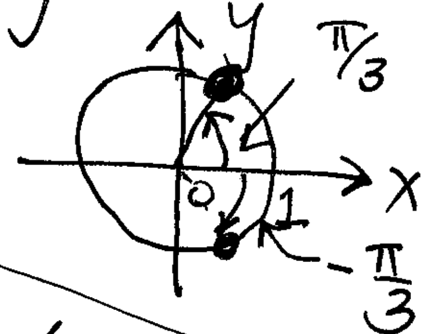
Lesson 13

§§ 3.6+3.7 - Rates of Change + Chain Rule

$\boxed{\text{EX1}}$ Find all x where $f(x) = x - 2\sin x$ has horizontal tangent. (i.e where is rate of change of f zero?)



Soln: $f'(x) = 1 - 2\cos x = 0$

$\cos x = \frac{1}{2} \Rightarrow$ $x = \frac{\pi}{3} + 2\pi k$, $k = 0, \pm 1, \pm 2, \dots$

$x = -\frac{\pi}{3} + 2\pi m$, $m = 0, \pm 1, \dots$

i.e $x = \pm \frac{\pi}{3} + 2\pi n$, $n = 0, \pm 1, \pm 2, \dots$

Ex2

ⓐ $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{\sin 4(x-\frac{\pi}{6})}{x-\pi/6} = \lim\limits_{t\to 0} \dfrac{\sin 4t}{t}$

let $t = x - \frac{\pi}{6}$

$= \lim\limits_{t\to 0} \left(\dfrac{\sin 4t}{4t}\right)\left(\dfrac{4\cancel{t}}{\cancel{t}}\right)$

$= 4$ ✓

ⓑ $\lim\limits_{x\to\frac{\pi}{6}} \dfrac{\sin x - \frac{1}{2}}{x-\frac{\pi}{6}} = f'\left(\frac{\pi}{6}\right) = \cos x \Big|_{x=\frac{\pi}{6}}$

where $f(x) = \sin x$

$= \dfrac{\sqrt{3}}{2}$ ✓

If $s(t)$ = position of object then average velocity over $[a, a+\Delta t]$

$v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(a+\Delta t)-s(a)}{\Delta t}$ average rate of change of $s(t)$ over $[a, a+\Delta t]$

<u>velocity</u> $= v = \dfrac{ds}{dt} = s'(t)$

<u>speed</u> $= |v| = |s'(t)|$ (≥ 0 always)

<u>acceleration</u> $= a = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2} = s''(t)$

<u>jerk</u> $= j = \dfrac{da}{dt} = s'''(t)$

$\boxed{\text{Ex3}}$ If height of object after t secs is $s = -16t^2 + 64t + 200$ ft., find

(a) average velocity over $[0, 2]$

$\therefore\ v_{av} = \dfrac{s(2) - s(0)}{2 - 0} = 64$ ft/sec

(b) speed at $t = 5$ secs

$\therefore\ v = s'(t) = -32t + 64$

$v(5) = -96$ ft/sec $\leftarrow$ velocity

speed is $|v(5)| = 96$ ft/sec

Car Scooter Runner

travels 5 times as fast as

travels 2 times as fast as runner

$5 \times 2 = 10$ times as fast runner

CHAIN RULE If f, g are differentiable functions and $f(g(x))$ is defined, then

$$\boxed{\frac{d}{dx}\{f(g(x))\} = f'(g(x)) \cdot g'(x)} \quad (*)$$

inner fcn

outer fcn

Or, if $y = f(u)$ and $u = g(x)$ then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}} \quad (**)$$

Ex4 Find derivative $\frac{dy}{dx}$

(a) $y = \sqrt{6x - x^3} = (6x - x^3)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(6x - x^3)^{\frac{1}{2}-1} \cdot (6 - 3x^2)$$

$$= \frac{1}{2}(6x - x^3)^{-1/2}(6 - 3x^2) \checkmark$$

Or, $y = (6x - x^3)^{1/2}$: $y = u^{1/2}$ and $u = (6x - x^3)$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{2}u^{-1/2} \cdot (6x - x^3)'$$

$$= \frac{1}{2}(6x - x^3)^{-1/2} \cdot (6 - 3x^2)$$

(b) $y = u^2 + e^u$ where $u = \sin x + 3$

find $\left.\dfrac{dy}{dx}\right|_{x=0}$.

Soln:

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

$$= (2u + e^u)(\cos x + 0)$$

when $x = 0$, $u = 3$

$$\therefore \left.\frac{dy}{dx}\right|_{\substack{x=0\\u=3}} = (6 + e^3)(1) \checkmark$$