Lesson 11

§3.4 – Product + Quotient Rules

$$\frac{d}{dx}\{f(x)g(x)\} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= f(x)\,g'(x) + g(x)\,f'(x)$$ Hence

Product Rule: If f, g are differentiable

then $$\boxed{\frac{d}{dx}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)}$$

Ex 1 Compute $\dfrac{d^2y}{dx^2}$ if $y = (x^2 + e^x)(4x - 1)$

Soln: $y' = (x^2+e^x)(4) + (4x-1)(2x+e^x)$ ②

$$y'' = 4(2x+e^x) + (4x-1)(2+e^x) + (2x+e^x)(4)$$

Quotient Rule: If f, g are differentiable

then
$$\boxed{\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}}$$

Provided $g(x) \neq 0$

Reason: $$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - g(x+h)f(x)}{h\, g(x+h)\, g(x)}$$

$$= \lim_{h\to 0} \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{h\, g(x+h)\, g(x)}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$\boxed{\text{Ex2}}$ Find derivative :

(a) $y = \dfrac{2x^3+1}{x^2-e^x}$

$$\Rightarrow y' = \frac{(x^2-e^x)(6x^2) - (2x^3+1)(2x-e^x)}{(x^2-e^x)^2}$$

(b) $g(x) = \dfrac{1}{x^p}$, $p = 1, 2, 3, \ldots$

$$\Rightarrow g'(x) = \frac{x^p(0) - (1)\,p\,x^{p-1}}{(x^p)^2} = \frac{-p\,x^{p-1}}{x^{2p}}$$

$$= -p\,x^{-p-1}$$

$g(x) = x^{-p}$

General Power Rule: If n is any real number then
$$\boxed{\frac{d(x^n)}{dx} = n x^{n-1}}$$

Ex3 Compute $f'(x)$

(a) $f(x) = 4x + \frac{1}{\sqrt{x}} = 4x + x^{-\frac{1}{2}}$

$\Longrightarrow f'(x) = 4 + \left(-\frac{1}{2}\right) x^{-\frac{1}{2}-1} = 4 - \frac{1}{2} x^{-\frac{3}{2}}$ ✓

(b) $f(x) = \dfrac{x^2 e^x}{1+\sqrt[3]{x}}$ ← $x^{1/3}$

$$\Longrightarrow f'(x) = \frac{\left(1+x^{1/3}\right)\left[x^2e^x + e^x(2x)\right] - (x^2e^x)\left(\frac{1}{3}x^{-2/3}\right)}{\left(1+\sqrt[3]{x}\right)^2}$$

Ex4 Compute $\lim_{x\to 2} \frac{3x^5-3(32)}{x-2} = f'(2)$

where $f(x)=3x^5$

$= 15x^4\Big|_{x=2}$

$= 15(2^4)$

$= \underline{240}$

Ex5 The # of students (in hundreds) with the flu after t months is

$$F(t)=4t^2-\frac{1}{3}t^3+1$$

Find Rate of Change of F(t) when $t=5$.

<u>Soln</u>: $F'(5)=8t-t^2\Big|_{t=5} = 40-25=15$