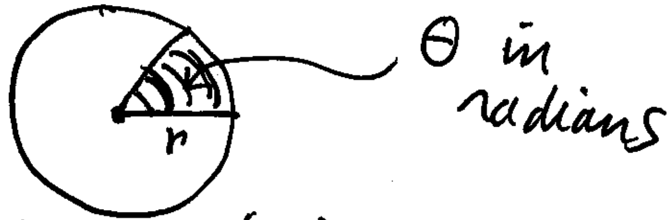
Lesson 12

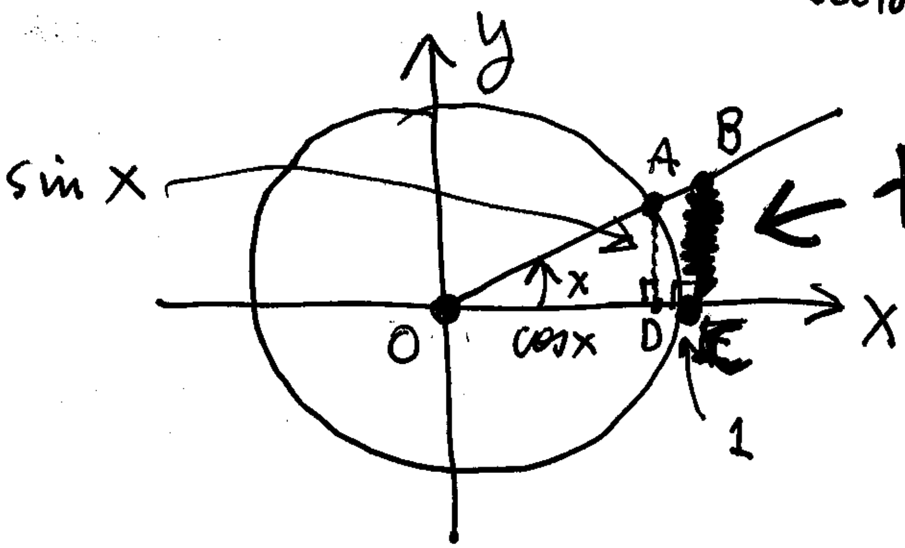
§3.5 – Derivatives of Trig Functions

Note:

$A = \pi r^2$



$$A_{sector} = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{1}{2} r^2 \theta$$



$\leftarrow \tan x = \dfrac{\sin x}{\cos x}$

area of ΔAOD < Asector < area of ΔBOC

$$\cancel{\tfrac{1}{2}}(\cos x)(\sin x) < \cancel{\tfrac{1}{2}}(1)^2 x < \cancel{\tfrac{1}{2}}(1)\frac{\sin x}{\cos x}$$

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad (\text{if } x > 0)$$

$$\frac{1}{\cos x} > \frac{\sin x}{x} > \cos x \qquad \text{As } x \to 0^+$$

$$\downarrow \qquad\quad \downarrow \qquad\quad \downarrow$$

$$1 \quad \therefore \quad 1 \qquad 1 \qquad \text{Hence}$$

Thm: If x is in radians then

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \qquad \boxed{\lim_{x\to 0} \frac{x}{\sin x} = 1}$$

$$\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0}$$

Ex.2 Compute these limits:

(a) $\displaystyle \lim_{x\to 0} \frac{\sin 5x}{3x} = \lim_{x\to 0} \left(\frac{\sin 5x}{\mathbf{5x}}\right)\left(\frac{\mathbf{5\cancel{x}}}{3\cancel{x}}\right)$

let $t = 5x$

$\displaystyle = \lim_{t\to 0} \left(\frac{\sin t}{t}\right)\left(\frac{5}{3}\right) = \frac{5}{3}$ ✓

(b) $\lim_{x \to 0} \frac{\tan 3x}{\sin \pi x} = \lim_{x \to 0} \frac{\frac{\sin 3x}{\cos 3x}}{\sin \pi x}$

$= \lim_{x \to 0} \frac{\sin 3x}{(\sin \pi x)(\cos 3x)}$

$= \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)\left(\frac{\pi x}{\sin \pi x}\right)\left(\frac{1}{\cos 3x}\right)\left(\frac{3x}{\pi x}\right)$

$t = 3x$ $\quad$ $t = \pi x$

$= \frac{3}{\pi}$ ✓

(c) $\lim_{x \to 3} \frac{\sin 2(x-3)}{8(3-x)} = \lim_{t \to 0} \frac{\sin 2t}{-8t}$

let $\boxed{t = x-3}$

$= \lim_{t \to 0} \left(\frac{\sin 2t}{2t}\right)\left(\frac{2t}{-8t}\right)$

$= (1)\left(-\frac{2}{8}\right) = -\frac{1}{4}$

Thm: If x is in radians then

$$\boxed{\frac{d(\sin x)}{dx} = \cos x} \qquad \boxed{\frac{d(\cos x)}{dx} = -\sin x}$$

Reason: $\frac{d(\sin x)}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$

$= \lim_{h \to 0} (\sin x)\left(\frac{\cos h - 1}{h}\right) + (\cos x)\left(\frac{\sin h}{h}\right)$

$= \cos x$

$\boxed{\text{Ex 2}}$ Find derivative

(a) $y = x \sin x + \cos x$

$\Rightarrow y' = x(\cos x) + (\sin x)(1) + (-\sin x)$

$y' = x \cos x$

(b) $y = \tan x = \dfrac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{(\cos x)^2} = \sec^2 x$$

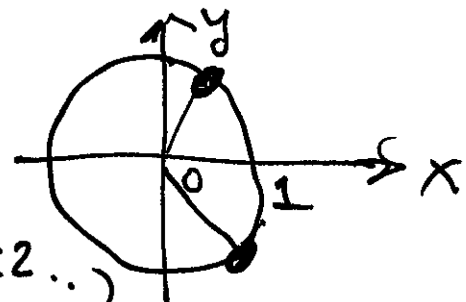
Hence: if x is in radians then

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

$\boxed{\text{Ex 3}}$ Find __all__ horizontal tangents for $f(x) = x - 2\sin x$.

__Soln__: $m_{tan} = f'(x) = 1 - 2\cos x = 0$

$\therefore \cos x = \frac{1}{2}$



$x = \frac{\pi}{3} + 2\pi k$ $(k = 0, \pm 1, \pm 2 \ldots)$

$x = -\frac{\pi}{3} + 2\pi m$ $(m = 0, \pm 1, \pm 2, \ldots)$

Or simply

$x = \pm \pi/3 + 2\pi n$

$n = 0, \pm 1, \pm 2 \ldots$

$n = 0, \pm 1, \pm 2, \ldots$