Lesson 14

§3.7 – Chain Rule (cont'd)

CHAIN RULE

$$\boxed{\frac{d}{dx}\{f(g(x))\} = f'(g(x))\, g'(x)}$$

or if $y = f(u)$ and $u = g(x)$ then

$$\boxed{\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)}$$

Tree diagram

$$\begin{tikzcd} y \arrow[d, dash, "\frac{dy}{du}"] \arrow[dd, bend right] \\ u \arrow[d, dash, "\frac{du}{dx}"] \\ x \end{tikzcd}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

For eg, $y = \sin^3 x = (\sin x)^3$

$\therefore \frac{dy}{dx} = 3(\sin x)^2 \cdot \cos x$ ✓

$\boxed{\text{Ex1}}$ Compute $(f \circ g)'(1)$ if

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	5	3	8	2
0	7	0	-1	0
1	0	6	-2	4

Soln: $(f \circ g)'(1) = \frac{d}{dx}\{f(g(x))\}\Big|_{x=1}$

$$= f'(g(x))\,g'(x)\Big|_{x=1}$$

$$= f'(g(1))\,g'(1)$$

$$= f'(-2)\,g'(1)$$

$$= (3)(4) = 12$$

$\boxed{\text{Ex2}}$ Find $\frac{dw}{dp}$ if $w = f(p)e^{f(p)}$

Soln: $\frac{dw}{dp} = f(p)\left[e^{f(p)} \cdot f'(p)\right] + e^{f(p)}\left[f'(p)\right]$

Motivation for Chain Rule:

$$\frac{d}{dx}\left\{f(g(a))\right\} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x-a}$$

let $y = g(x)$ and $b = g(a)$

$$= \left(\lim_{y \to b} \frac{f(y) - f(b)}{y-b}\right)\left(\lim_{x \to a} \frac{g(x) - g(a)}{x-a}\right)$$

$$= f'(b)\,g'(a) = f'(g(a))\,g'(a)$$

<u>Note</u>: $y = f(u)$, $u = g(w)$, $w = h(x)$

$\therefore\ y = f\Big(g\big(h(x)\big)\Big)$

$$\Rightarrow \frac{dy}{dx} = f'\big(g(h(x))\big) \cdot g'\big(h(x)\big) \cdot h'(x)$$

$$\begin{tikzcd}
y \arrow[d, "\frac{dy}{du}"] \arrow[ddd, bend right] \\
u \arrow[d, "\frac{du}{dw}"] \\
w \arrow[d, "\frac{dw}{dx}"] \\
x
\end{tikzcd}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dw}\,\frac{dw}{dx}$$

$\boxed{\text{Ex3}}$ Find derivative:

(a) $y = e^{\cos\sqrt{x}}$ $\qquad\qquad \sqrt{x} = x^{1/2}$

<u>Soln</u>: $\dfrac{dy}{dx} = e^{\cos\sqrt{x}} \cdot \dfrac{d}{dx}\left(\cos\sqrt{x}\right)$

$$= e^{\cos\sqrt{x}}\left\{-\sin\sqrt{x}\right\}\left\{\frac{1}{2}x^{-1/2}\right\}$$

(b) $y = \sin\left(\sec\left(\theta^2 - \theta\right)\right)$

Soln: $\dfrac{dy}{d\theta} = \cos\left(\sec\left(\theta^2 - \theta\right)\right)\left\{\sec(\theta^2-\theta)\tan(\theta^2-\theta)\right\}$

$\cdot \left\{2\theta - 1\right\}$

Question: Find $\dfrac{dw}{dt}$ if $w = \left[(1-t^2)(2t+t^3)\right]^{10}$

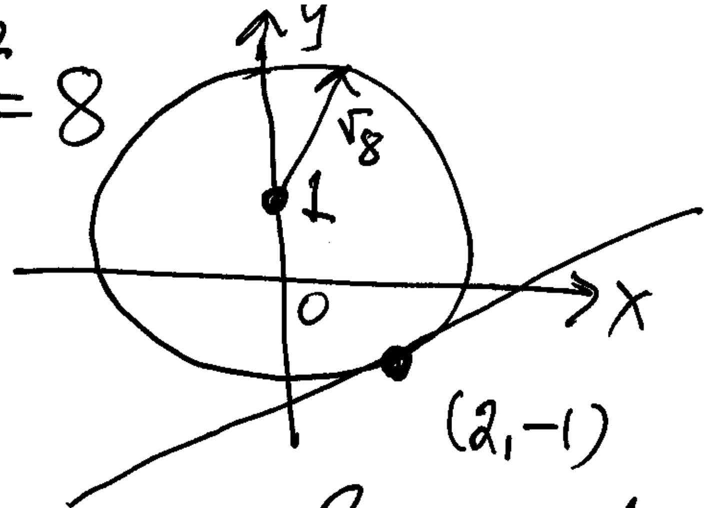
$\boxed{\text{Ex4}}$ Compute $\dfrac{dy}{dx}$ if $y = e^{u^3} + (u^2-1)^{10}$

where $u = \frac{1}{2}(e^x + e^{-x})$.

Soln: $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$

$= \left[\left(e^{u^3}\right)\left(3u^2\right) + 10\left(u^2-1\right)^9(2u)\right] \cdot \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]$

$x^2+(y-1)^2=8$



slope = ? i.e. $\left.\dfrac{dy}{dx}\right|_{(2,-1)} = ?$

Soln 1 - Explicit differentiation : find y then differentiate

Next time