RESEARCH PROPOSAL
RIEMANN HYPOTHESIS
LOUIS DE BRANGES

A proof of the Riemann hypothesis is proposed for the Euler zeta function $\zeta(s)$ which is defined in the half-plane $Rs > 1$ as a product

$$\zeta(s) = \Pi(1 - p^{-s})^{-1}$$

over the primes p and which is analytic in the complex plane with the exception of a simple pole at $s = 1$. The Euler product denies zeros in the half-plane of convergence. The Riemann hypothesis [7] is the conjecture that the zeta function has no zeros in the larger half-plane $Rs > \frac{1}{2}$. The conjecture validates the Legendre estimation of the number of primes less than a given positive number.

The extension problem is treated by change of variable in another half-plane. An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half-plane.

An Euler weight function is an analytic weight function $W(z)$ such that for every h in the interval $-1 \leq h \leq 1$ the function

$$W(z + \tfrac{1}{2}ih) \big/ W(z - \tfrac{1}{2}ih)$$

of z is analytic and has nonnegative real part in the upper half-plane.

An Euler weight function has an analytic extension without zeros to the larger half-plane $iz^- - iz > -1$. The analytic weight function

$$W(z) = \zeta(1 - iz)$$

is not an Euler weight function since it has a singularity at the origin. However factors of the zeta function do define Euler weight functions. The analytic weight function

$$W(z) = (1 - p^{-1+2iz})^{-1}$$

is an Euler weight function for every prime p. It appears in Fourier analysis on a p-adic plane. There is an analogy with Fourier analysis on the complex plane:

The Euler weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

defined by the gamma function appears in Fourier analysis on the complex plane. A problem which originates with N. Sonine [6] is to find all functions which vanish in a given neighborhood of the origin and whose Fourier transform vanishes in the same neighborhood.

A function $f(z)$ of z in the complex plane is said to be of order ν if

$$f(\omega z) = \omega^\nu f(z)$$

for every element ω of the unit circle $\omega^- \omega = 1$. The Fourier transformation for the complex plane takes functions of order ν into functions of order ν and commutes with the transformation $f(z)$ into $f(z^-)$ which takes functions of order ν into functions of order $-\nu$. It is sufficient to treat functions of nonnegative order ν.

A function

$$f(z) = h(z^- z) z^\nu$$

of nonnegative order ν is parametrized by a function $h(t)$ of $t > 0$. The Hankel transformation of order ν reformulates the Fourier transformation as it acts on functions of order ν. The Hankel transformation of order ν takes functions of a positive variable into functions of a positive variable and is its own inverse.

Hardy and Titchmarsh [4] define a self-reciprocal function of order ν as a function which is its own Hankel transform of order ν, a skew-reciprocal function of order ν as a function which is minus its own Hankel transform of order ν. The results of N. Sonine are presented as examples of self-reciprocal functions of order ν and skew-reciprocal functions of order ν which vanish in a given interval $(0, a)$.

The Sonine problem is treated for the Hankel transformation of order zero by Virginia Rovnyak in her doctoral dissertation [5]. The Fourier transformation is applied to functions which are square integrable with respect to plane measure. Functions of order zero are parametrized by functions which are square integrable with respect to Lebesgue measure. The formal Sonine construction is implemented by isometric transformations in Hilbert spaces of square integrable functions. A construction is made of all self-reciprocal and all skew-reciprocal functions which are square integrable with respect to Lebesgue measure and which vanish in a given interval $(0, a)$.

Her results clarify a solution of the Sonine problem for the Hankel transformation of order zero, obtained by the present investigator [1], as an application of the theory [2] of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever an entire function $F(z)$ of z belongs to the space and has a nonreal zero w, the entire function

$$F(z)(z - w^-)/(z - w)$$

of z belongs to the space and has the same scalar self-product as $F(z)$.

(H2) A continuous linear functional is defined on the space by taking a function $F(z)$ of z into its value $F(w)$ at w for every nonreal number w.

(H3) The conjugate function

$$F^*(z) = F(z^-)^-$$

of z belongs to the space whenever the function $F(z)$ of z belongs to the space, and it has the same scalar self-product as $F(z)$.

A Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3) is said to be a Stieltjes space since such spaces first appeared in the Stieltjes integral representation of positive linear functionals on polynomials.

An example of a Stieltjes space is defined by an entire function

$$E(z) = A(z) - iB(z)$$

of z which satisfies the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for all real x when y is positive. The entire functions $A(z)$ and $B(z)$ of z are self-conjugate. The space is the set of entire functions $F(z)$ of z such that the integral

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

converges and which satisfy the inequality

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

for all complex numbers z, where

$$K(w, z) = [B(z)A(w^-) - A(z)B(w^-)]/[\pi(z - w^-)]$$

for all complex numbers z and w.

The function $K(w, z)$ of z belongs to the space for all complex numbers w and acts as reproducing kernel function for function values at w.

Every Stieltjes space which contains a nonzero element is isometrically equal to the Stieltjes space defined by some entire function $E(z)$. The function is not unique.

The function $E(z)$ and the elements of the defined space are polynomials in the application due to Stieltjes. In this case the Stieltjes space determines a nested family of Stieltjes spaces which are contained isometrically in the given space. The Stieltjes spaces determined by a positive linear functional on polynomials belong to a maximal totally ordered family of Stieltjes spaces. There may be no greatest member of the family. There always exists a nonnegative measure on the Baire subsets of real line such that the Stieltjes spaces are contained isometrically in the Hilbert space of square integrable functions with respect to the measure. The measure need not be unique. Nonuniqueness occurs when, and only when, the family of Stieltjes spaces has a greatest element.

The axiomatization of Stieltjes spaces permits generalization to spaces of infinite dimension whose elements are entire functions.

When a Stieltjes space is defined by an entire function $E(z)$, multiplication by z is the transformation which takes $F(z)$ into $zF(z)$ whenever the functions of z belong to the space.

The closure of the domain of multiplication by z is a Stieltjes space which is contained isometrically in the given space. Notation is required when a new Stieltjes space containing a nonzero element is created.

There is then a Stieltjes space with defining function

$$E(a, z) = A(a, z) - iB(a, z)$$

which is contained isometrically in a Stieltjes space with defining function

$$E(b, z) = A(b, z) - iB(b, z)$$

and whose orthogonal complement has dimension one.

The defining functions of the Stieltjes spaces can be chosen so that an element

$$A(a, z)u + B(a, z)v = A(b, z)u + B(b, z)v$$

of norm one in the orthogonal complement is defined by the some complex numbers u and v. The product

$$v^- u = u^- v$$

is then real.

Orthogonal polynomials define continued fractions, which are best treated as infinite products of two-by-two matrices.

A matrix

$$M(a,b,z) = \begin{pmatrix} A(a,b,z) & B(a,b,z) \\ C(a,b,z) & D(a,b,z) \end{pmatrix}$$

appears when the defining functions of the Stieljes spaces are chosen with value one at the origin. The identity

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

is satisfied with

$$A(a,b,z) = 1 - \beta z, \quad B(a,b,z) = \alpha z$$

and

$$C(a,b,z) = \gamma z, \quad D(a,b,z) = 1 + \beta z$$

where

$$\alpha = \pi u^- u, \quad \beta = \pi u^- v = \pi v^- u, \quad \gamma = \pi v^- v.$$

Stieltjes spaces now appear which are contained in the Stieltjes space defined by $E(b,z)$ and which contain the Stieltjes space defined by $E(a,z)$. If the parameters a and b are positive numbers such that $a < b$, a Stieltjes space is defined by an entire function

$$E(t,z) = A(t,z) - iB(t,z)$$

when $a \leq t \leq b$. The functions $A(t,z)$ and $B(t,z)$ are defined by linearity in t to have the given values when $t = a$ and when $t = b$.

The recurrence relations for Stieltjes spaces of finite dimension are replaced by differential equations which apply in arbitrary dimensions. The differential equations are stated for a family of Stieltjes spaces with defining functions

$$E(t,z) = A(t,z) - iB(t,z)$$

which are parametrized by positive numbers t so that the space with parameter a is contained in the space with parameters b when $a < b$. The inclusion is contractive and is isometric on the closure of the domain of multiplication by z.

The functions $A(t,z)$ and $B(t,z)$ are absolutely continuous functions of t for every complex number z which satisfy the differential equations

$$B'(t,z) = zA(t,z)\alpha'(t) + zB(t,z)\beta'(t)$$

and

$$-A'(t,z) = zA(t,z)\beta'(t) + zB(t,z)\gamma'(t)$$

for absolutely continuous functions $\alpha(t), \beta(t)$ and $\gamma(t)$ of positive t with real values. The matrix

$$m'(t) = \begin{pmatrix} \alpha'(t) & \beta'(t) \\ \beta'(t) & \gamma'(t) \end{pmatrix}$$

is nonnegative for almost all t. The solution $B(t,z)$ has derivative zero at the origin as a function of z. The solution $A(t,z)$ value one at the origin.

Matrix notation is advantageous for the treatment of the differential equations. Two-by-two matrices with complex entries act on the right of row vectors with two complex entries and on the left of column vectors with two complex entries. A bar denotes the conjugate transpose of a row or column vector as well as of a matrix. The matrix

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a generalization of the imaginary unit.

The differential equation reads

$$(A'(t,z), B'(t,z))I = z(A(t,z), B(t,z))m'(t).$$

The reproducing kernel function

$$[B(t,z)A(t,w)^- - A(t,z)B(t,w)^-]/[\pi(z-w^-)]$$

is an absolutely continuous function of t with derivative

$$\pi^{-1}(A(t,z), B(t,z))m'(t)(A(t,w), B(t,w))^-.$$

The increment

$$\begin{gathered}[B(b,z)A(b,w)^- - A(b,z)B(b,w)^-]/[\pi(z-w^-)] \\ -[B(a,z)A(a,w)^- - A(a,z)B(a,w)^-]/[\pi(z-w^-)]\end{gathered}$$

in reproducing kernel functions is the integral

$$\int_a^b (A(t,z), B(t,z))dm(t)(A(t,w), B(t,w))^-.$$

A Hilbert space $L^2(m)$ is defined whose elements are equivalence classes of pairs $(f(t), g(t))$ of Baire functions of positive t such that the integral

$$\int_0^\infty (f(t), g(t))dm(t)(f(t), g(t))^- < \infty$$

converges. The integral defines the scalar self-product. Equivalence of pairs means that the scalar self-product of the difference vanishes.

The elements of the Stieltjes space with parameter a are entire functions

$$F(z) = \int_0^a (A(t,z)B(t,z))m'(t)(f(t)^-, g(t)^-)^- dt$$

of z which are represented by pairs $(f(t), g(t))$ of Baire functions of t such that the integral

$$\pi \int_0^a (f(t), g(t))m'(t)(f(t), g(t))^- dt < \infty$$

converges. The integral is equal to the scalar self-product of the entire function in the Stieltjes space defined by $E(a,z)$.

The representation of elements of a Stieltjes space as integrals is a generalization of the Fourier transformation for the real line. The Stieltjes space defined by

$$E(a,z) = \exp(-iaz)$$

is the Paley-Wiener space of entire functions $F(z)$ of exponential type at most a which are square integrable on the real axis. An element of the space is represented as the Fourier integral

$$F(z) = \int_{-a}^{a} \exp(itz) f(t) dt$$

of a square integrable functions $f(t)$ of real t which vanishes outside of the interval $(-a, a)$. The isometric property of the Fourier transformation reads

$$2\pi \int_{-\infty}^{\infty} |F(t)|^2 dt = \int_{-a}^{a} |f(t)|^2 dt.$$

An analytic weight function is a function $W(z)$ of z which is analytic and without zeros in the upper half-plane. The weighted Hardy space $\mathcal{F}(W)$ is the set of functions $F(z)$ of z analytic in the upper half-plane such that the integral

$$\int_{-\infty}^{\infty} |F(t+iy)/W(t+iy)|^2 dt < \infty$$

converges when y is positive and is a bounded function of y. As y decreases to zero, the integral increases to the scalar self-product of the function in the weighted Hardy space.

When an analytic weight function $W(z)$ is given, there may exist a nontrivial entire function $F(z)$ of z such that for some real number τ the functions

$$\exp(i\tau z) F(z)$$

and

$$\exp(i\tau z) F^*(z)$$

of z in the upper half-plane belong to the weighted Hardy space. The set of all such functions is then a Stieltjes space which is mapped isometrically into the weighted Hardy space on multiplication by $\exp(i\tau z)$.

A parametrized family of Stieltjes spaces is defined by entire functions

$$E(t, z) = A(t, z) - iB(t, z)$$

of z such that for every positive number t a least real number $\tau(t)$ exists such that multiplication by $\exp(i\tau(t)z)$ is a contractive transformation of the Stieltjes space defined by $E(t, z)$ into the weighted Hardy space and is isometric on the closure of the domain of multiplication by z in the space. The image of the Stieltjes space defined by $E(a, z)$ is contained in the image of the Stieltjes space defined by $E(b, z)$ when $a < b$. The function $\tau(t)$ of t, which is nondecreasing and absolutely continuous, satisfies the differential equation

$$\tau'(t)^2 = \alpha'(t)\gamma'(t) - \beta'(t)^2.$$

An analytic weight function

$$W_\infty(z) = \lim \exp(i\tau(t)z) E(t, z)$$

is obtained as a limit uniformly on compact subsets of the upper half-plane as t increases to infinity. Multiplication by

$$\exp(i\tau(t)z)$$

is a contractive transformation of the Stieltjes space defined by $E(t, z)$ into the weighted Hardy space defined by $W_\infty(z)$ and is isometric on the closure of the domain of multiplication by z. The weighted Hardy space $\mathcal{F}(W_\infty)$ is contained isometrically in the weighted Hardy space $\mathcal{F}(W)$ but can be a proper subspace of $\mathcal{F}(W)$. The analytic weight function $W_\infty(z)$ is said to be the scattering function of the parametrized family of Stieltjes spaces defined by the analytic weight function $W(z)$.

Two fundamental problems arise in the construction of Stieltjes spaces by analytic weight functions: 1) When does an analytic weight function define a parametrized family of Stieltjes spaces? 2) When is an analytic weight function the scattering function of the parametrized family of Stieltjes spaces which it defines?

Both problems are solved under a hypothesis which is relevant to the Riemann hypothesis. An analytic weight function $W(z)$ is said to be an Euler weight function if for every h in the interval $-1 \le h \le 1$ a maximal accretive transformation is defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

A linear relation with domain and range in a Hilbert space is said to be accretive if the sum

$$< a, b > + < b, a > \ \ge \ 0$$

of conjugate scalar products is nonnegative for every element (a, b) of its graph.

An accretive linear relation is said to be maximal accretive if its graph is not a proper vector subspace of the graph of an accretive linear relation with domain and range in the same Hilbert space.

A linear transformation with domain and range in a Hilbert space is said to be maximal accretive if it is maximal accretive as a linear relation.

Every Euler weight function defines a parametrized family of Stieltjes spaces and is the scattering function of the spaces which it defines.

The defined Stieltjes spaces inherit maximal accretive transformations. When $-1 \le h \le 1$, a maximal accretive transformation in the Stieltjes space defined by $E(t, z)$ takes $F(z)$ into $F(z + ih)$ whenever the functions of z belong to the space.

An Euler weight function $W(z)$ has an analytic extension to the half-plane $iz^- - iz > -1$ which satisfies the recurrence relation

$$W(z + \tfrac{1}{2}i) = W(z - \tfrac{1}{2}i)\phi(z)$$

for a function $\phi(z)$ of z which is analytic and has nonnegative real part in the upper half-plane. The recurrence relation denies the existence of zeros in the half-plane.

If a nontrivial function $\phi(z)$ of z is given which is analytic and has nonnegative real part in the upper half-plane, an Euler weight function exists which satisfies the recurrence relation for the given function $\phi(z)$. The Euler weight function is unique within a constant factor.

Euler weight functions construct Stieltjes spaces relevant to generalizations of the Riemann hypothesis [3].

The solution of the inverse problem for the Euler weight function

$$W(z) = \Gamma(\tfrac{1}{2} - iz)$$

is the original solution of the Sonine problem for the Hankel transformation of order zero.

The solution of the inverse problem for the Euler weight function

$$W(z) = (1 - p^{-1+2iz})^{-1}$$

is the solution of the analogous problem in Fourier analysis on the p-adic plane.

The solution of the inverse problem for the analytic weight function

$$W(z) = \Gamma(\tfrac{1}{2} - \tfrac{1}{2}iz)\zeta(1 - iz)$$

is the solution of the resulting problem in Fourier analysis on the composite plane.

The analytic weight function constructed from the zeta function is not an Euler weight function. The transformations defined in the weighted Hardy space $\mathcal{F}(W)$ by taking $F(z)$ into $F(z + ih)$ when $-1 \le h \le 1$ are not maximal accretive.

Since the analytic weight function $W(z)$ has real values on the imaginary axis, it is symmetric

$$W(z) = W(-z^-)^-$$

about the imaginary axis. The weighted Hardy space is mapped isometrically into itself by the conjugation $F(z)$ into $F(-z^-)^-$.

The space is the orthogonal sum of a subspace of self-conjugate functions and a subspace of skew-conjugate functions, each of which is an invariant subspace for the transformation $F(z)$ into $F(z + ih)$ when $-1 \le h \le 1$.

The restriction of each transformations to the subspace of skew-conjugate functions is maximal accretive. This information denies zeros of $W(z)$ in the half-plane $iz^- - iz > -1$.

References

[1] L. de Branges, *Self-reciprocal functions*, J. Math. Anal. Appl. **9** (1964), 433-457.
[2] L. de Branges, *Hilbert Space of Entire Functions*, Prentice-Hall, Englewood Cliffs, N.J., 1968
[3] L. de Branges, *The Riemann hypothesis for Hilbert spaces of entire functions*, Bull. Amer. Math. Soc. **15** (1986), 1-17.
[4] G.H. Hardy and E.C. Titchmarsh, *Self-reciprocal functions*, Quart. J. Math. (Oxford) **1** (1930), 196-231.
[5] V. Rovnyak, *Self-reciprocal functions*, Duke Math. J. **33** (1966), 363-378.
[6] N. Sonine, *Recherches sur les fonctions cylindriques et le développement des fonctions continues en séries*, Math. Annalen **16** (1880), 1-80.
[7] E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Cambridge at the University Press, 1902.