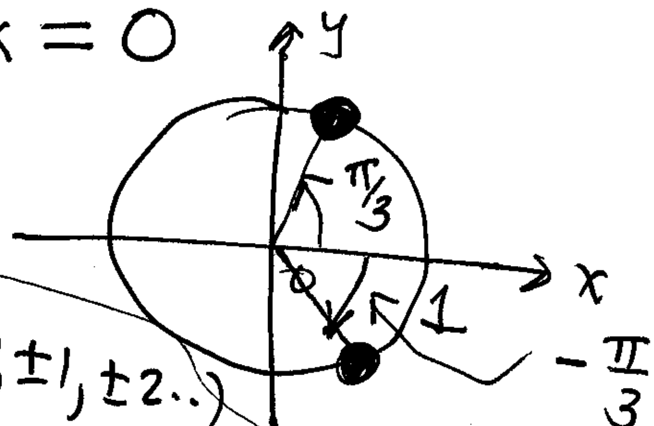
Lesson 13

§§3.6 + 3.7 – Rates of Change + Chain Rule

Ex 1 Find all x where $f(x) = x - 2\sin x$ has a horizontal tangent (i.e. where is rate of change of f zero?)

Soln: $f'(x) = 1 - 2\cos x = 0$

$\cos x = \frac{1}{2}$



$\Rightarrow$ $x = \frac{\pi}{3} + 2\pi k$ $(k = 0, \pm 1, \pm 2 ..)$

$x = -\frac{\pi}{3} + 2\pi m$ $(m = 0, \pm 1, ...)$

or, $x = \pm \frac{\pi}{3} + 2\pi n$ $\quad n = 0, \pm 1, \pm 2, ...$

$\boxed{\text{Ex 2}}$ Compute limits

ⓐ $\lim_{x \to \frac{\pi}{6}} \dfrac{\sin 4\left(x-\frac{\pi}{6}\right)}{x-\frac{\pi}{6}} = \lim_{t \to 0} \dfrac{\sin 4t}{t}$

let $t = x - \frac{\pi}{6}$

$= \lim_{t \to 0} \left(\dfrac{\sin 4t}{4t}\right)\left(\dfrac{4\cancel{t}}{\cancel{t}}\right)$

$= 4$ ✓

ⓑ $\lim_{x \to \frac{\pi}{6}} \dfrac{\sin x - \frac{1}{2}}{x-\frac{\pi}{6}} = f'\left(\frac{\pi}{6}\right) = \cos x \Big|_{x=\frac{\pi}{6}}$

where $f(x) = \sin x$

$= \dfrac{\sqrt{3}}{2}$ ✓

If $s(t)$ = position of object, then average velocity over $[a, a+\Delta t]$ is $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(a+\Delta t) - s(a)}{\Delta t}$ = average rate of change of $s(t)$ over $[a, a+\Delta t]$

velocity $= v = \dfrac{ds}{dt} = s'(t)$

speed $= |v| = |s'(t)|$ (≥ 0 always)

acceleration $= a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = s''(t)$

jerk $= j = \dfrac{da}{dt} = s'''(t)$

$\boxed{\text{Ex3}}$ If height of object after t secs is
$s = -16t^2 + 64t + 200$ ft, find

(a) average velocity over $[0,2]$

$\therefore v_{av} = \dfrac{s(2) - s(0)}{2 - 0} = 64$ ft/sec

(b) <u>speed at $t = 5$ secs</u>

$v = s'(t) = -32t + 64$

$v(5) = -96$ ft/sec $\Leftarrow$ velocity

speed is $|v(5)| = 96$ ft/sec.

Car Scooter Runner

~~5~~ travels 5 times as fast as scooter

travels 2 times as fast as runner

$$5 \times 2 = 10$$

CHAIN RULE If f, g are differentiable and $f(g(x))$ is defined, then

$$\boxed{\frac{d}{dx}\{f(g(x))\} = f'(g(x)) \cdot g'(x)} \quad (*)$$

inner fcn

outer fcn

Or, if $y = f(u)$ and $u = g(x)$, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}} \quad (**)$$

Ex4 Find derivative $\frac{dy}{dx}$

ⓐ $y = \sqrt{6x - x^3} = (6x - x^3)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(6x - x^3)^{1/2 - 1}(6 - 3x^2)$$
$$= \frac{1}{2}(6x - x^3)^{-1/2}(6 - 3x^2) \checkmark$$

or, $y = u^{1/2}$ and $u = (6x - x^3)$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= \left(\frac{1}{2}u^{-1/2}\right)(6 - 3x^2)$$
$$= \left(\frac{1}{2}(6x - x^3)^{-1/2}\right)(6 - 3x^2) \checkmark$$

⑤ $y = u^2 + e^u$, where $u = \sin x + 3$

find $\frac{dy}{dx}$ when $x = 0$.

Soln: $$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$= (2u + e^u)(\cos x)$$

when $x = 0$, $u = 3$

$$\therefore \left.\frac{dy}{dx}\right|_{\substack{x=0 \\ u=3}} = (6 + e^3)(1)$$ ✓