Lesson 14

§3.7 – Chain Rule (cont'd)

CHAIN RULE $\boxed{\dfrac{d}{dx}\{f(g(x))\} = f'(g(x))\, g'(x)}$

or if we let $y = f(u)$ and $u = g(x)$ then

Tree diagram

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}}$$

$$\begin{tikzcd} y \arrow[d, dash, "\frac{dy}{du}"] \\ u \arrow[d, dash, "\frac{du}{dx}"] \\ x \end{tikzcd}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

For eg: if $y = \sin^3 x = (\sin x)^3$

$$\therefore \frac{dy}{dx} = 3(\sin x)^2 (\cos x) \checkmark$$

$\boxed{\text{Ex 1}}$ Compute $(f\circ g)'(1)$ if

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	5	3	8	2
0	7	0	-1	0
1	0	6	-2	4

Soln: $(f\circ g)'(1) = \dfrac{d}{dx}\left\{ f(g(x)) \right\}\Big|_{x=1}$

$$= f'(g(x))\,g'(x)\Big|_{x=1}$$

$$= f'(g(1))\,g'(1)$$

$$= f'(-2)\,g'(1)$$

$$= (3)(4) = 12$$

$\boxed{\text{Ex 2}}$ Find $\frac{dw}{dp}$ if $w = f(p)e^{f(p)}$

Soln: $\frac{dw}{dp} = f(p)\left[e^{f(p)} f'(p)\right] + e^{f(p)}\left[f'(p)\right]$

Motivation for Chain Rule:

$$\frac{d}{dx}\{f(g(a))\} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x-a}$$

$$= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x-a}$$

let $y = g(x)$, $b = g(a)$

$$= \left(\lim_{y \to b} \frac{f(y) - f(b)}{y-b}\right)\left(\lim_{x \to a} \frac{g(x) - g(a)}{x-a}\right)$$

$$= f'(b)\, g'(a) = f'(g(a))\, g'(a)$$

Note: $y = f(u)$, $u = g(w)$, $w = h(x)$

$$\therefore\ y = f(g(h(x)))$$

$$\Rightarrow \frac{dy}{dx} = f'(g(h(x)))\,g'(h(x))\,h'(x)$$

$$\begin{array}{cc} y & \\ | & \frac{dy}{du} \\ u & \\ | & \frac{du}{dw} \\ w & \\ | & \frac{dw}{dx} \\ x & \end{array} \qquad \therefore\ \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dw}\,\frac{dw}{dx}$$

$\boxed{\text{Ex3}}$ Find derivative:

(a) $y = e^{\cos\sqrt{x}}$

$\sqrt{x} = x^{1/2}$

Soln: $$\frac{dy}{dx} = e^{\cos\sqrt{x}} \cdot \frac{d}{dx}\{\cos\sqrt{x}\}$$
$$= e^{\cos\sqrt{x}}\left[(-\sin\sqrt{x})\left(\tfrac{1}{2}x^{-1/2}\right)\right] \checkmark$$

(b) $y = \sin(\sec(\theta^2 - \theta))$

Soln:

$$\frac{dy}{d\theta} = \left[\cos(\sec(\theta^2 - \theta))\right]\left[\sec(\theta^2 - \theta)\tan(\theta^2 - \theta)\right](2\theta - 1)$$

Question: Find $\frac{dw}{dt}$ if $w = \left[(1 - t^2)(2t + t^3)\right]^{10}$

$\boxed{\text{Ex4}}$ Compute $\frac{dy}{dx}$ if $y = e^{u^3} + (u^2 - 1)^{10}$

where $u = \frac{1}{2}(e^x + e^{-x})$.

Soln:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \left[e^{u^3} \cdot 3u^2 + 10(u^2 - 1)^9(2u)\right]\left[\frac{1}{2}(e^x - e^{-x})\right]$$