THE RIEMANN HYPOTHESIS FOR DIRICHLET ZETA FUNCTIONS

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A Dirichlet zeta function
\[ \zeta(s) = \sum \chi(n)n^{-s} \]
is defined in the half-plane \( \Re s > 1 \) as a sum over the positive integers \( n \) with \( \chi \) a primitive character modulo \( \rho \) other than the principal character modulo one. The zeta function has an analytic extension to the complex plane. The functional identity for the zeta function states that the entire functions
\[ (\rho/\pi)^{1/2 + \frac{s}{2}} \Gamma(\frac{1}{2} + \frac{s}{2}) \zeta(s) \]
and
\[ (\rho/\pi)^{1/2 + \frac{s}{2} - \frac{1}{2}} \Gamma(\frac{1}{2} + \frac{1}{2} - \frac{s}{2}) \zeta(1 - s)^{-1} \]
are linearly dependent with \( \nu \) equal to zero or one of the same parity as \( \chi \). The entire function
\[ E(z) = (\rho/\pi)^{1/2 + \frac{s}{2} - \frac{1}{2}i} \Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}iz) \zeta(1 - iz) \]
satisfies the inequality
\[ |E(x - iy)| < |E(x + iy)| \]
for \( y \) positive which permits the construction of an associated Hilbert space \( \mathcal{H}(E) \) of entire functions. A maximal dissipative transformation in the space implies the simplicity of zeros of \( E(z) \) and their position on the line \( iz^{-} - iz = -1 \). The space is constructed from Hilbert spaces of entire functions appearing the theory of the Hankel transformation of order \( \nu \).

The Laplace transformation of order \( \nu \) for the complex plane is defined when \( \nu \) is a nonnegative integer. The domain of the transformation is the set of functions \( f(\xi) \) of a complex variable \( \xi \) which satisfy the identity
\[ f(\omega \xi) = \omega^{\nu} f(\xi) \]
for every element \( \omega \) of the unit circle and which are square integrable with respect to Lebesgue measure for the plane. The Laplace transform of order \( \nu \) of the function \( f(\xi) \) of \( \xi \) in the plane is the analytic function \( g(z) \) of \( z \) in the upper half-plane defined by the integral
\[ 2\pi g(z) = \int (\xi^{\nu})^{-} f(\xi) \exp(\pi i \xi^{-} z \xi / \rho) d\xi \]
with respect to Lebesgue measure for the plane. When \( \nu \) is zero, the identity
\[
\sup_{y} \int_{-\infty}^{+\infty} |g(x + iy)|^2 dx = (2\pi/\rho) \int |f(\xi)|^2 d\xi
\]
holds with the least upper bound taken over all positive numbers \( y \). The identity
\[
\int_{0}^{\infty} \int_{-\infty}^{+\infty} |g(x + iy)|^2 y^{\nu-1} dx dy = (2\pi/\rho)^{\nu} \Gamma(\nu) \int |f(\xi)|^2 d\xi
\]
holds when \( \nu \) is positive. Integration is with respect to Lebesgue measure for the plane.

A Hilbert space of functions analytic in the upper half–plane is obtained. The elements of the space are characterized by convergent integrals. A maximal dissipative transformation in the space is defined by taking \( f(z) \) into \( (i/z)^{\frac{\nu}{2}} f(z) \) when both functions of \( z \) belong to the space. The square root is taken with positive real part.

The domain of the Hankel transformation of order \( \nu \) for the complex plane is the domain of the Laplace transformation of order \( \nu \) for the complex plane. The Hankel transformation of order \( \nu \) for the complex plane is its own inverse and satisfies the identity
\[
\int |f(\xi)|^2 d\xi = \int |g(\xi)|^2 d\xi
\]
when it takes a function \( f(\xi) \) of \( \xi \) in the complex plane into a function \( g(\xi) \) of \( \xi \) in the complex plane. The identity
\[
\int (\xi^{\nu})^{-1} g(\xi) \exp(\pi i \xi^{-z} \xi/\rho) d\xi = (i/z)^{1+\nu} \int (\xi^{\nu})^{-1} f(\xi) \exp(-\pi i \xi^{-z-1} \xi/\rho) d\xi
\]
with \( z \) in the upper half–plane defines the Hankel transformation of order \( \nu \) for the complex plane. Integration is with respect to Lebesgue measure for the plane. A fundamental property of the Hankel transformation of order \( \nu \) for the complex plane was discovered in 1880 by Nikolai Sonine. If \( a \) is a positive number, a nontrivial function \( f(\xi) \) of \( \xi \) in the complex plane exists which is in the domain of the Hankel transformation of order \( \nu \) for the complex plane, which vanishes in the neighborhood \( |\xi| < a \) of the origin, and whose Hankel transform of order \( \nu \) for the complex plane vanishes in the same neighborhood.

The Mellin transformation of order \( \nu \) for the complex plane is a spectral theory for the Laplace transformation of order \( \nu \) for the complex plane. The domain of the transformation is the domain of the Laplace transformation of order \( \nu \) for the complex plane. If a function \( g(z) \) of \( z \) in the upper half–plane is the Laplace transform of order \( \nu \) for the complex plane of a function \( f(\xi) \) of \( \xi \) in the plane which vanishes when \( |\xi| < a \), then its Mellin transform of order \( \nu \) for the complex plane is the analytic function \( F(z) \) of \( z \) in the upper half–plane defined by the integral
\[
F(z) = \int_{0}^{\infty} g(it)t^{\frac{\nu}{2}-\frac{1}{2}-\frac{i}{2}z} dt.
\]
Mellin transforms of order \( \nu \) for the complex plane are characterized using weighted Hardy spaces.

An analytic weight function is a function \( W(z) \) of \( z \) in the upper half-plane which is analytic and without zeros in the half-plane. The weighted Hardy space \( F(W) \) is the set of analytic functions \( F(z) \) of \( z \) in the upper half-plane such that a finite least upper bound

\[
\sup_{y>0} \int_{-\infty}^{+\infty} |F(x+iy)/W(x+iy)|^2 \, dx
\]

is obtained over all positive numbers \( y \). Boundary values

\[
F(x)/W(z) = \lim_{y \to 0} F(x+iy)/W(x+iy)
\]

exist almost everywhere with respect to Lebesgue measure for the real line. The least upper bound is equal to the integral

\[
\|F(t)\|^2_{F(W)} = \int_{-\infty}^{+\infty} |F(t)/W(t)|^2 \, dt
\]

with respect to Lebesgue measure for the real line.

The analytic weight function

\[
W(z) = (\rho/\pi)^{1/2-\nu} \Gamma(1/2+1/2-\nu)\Gamma(1/2\nu + 1/2 - \nu/2 iz)
\]

is applied in the characterization of Mellin transforms of order \( \nu \) for the complex plane. The Mellin transform of order \( \nu \) for the complex plane of a function \( f(\xi) \) of \( \xi \) in the complex plane which vanishes when \( |\xi| < a \) is an analytic function \( F(z) \) of \( z \) in the upper half-plane such that the function

\[
a^{-iz} F(z)
\]

belongs to the space \( F(W) \). Every element of the space \( F(W) \) is of this form. A maximal dissipative transformation in the space \( F(W) \) is defined by taking \( F(z) \) into \( F(z+i) \) whenever the functions \( F(z) \) and \( F(z+i) \) of \( z \) belong to the space \( F(W) \). The set of entire functions \( F(z) \) such that \( F(z) \) and

\[
F^*(z) = F(z^-)
\]

belong to the space \( F(W) \) is a Hilbert space which is contained isometrically in the space \( F(W) \). The elements of the space are the Mellin transforms of order \( \nu \) for the complex plane of functions in the domain of the Hankel transformation of order \( \nu \) for the complex plane which vanish in the neighborhood \( |\xi| < a \) of the origin and whose Hankel transform of order \( \nu \) for the complex plane vanishes in the same neighborhood. The Sonine spaces of order \( \nu \) for the complex plane, which are so obtained, are fundamental examples of Hilbert spaces of entire functions which have an axiomatic characterization.

Consider a Hilbert space \( \mathcal{H} \) whose elements are entire functions and which has these properties.
(H1) Whenever $F(z)$ belongs to the space and has a nonreal zero $w$, the function

$$F(z)(z-w^-)/(z-w)$$

belongs to the space and has the same norm as $F(z)$.

(H2) The linear functional on the space which takes $F(z)$ into $F(w)$ is continuous for every nonreal number $w$.

(H3) The function $F^*(z)$ belongs to the space whenever $F(z)$ belongs to the space and it always has the same norm as $F(z)$.

An example of such a space is obtained when $E(z)$ is an entire function which satisfies the inequality

$$|E(x-iy)| < |E(x+iy)|$$

for $y$ positive. A weighted Hardy space $\mathcal{F}(E)$ exists since $E(z)$ has no zeros in the upper half-plane. The space $\mathcal{H}(E)$ is the set of entire functions $F(z)$ such that $F(z)$ and $F^*(z)$ belong to the space $\mathcal{F}(E)$. The space $\mathcal{H}(E)$ is a Hilbert space which is contained isometrically in the space $\mathcal{F}(E)$. The space satisfies the axioms (H1), (H2), and (H3). The function

$$K(w,z) = [E(z)E(w^-) - E^*(z)E(w^-)]/[2\pi i(w^- - z)]$$

do not belongs to the space for every complex number $w$. The identity

$$F(w) = \langle F(t), K(w,t) \rangle_{\mathcal{H}(E)}$$

holds for every element $F(z)$ of the space. A Hilbert space whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element, is isometrically equal to a space $\mathcal{H}(E)$.

The Sonine spaces of order $\nu$ for the Euclidean plane are Hilbert spaces of entire functions which satisfy the axioms (H1), (H2), and (H3). The space of parameter $a$ contains a nonzero element for every positive number $a$. A maximal dissipative transformation in the space is defined by taking $F(z)$ into $F(z+i)$ when the functions $F(z)$ and $F(z+i)$ of $z$ belong to the space.

The Hilbert space of entire functions associated with a Dirichlet zeta function is also a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). The space is a space $\mathcal{H}(E)$ with

$$E(z) = (\rho/\pi)^{\nu+\frac{1}{2}} - \frac{1}{2}iz \Gamma(\frac{1}{2} \nu + \frac{1}{2} - i\frac{1}{2}iz) \zeta(1 - iz).$$

The functional identity for the zeta function implies that the entire functions $E(z-i)$ and $E^*(z)$ are linearly dependent. A maximal dissipative transformation in the space exists which implies that the zeros of $E(z)$ are simple and lie on the line $iz^- - iz = -1$. A construction of the space $\mathcal{H}(E)$ from the Sonine spaces of order $\nu$ for the complex plane is an interpretation of convergence of the Euler product. The maximal dissipative
transformation in the space $\mathcal{H}(E)$ is derived from the maximal dissipative transformations in the Sonine spaces of order $\nu$ for the complex plane.

The Euler product

$$\zeta(s)^{-1} = \prod (1 - \chi(p)p^{-s})$$

for the zeta function is taken over the primes which are not divisors of $\rho$. The entire function

$$1 - \chi(p)p^{-s}$$

has its zeros on the imaginary axis. The function

$$p^{\frac{1}{2} s} - \chi(p)p^{-\frac{1}{2} s}$$

is a limit of polynomials whose zeros lie on the imaginary axis. Convergence is uniform on compact subsets of the complex plane. The Euler product converges uniformly on compact subsets of the half-plane $\Re s > 1$. A construction of the space $\mathcal{H}(E)$ results from the Sonine spaces of order $\nu$ for the complex plane. The main step in the construction will be indicated.

If a space $\mathcal{H}(E_0)$ has dimension greater than $r$ and if $S(z)$ is a polynomial of degree $r$ whose zeros lie at distance at least one from the upper half-plane, then a space $\mathcal{H}(E_r)$ exists, whose elements are the entire functions $F(z)$ such that $S(z)F(z)$ belongs to the space $\mathcal{H}(E_0)$, such that multiplication by $S(z)$ is an isometric transformation of the space $\mathcal{H}(E_0)$. If a maximal dissipative transformation in the space $\mathcal{H}(E_0)$ is defined by taking $F(z)$ into $F(z + i)$ whenever the functions $F(z)$ and $F(z + i)$ of $z$ belong to the space, then a maximal dissipative transformation in the space $\mathcal{H}(E_r)$ is defined by taking $F(z)$ into $G(z + i)$ whenever the functions $F(z)$ and $G(z + i)$ are elements of the space such that the identity

$$S(z)G(z + i) = H(z + i)$$

holds for an element $H(z)$ of the space $\mathcal{H}(E_0)$ nearest $S(z)F(z)$. Entire functions $P(z)$ and $Q(z)$ exist such that the function

$$[Q(z)P(w) - P(z)Q(w)]/[\pi(z - w^{-})]$$

of $z$ belongs to the space $\mathcal{H}(E_r)$ for every complex number $w$ and such that the identity

$$G(w) = \langle F(t), [Q(t)P(w) - P(t)Q(w)]/[\pi(t - w^{-})]\rangle_{\mathcal{H}(E_r)}$$

holds for all complex numbers $w$ whenever the transformation in the space $\mathcal{H}(E_r)$ takes $F(z)$ into $G(z + i)$.

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