equivalence classes of elements of the infinite group with respect to the finite group whose inverse is represented by a fraction with integral numerator and denominator n.

Hecke operators act as normal transformations in Hilbert spaces of finite dimension appearing in Fourier analysis on locally compact Abelian groups constructed from the given discrete algebra. The hypercomplex algebra is a vector space over the hypercomplex skew-field of quaternions with real numbers as coordinates which contains the discrete algebra and extends addition, multiplication, and conjugation. The discrete algebra has a basis as a vector space over the discrete skew-field whose elements belong to the finite group. The hypercomplex algebra is a vector space over the hypercomplex skew-field with the same basis.

The hypercomplex algebra is locally compact since it is a vector space of finite dimension over the locally compact hypercomplex skew-field. The canonical measure with respect to addition is normalized so that the Fourier transformation is isometric as a transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself.

The multiplicative group of elements of the hypercomplex algebra with conjugate as inverse is compact. Its elements act as isometric transformations of the Hilbert space of square integrable functions into itself which commute with the Fourier transformation. The Hilbert space decomposes as the orthogonal sum of invariant subspaces of finite dimension which are mapped into themselves by the Fourier transformation.

The adic algebra is constructed from the ring of integral elements of the discrete algebra. The adic topology of the ring of integral elements of the discrete skew-field is the weakest topology with respect to which every homomorphism onto a finite ring is continuous, a finite ring being treated as a compact Hausdorff space in the discrete topology. The adic topology of the ring of integral elements of the discrete algebra is the Cartesian product topology of the adic topology of the ring of integral elements of the discrete skew-field with respect to a canonical basis.

The ring of integral elements of the adic algebra is the completion of the ring of integral elements of the discrete algebra in the adic topology. The adic algebra is the ring of quotients of the ring of its integral elements with positive integers as denominators.

The ring of integral elements of the adic algebra is compact in the adic topology. The adic algebra is locally compact in an adic topology which has the adic topology of the ring of integral elements as subspace topology and which is otherwise defined so that multiplication by a positive integer is a homeomorphism.

The canonical measure for the adic algebra is normalized so that the ring of integral elements has measure one. The Fourier transformation maps the Hilbert space of square integrable functions with respect to the canonical measure isometrically into itself.

The multiplicative group of elements of the adic algebra with conjugate as inverse is compact. Its elements act as isometric transformations of the Hilbert space of square integrable functions into itself which commute with the Fourier transformation. The Hilbert space decomposes as the orthogonal sum of invariant subspaces of finite dimension which are mapped into themselves by the Fourier transformation.

The adelic algebra is the locally compact ring which is the Cartesian product of the hypercomplex algebra and the adic algebra. The canonical measure for the adelic algebra is the Cartesian product measure of the canonical measure for the hypercomplex algebra and the canonical measure for the adic algebra. The Fourier transformation for the adelic algebra is the tensor product of the Fourier transformation for the hypercomplex algebra and the Fourier transformation for the adic algebra.The Fourier transformation for the adelic algebra is isometric in the Hilbert space of square integrable functions with respect to the canonical measure.

The group of elements of the adelic algebra with conjugate as inverse is compact since it is the Cartesian product of the group of elements of the hypercomplex algebra with conjugate as inverse and the group of elements of the adic algebra with conjugate as inverse. The elements of the group act as isometric transformations of the Hilbert space of square integrable functions into itself which commute with the Fourier transformation.

The Hilbert space is the orthogonal sum of invariant subspaces of finite dimension which are mapped into themselves by the Fourier transformation and by the transformations defined by elements with conjugate as inverse. An invariant subspace for the adelic algebra is the Cartesian product of an invariant subspace for the hypercomplex algebra and an invariant subspace for the adic algebra.

Multiplication by an invertible element of the discrete algebra is measure preserving on Baire subsets of the adelic algebra. The element of the discrete algebra defines an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself. Hecke operators are commuting normal transformations of the Hilbert space into itself which map invariant subspaces for the group of invertible elements into themselves when elements of the discrete algebra with conjugate as inverse act trivially. Heckers operators commute with the Fourier transformation.

Invariant subspaces for the group of invertible elements of the adelic algebra are defined as the closed span of eigenfunctions of Hecke operators for eigenvalues which are coefficients of a zeta function having an Euler product and satisfying a functional identity with a gamma function factor. With notable exceptions the zeta function has an analytic extension to the complex plane from the half-plane to the right of the unit where it is defined by the Euler product. Exceptional zeta functions have an analytic extension to the complex plane with the exception of a simple pole at the unit. The vertical line of symmetry for the functional identity lies half way between the imaginary axis and the boundary of the half-plane of convergence for the Euler product. The Riemann hypothesis is generalized as the conjecture that the zeta function has no zeros to the right of the line of symmetry.

A proof of the conjecture is proposed for nonsingular zeta functions and also in modified form for singular zeta functions.

The proposed proof does not apply to the Euler zeta function, which is obtained when the discrete algebra is replaced by the field of rational numbers. It is not known whether the proof can be adapted to the less structured algebra. Such an argument is unnecessary since the conjecture made by Riemann is a consequence of its generalization.

The substance of the proposed proof lies in analysis rather than in algebra. An application is made of the theory [1] of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z-\overline{w})/(z-w)$$

belongs to the space and has the same scalar self-product as $F(z)$.

(H2) A continuous linear functional is defined on the space by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The conjugate function

$$F^*(z) = F(\overline{z})^-$$

of z belongs to the space whenever the function $F(z)$ of z belongs to the space, and has the same scalar self-product as $F(z)$.

A Hilbert space of entire functions which satisfies the axioms (H1),(H2), and (H3) is said to be a Stieltjes space since such spaces first appeared in the Stieltjes structure theory of positive linear functionals on polynomials.

An example of a Stieltjes space is defined by an entire function

$$E(z) = A(z) - iB(z)$$

of z which satisfies the inequality

$$|E(x-iy)| < |E(x+iy)|$$

for all real x when y is positive. The entire functions $A(z)$ and $B(z)$ of z are self-conjugate. The space is the set of entire functions $F(z)$ of z such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

converges and which satisfy the inequality

$$|F(z)|^2 \leq \|F\|^2 K(z,z)$$

for all complex numbers z where

$$K(w,z) = [B(z)A(\overline{w}) - A(z)B(\overline{w})]/[\pi(z-\overline{w})]$$

for all complex numbers z and w.

The function $K(w,z)$ of z belongs to the space for all complex numbers w and acts as reproducing kernel function for function values at w.

Every Stieltjes space which contains a nonzero element is isometrically equal to the Stieltjes space defined by some entire function $E(z)$. The function is not unique.

The function $E(z)$ and the elements of the defined space are polynomials in the application due to Stieltjes. In this case the Stieltjes space determines a nested family of Stieltjes spaces which are contained isometrically in the given space. The Stieltjes space determined by a positive linear functional on polynomials belongs to a maximal totally ordered family of Stieltjes spaces. There may be no greatest member of the family. There always exists a nonnegative measure on the Baire subsets of real line such that the Stieltjes spaces are contained isometrically in the Hilbert space of square integrable functions with respect to the measure. The measure need not be unique. Nonuniqueness occurs when, and only when, the family of Stieltjes spaces has a greatest element.

The axiomatization of Stieltjes spaces permits generalization to spaces of infinite dimension whose elements need not be polynomials. It is sufficient to have entire functions. In this case it is essential to treat a phenomenon which appears in spaces of finite dimension but seems of no importance in these examples.

When a Stieltjes space is defined by an entire function $E(z)$, multiplication by z is the transformation in the space which takes $F(z)$ into $zF(z)$ whenever the functions of z belong to the space.

The closure of the domain of multiplication by z is a Stieltjes space which is contained isometrically in the given space. Notation is required when a new Stieltjes space containing a nonzero element is created.

There is then a Stieltjes space with defining function

$$E(a, z) = A(a, z) - iB(a, z)$$

which is contained isometrically in a Stieltjes space with defining function

$$E(b, z) = A(b, z) - iB(b, z)$$

and whose orthogonal complement has dimension one.

The defining functions of the Stieltjes spaces can be chosen so that an element

$$A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$$

of norm one in the orthogonal complement is defined by the same complex numbers u and v. The product

$$\overline{v}u = \overline{u}v$$

is then real.

It is common knowledge that orthogonal polynomials define continued fractions. Continued fractions are best treated as infinite products of two-by-two matrices.

A matrix

$$M(a, b, z) = \begin{pmatrix} A(a, b, z) & B(a, b, z) \\ C(a, b, z) & D(a, b, z) \end{pmatrix}$$

appears when the defining functions of the Stieljes spaces are chosen with value one at the origin. The identity

$$(A(b, z), B(b, z)) = (A(a, z), B(a, z))M(a, b, z)$$

is satisfied with

$$A(a, b, z) = 1 - \beta z, \quad B(a, b, z) = \alpha z$$

and

$$C(a,b,z) = \gamma z, \quad D(a,b,z) = 1 + \beta z$$

where

$$\alpha = \pi \bar{u} u, \quad \beta = \pi \bar{u} v = \pi \bar{v} u, \quad \gamma = \pi \bar{v} v$$

Stieltjes spaces now appear which are contained contractively in the Stieltjes space defined by $E(b,z)$ and which contain contractively the Stieltjes space defined by $E(a,z)$. If the parameters a and b are positive numbers such that $a < b$, a Stieltjes space is defined by an entire function

$$E(t,z) = A(t,z) - iB(t,z)$$

when $a \leq t \leq b$. The functions $A(t,z)$ and $B(t,z)$ are defined by linearity in t to have the given values when $t = a$ and when $t = b$.

The recurrence relations for Stieltjes spaces of finite dimension are replaced by differential equations which apply in arbitrary dimensions. The differential equations are stated for a family of Stieltjes spaces with defining functions

$$E(t,z) = A(t,z) - iB(t,z)$$

which are parametrized by positive numbers t so that the space with parameter a is contained in the space with parameters b when $a < b$. The inclusion is contractive and is isometric on the closure of the domain of multiplication by z.

The functions $A(t,z)$ and $B(t,z)$ are absolutely continuous functions of t for every complex number z which satisfy the differential equations

$$B'(t,z) = zA(t,z)\alpha'(t) + zB(t,z)\beta'(t)$$

and

$$-A'(t,z) = zA(t,z)\beta'(t) + zB(t,z)\gamma'(t)$$

for coefficients defined by absolutely continuous functions and $\alpha(t), \beta(t)$ and $\gamma(t)$ of positive t with real values. The matrix

$$m'(t) = \begin{pmatrix} \alpha'(t) & \beta'(t) \\ \beta'(t) & \gamma'(t) \end{pmatrix}$$

is nonnegative for almost all t. The solution $B(t,z)$ has value zero at the origin. The solution $A(t,z)$ value one at the origin.

Matrix notation is advantageous for the treatment of the differential equations. Two-by-two matrices with complex entries act on the right of row vectors with two complex entries and on the left of column vectors with two complex entries. A bar denotes the conjugate transpose of a row or column vector as well as a matrix. The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a generalization of the imaginary unit.

The differential equation reads

$$(A'(t,z), B'(t,z))I = z(A(t,z), B(t,z))m'(t).$$

The reproducing kernel function

$$[B(t,z)A(t,w)^{-} - A(t,z)B(t,w)^{-}]/[\pi(z - \overline{w})]$$

is an absolutely continuous function of t with derivative

$$\pi^{-1}(A(t,z),B(t,z))m'(t)(A(t,w),B(t,w))^{-}.$$

The increment

$$[B(b,z)]A(b,w)^{-} - A(b,z)B(b,w)^{-}]/[\pi(z-\overline{w})]$$
$$-[B(a,z)A(a,w)^{-} - A(a,z)B(a,w)^{-}]/[\pi(z-\overline{w})]$$

in reproducing kernel functions is the integral

$$\int_a^b (A(t,z),B(t,z))dm(t)(A(t,w),B(t,w))^{-}.$$

A Hilbert space $L^2(m)$ is defined whose elements are equivalence classes of pairs $(f(t),g(t))$ of Baire functions of positive t such that the integral

$$\int_0^\infty (f(t),g(t))dm(t)(f(t),g(t))^{-} < \infty$$

converges. The integral defines the scalar self-product. Equivalence of pairs means that the scalar self-product of the difference vanishes.

The elements of the Stieltjes space with parameter c are entire functions

$$F(z) = \int_0^c (A(t,z)B(t,z))m'(t)(f(t)^{-},g(t)^{-})^{-}$$

of z which are represented by pairs $(f(t),g(t))$ of Baire functions of t such that the integral

$$\int_0^c (f(t),g(t))m'(t)(f(t),g(t))^{-} < \infty$$

converges. The integral is equal to π times the scalar self-product of the entire function in the Stieltjes space defined by $E(c,z)$.

The representation of elements of a Stieltjes space as integrals is a generalization of the Fourier transformation for the real line. The Stieltjes space defined by

$$E(c,z) = \exp(-icz)$$

is the Paley-Wiener space of entire functions $F(z)$ of exponential type at most c which are square integrable on the real axis. An element of the space is represented as the Fourier integral

$$F(z) = \int_{-c}^{c} \exp(-itz)f(t)dt$$

of a square integrable functions $f(t)$ of real t which vanishes outside of the interval $(-c,c)$. The Plancherel formula

$$\int_{-\infty}^{\infty} |F(t)|^2 dt = \pi \int_{-c}^{c} |f(t)|^2 dt$$

holds.

The Paley-Wiener formulation of the Fourier transformation is a variant of a formulation due to Hardy, who introduced the unweighted Hardy space for that purpose. The elements of the space are the analytic functions $F(z)$ of z in the upper half-plane such that the integral

$$\int_{-\infty}^{+\infty} |F(t+iy)|^2 dt$$

converges when y is positive and is a bounded function of y. As y decreases to zero, the integral increases to the scalar self-product of the element of the Hardy space. The element of the space is represented as the Fourier integral

$$F(z) = \int_0^\infty \exp(-itz) f(t) dt$$

of a square integrable function$f(t)$ of positive t. The integral

$$\pi \int_0^\infty |f(t)|^2 dt$$

is equal to the scalar self-product of the element of the Hardy space.

The Hardy formulation of the Fourier transformation admits a generalization for weighted Hardy spaces. An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half-plane. The weighted Hardy space $\mathcal{F}(W)$ is the set of functions $F(z)$ of z analytic in the upper half-plane such that the integral

$$\int_{-\infty}^\infty |F(t+iy)/W(t+iy)|^2 dt$$

converges when y is positive and is a bounded function of y. As y decreases to zero, the integral increases to the scalar self-product of the function in the weighted Hardy space. Multiplication by $W(z)$ is an isometric transformation of the unweighted Hardy space onto the weighted Hardy space.

When an analytic weight function $W(z)$ is given, there may exist a nontrivial entire function $F(z)$ of z such that for some real number τ the functions

$$\exp(i\tau z)F(z)$$

and

$$\exp(i\tau z)F^*(z)$$

of z in the upper half-plane belong to the weighted Hardy space. The set of all such functions is then a Stieltjes space which is mapped isometrically into the weighted Hardy space on multiplication by $W(z)$. A least real number τ exists which defines such a multiplication.

A parametrized family of Stieltjes spaces defined by entire functions

$$E(t,z) = A(t,z) - iB(t,z)$$

exists such that for every positive number t a least real number $\tau(t)$ exists such that multiplication by $\exp(i\tau(t)z)$ is a contractive transformation of the Stieltjes space defined by $E(t,z)$ into the weighted Hardy space and is isometric on the closure of the domain of multiplication by z in the space. The image of the Stieltjes space defined by $E(a,z)$ is contained in the image of the Stieltjes space defined by $E(b,z)$ when $a < b$. The function $\tau(t)$ of t, which is nondecreasing and absolutely continuous, is a solution of the differential equation

$$\tau'(t)^2 = \alpha'(t)\gamma'(t) - \beta'(t)^2.$$

An analytic weight function

$$W_\infty(z) = \lim \exp(i\tau(t)z)E(t,z)$$

is obtained as a limit uniformly on compact subsets of the upper half-plane as t increases to infinity. Multiplication by

$$\exp(i\tau(t)z)$$

is a contractive transformation of the Stieltjes space defined by $E(t, z)$ into the weighted Hardy space defined by $W_\infty(z)$ and is isometric on the closure of the domain of multiplication by z. The weighted Hardy space $\mathcal{F}(W_\infty)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ but can be a proper subspace of $\mathcal{F}(W)$. The analytic weight function $W_\infty(z)$ is said to be the scattering function of the parametrized family of Stieltjes spaces defined by the analytic weight function $W(z)$.

Two fundamental problems arise in the construction of Stieltjes spaces by analytic weight functions: 1) When does an analytic weight function define a parametrized family of Stieltjes spaces? 2) When is an analytic weight function the scattering function of the parametrized family of Stieltjes spaces which it defines?

Both problems are solved under a hypothesis which is relevant to the Riemann hypothesis. An analytic weight function $W(z)$ is said to be an Euler weight function if for every h in the interval $-1 \leq h \leq 1$ a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$< a, b > + < b, a > \geq 0$$

of conjugate scalar products is nonnegative for every element (a, b) of its graph.

An accretive linear relation is said to be maximal accretive if its graph is not a proper vector subspace of the graph of an accretive linear relation with domain and range in the same Hilbert space.

A linear transformation with domain and range in a Hilbert space is said to be maximal accretive if it is maximal accretive as a linear relation.

Every Euler weight function defines a parametrized family of Stieltjes spaces and is the scattering function of the family of spaces which it defines.

The defined Stieltjes spaces inherit maximal accretive transformations. When $-1 \leq h \leq 1$, a maximal accretive transformation is defined in the Stieltjes space defined by $E(t, z)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

An Euler weight function is relevant to the Riemann hypothesis for a nonsingular zeta function because the weight function has an analytic extension without zeros to the half-plane $iz^- - iz > -1$.

An Euler weight function is not relevant to a singular zeta function since the singularity of the zeta function at the unit is a denial of analyticity in a half-plane containing the unit.

A generalization of the concept of Euler weight function applies when an analytic weight function $W(z)$ satisfies the symmetry condition

$$W(z) = W^*(-z)$$

which results from real values on the imaginary axis.

An accretive transformation which is almost maximal is defined in the weighted Hardy space $\mathcal{F}(W)$ when $-1 \leq h \leq 1$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

Almost maximal means that the graph of the transformation is a closed vector subspace of codimension at most one in the graph of a maximal accretive relation.

The analytic weight function has an analytic extension without zeros to the region obtained from the half-plane $i\overline{z} - iz > -1$ by omitting the origin.

These properties of analytic weight functions indicate a strengthening of the Riemann conjecture. A linear change of the independent variable is made so that the half-plane to the right of the unit is mapped onto the upper half-plane so as to map the unit into the origin.

The zeta function is multiplied by the gamma function factor required for the functional identity. The product is mapped into an analytic weight function since the half-plane to the right of the unit is the region of convergence for the Euler product. The analytic weight function has real values on the imaginary axis when the zeta function has real values on the real axis. The line of symmetry for the functional identity is mapped into the line $i\overline{z} - iz = -1$. Since the zeta function is analytic in the complex plane (with the possible exception of a singularity at the unit), the analytic weight function is analytic in the complex plane (with the possible exception of a singularity at the origin). The Riemann hypothesis is equivalent to the conjecture that the analytic weight function has no zeros in the half-plane $i\overline{z} - iz > -1$.

The proposed proof of the Riemann hypothesis asserts that the analytic weight function is an Euler weight function when the zeta function is nonsingular and that it is a generalization of an Euler weight function when the zeta function is singular.

The conjectured properties of these analytic weight functions are treated as properties of the Stieltjes spaces which they define. The Stieltjes spaces apply to Fourier analysis on the adelic algebra.

The Hilbert space of square integrable functions with respect to the canonical measure for the adelic algebra is decomposed into invariant subspaces under the action of the compact group of elements with conjugate as inverse. The Stieltjes spaces apply to Fourier analysis on an invariant subspace.

A zeta function defines a smaller subspace which is invariant under the larger group of invertible elements. The subspace is the span of the eigenvectors of Hecke operators for the eigenvalues defining the Dirichlet coefficients of the zeta function. The subspace is mapped into itself by the Fourier transformation. The Stieltjes spaces apply to Fourier analysis in the subspace defined by the zeta function.

The origin of the adelic algebra is contained in a nested family of open sets which are invariant under invertible elements of the adelic algebra. Each open set is the Cartesian product of an open subset of the hypercomplex algebra and a open subset of the adic algebra of equal canonical measure. An element of the set is determined by its component in the hypercomplex algebra, which is a Hilbert space. A positive number c exists such that the elements of the hypercomplex algebra obtained are the elements whose scalar self-product is less than c. The open subset of the adelic algebra is said to be a disk of radius c.

For every positive rational number t an isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure for the adelic algebra into itself is defined by taking a function $f(\xi)$ of ξ into the function $f(t\xi)$ of ξ. The invariant subspace of functions which vanish in the disk of radius c is mapped into itself by the transformation when $t > 1$. The spectral analysis of the transformation in the subspace is made by an interpretation of the Mellin transformation which is due to Hardy.

For every positive number c the Mellin transformation maps the set of elements of the invariant subspace defined by c onto a Hilbert space of functions analytic in the upper half-plane which is mapped isometrically onto the weighted Hardy space on multiplication by c^{iz}.

The domain of the Mellin transformation is mapped isometrically into itself by the transformation which takes $f(\xi)$ into $f(t\xi)$ when $t > 1$. The range is mapped isometrically into itself on multiplication by t^{iz}. The transformations are unitarily equivalent under the Mellin transformation since it is a constant multiple of an isometric transformation.

A related construction of entire functions is another interpretation of the Mellin transformation which is due to Hardy. A closed invariant subspace is defined for every positive number c as the set of functions which vanish in the disk of radius c and whose Fourier transform also vanishes in the disk.

The Mellin transformation maps the subspace onto the set of entire functions $F(z)$ of z such that the functions

$$c^{iz}F(z)$$

and

$$c^{iz}F^*(z)$$

of z in the upper half-plane belong to the weighted Hardy space.

Hardy knew that nontrivial entire functions are constructed as Mellin transforms for every positive number c, but neither he nor his student Titchmarsh treated the consequences of the construction. An awareness of consequences is shown by Titchmarch in his treatment of eigenfunction expansions of ordinary differential equations of the second order. Stieltjes spaces of entire functions are constructed with applications to the spectral theory of differential equations.

The Mellin transformation maps the space of functions which vanish in the disk of radius c and whose Fourier transform vanishes in the disk onto a Stieltjes space which is mapped isometrically into the weighted Hardy space on multiplication by c^{iz}. The infinitessimal generator of the semigroup of isometric transformation in the domain is unitarily equivalent to the transformation which takes $F(z)$ into $zF(z)$ whenever the functions of z belongs to the Stieltjes space.

An issue underlying the Riemann hypothesis is the construction of all functions in the domain which vanish in the disk of radius c and whose Fourier transform vanishes in the disk. The construction of all such functions is equivalent to a solution of the structure problem for the constructed Stieltjes spaces.

For a proof of the Riemann hypothesis it remains to construct a transformation which is maximal accretive in the case of a nonsingular zeta function and which is nearly maximal accretive in the case of a singular zeta function.

The construction of the transformation is made in Fourier analysis on a locally compact Abelian group which is the Cartesian product of a locally compact Abelian group with itself. The complex plane is a fundamental example of such a group.

The complex plane is the Cartesian product of the real axis and the imaginary axis. The canonical measure for the complex plane is the Cartesian product measure of Lebesgue measure on the axes. The complex plane admits a conjugation analogous to the conjugation of the complex algebra. A simplification of multiplicative groups occurs since they are commutative.

The Fourier transformation for the complex plane is the unique isometric transformation of the Hilbert space of square integrable functions with respect to the canonical measure into itself which takes an integrable function $f(\xi)$ of ξ into the continuous function

$$g(\xi) = \int \exp(\pi i(\eta^- \xi + \xi^- \eta)) f(\eta) d\eta$$

of ξ defined by integration with respect to the canonical measure. The inverse transformation is the adjoint defined by conjugation of the Fourier kernel

$$\exp(\pi i(\eta^- \xi + \xi^- \eta)).$$

An isometric transformation of the Hilbert space of square integrable functions with respect the canonical measure into itself is defined by taking a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the complex plane with conjugate as inverse. The transformation commutes with the Fourier transformation.

An invariant subspace for the transformation is defined for every integer ν as the set of functions $f(\xi)$ of ξ which satisfy the identity

$$f(\omega\xi) = \omega^\nu f(\xi)$$

for every element ω of the complex plane with the conjugate as inverse. Such functions are said to be of order ν. The set of functions of order ν is mapped into itself by the Fourier transformation. It can be assumed that ν is nonnegative since an isometric transformation of the space of function of order ν onto the space of functions of order -ν is defined by taking a function $f(\xi)$ of ξ into the function $f(\xi^-)$ of ξ.

A function

$$f(\xi) = \xi^\nu h(\xi^- \xi)$$

of order ν is parametrized by a function $h(\xi)$ of ξ in the real line which satisfies the identity

$$h(\omega\xi) = \omega^\nu h(\xi)$$

for every element ω of the real line which is its own inverse. The identity

$$\int |f(\xi)|^2 d\xi = \pi \int |h(\xi)|^2 |\xi|^\nu d\xi$$

holds with integration on the left with respect to the canonical measure for the complex plane and integration on the right with respect to the canonical measure for the real line over the positive half-line.

A Radon transformation of order ν is defined in the Hilbert space of functions of order ν. The transformation commutes with the transformation which takes a function $f(\xi)$ of ξ into the function $f(\omega\xi)$ of ξ for every element ω of the complex plane with conjugate as inverse.

The transformation takes a function $f(\xi)$ of ξ into a function $g(\xi)$ of ξ which needs definition only on the positive half-line. When ξ is positive,

$$\xi^{-\nu} g(\xi) = \int (\xi + it)^{-\nu} f(\xi + it) dt$$

is an integral with respect to the canonical measure for the real line.

The definition is applied however only when the integral is absolutely convergent and the resulting function is square integrable with respect to the canonical measure for the complex plane. The transformation is otherwise defined to have a closed graph.

A fundamental example of a function

$$f(\xi) = \xi^\nu \exp(\pi i z \xi^- \xi)$$

of ξ which belongs to the domain of the Radon transformation of order ν is defined when z is in the upper half-plane. The function is an eigenfunction of the Randon transformation for the eigenvalue

$$(i/z)^{\frac{1}{2}}$$

for the square root having real part positive.

The maximal accretive property of the Radon transformation of order ν is verified by spectral decomposition. The Laplace transformation of order ν maps the Hilbert space of functions of order ν onto a Hilbert space of functions $F(z)$ of z analytic in the upper half-plane which belong to the unweighted Hardy space when $\nu = 0$ and which are otherwise characterized by convergence of the integral

$$\int_0^\infty \int_{-\infty}^{+\infty} |F(x+iy)|^2 y^{\nu-1} dx dy < \infty.$$

The Laplace transformation is a constant multiple of a isometry. The Radon transformation is unitarily equivalent to the maximal accretive transformation which takes $F(z)$ into $(i/z)^{\frac{1}{2}} F(z)$ whenever the functions of z belong to the space.

The construction of a maximal accretive transformation is similar for the hypercomplex algebra. The maximal accretive transformation for the adic algebra is more structured since it is self-adjoint and nonnegative. In the case of the adelic algebra the spectral analysis of the Radon transformation applies a sum of quadratic exponentials which is a generalization of a Jacobian theta function.

Complications of notation are an obstacle in the way of realizing the proposal for a proof of the Riemann hypothesis. The production of the manuscript is tedious since it has to be readable as algebra as well as analysis.

A preview of the proposed research was well received at the Workshop Hilbert Spaces of Entire Functions held at the Polish conference center in Bodlewo in the week 22-26 May 2017. The presentation was prepared in the week 8-12 May 2017 at the Chebyshev Laboratory of St. Petersburg State University. These invitations reveal international interest in applications of Stieljes spaces which support the present extension of the theory.

The axiomatization of integration on the real line which is due to Stieltjes and which is pursued in the Stieltes spaces creates interest in a measure problem due to Banach.

Banach conjectured that nonnegative measure which is defined and finite on all subsets of the real line vanishes identically if it vanishes on the complement of every countable set. No solution is known which makes no hypothesis in set theory.

The Banach problem and the Stieltjes spaces return to foundations of analysis laid in the twentieth century.

References

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