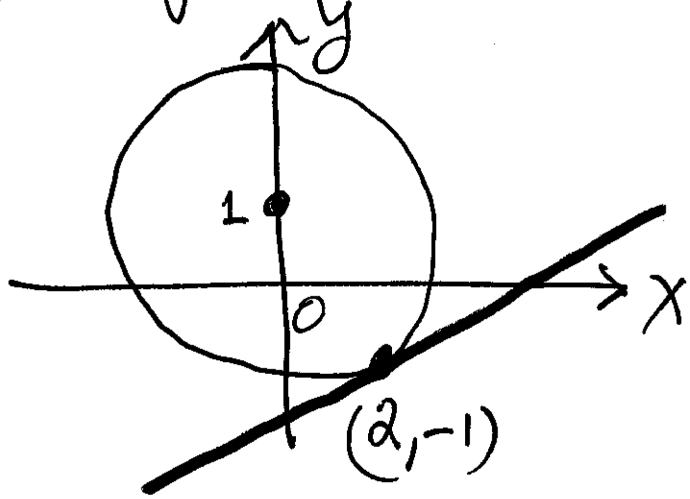
§3.8 Implicit Differentiation

Find slope of tangent line to $x^2+(y-1)^2=8$ at $(2,-1)$.



Soln 1 – Explicit diff. (solve for y, then diff.)

$(y-1)^2 = 8-x^2 \Rightarrow y-1 = \pm\sqrt{8-x^2}$

$y = 1 \pm \sqrt{8-x^2}$. Since $y=-1$ when $x=2$

$\Rightarrow y = 1-\sqrt{8-x^2}$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,-1)} = 0 - \frac{1}{2}(8-x^2)^{-1/2}(-2x)\Big|_{(2,-1)} = 1 \checkmark$$

Soln 2 – Implicit Differentiation

diff. w.r.t. x both sides of equation then solve for $\frac{dy}{dx}$ (Note: $y = y(x)$)

$$x^2 + (y-1)^2 = 8$$

$$\frac{d}{dx}\left\{ x^2 + (y(x)-1)^2 \right\} = \frac{d}{dx}\{8\}$$

$$2x + 2(y-1)\left(\frac{dy}{dx} - 0\right) = 0$$

$$\frac{dy}{dx}\left(2(y-1)\right) = -2x$$

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-2x}{2(y-1)} = \frac{-2(2)}{2(-1-1)} = 1 \checkmark$$

Implicit Differentiation Method

If $y = y(x)$ is defined implicitly by an equation $F(x,y) = 0$ then

Step 1 - Diff. $F(x,y) = 0$ w.r.t. x

Step 2 - Solve for $\frac{dy}{dx}$

Note: $\frac{d}{dx}\{y^n\} = n y^{n-1} \frac{dy}{dx}$

$\boxed{\text{Ex1}}$ Compute $\frac{dy}{dx}$ if

$$x^2 + y^2 + e^{3y} = x^2 y^4 + 3x$$

Soln: $\frac{d}{dx}\left\{x^2 + y^2 + e^{3y}\right\} = \frac{d}{dx}\left\{x^2 y^4 + 3x\right\}$

$$\Rightarrow 2x + 2y\frac{dy}{dx} + e^{3y}\left(3\frac{dy}{dx}\right) = x^2\left(4y^3\frac{dy}{dx}\right) + y^4(2x) + 3$$

$$\left(\frac{dy}{dx}\right)\left[2y + 3e^{3y} - 4x^2y^3\right] = 2xy^4 + 3 - 2x$$

$$\frac{dy}{dx} = \frac{2xy^4 + 3 - 2x}{2y + 3e^{3y} - 4x^2y^3}$$ ✓

$\boxed{\text{Ex 2}}$ Find eqn of tangent line + normal line @ $(1,-1)$:

$$x^3 + y^3 = 2xy + 2$$

Soln: $$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(2xy + 2\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = (2x)\frac{dy}{dx} + (y)(2)$$

$$\vdots$$

$$\frac{dy}{dx} = \frac{2y - 3x^2}{3y^2 - 2x}$$ ✓

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = -5 = m_{tan}$$

Tangent line: $y-(-1) = -5(x-1)$ ✓

Normal line: $y-(-1) = \frac{1}{5}(x-1)$ ✓

$$m_{normal} = -\frac{1}{m_{tan}}$$

$\boxed{\text{Ex3}}$ Compute $\frac{dw}{du}$ if $u^2 + \tan w = 3u$

Soln: Want $\frac{dw}{du}$ ($\therefore w = w(u)$) so $\frac{d}{du}\left\{u^2 + \tan w\right\} = \frac{d}{du}\left\{3u\right\}$

$$2u + (\sec^2 w)\frac{dw}{du} = 3$$

$$\therefore \frac{dw}{du} = \frac{3-2u}{\sec^2 w}$$ ✓

Suppose want $\frac{du}{dw}$?

($\therefore u = u(w)$)

$$\frac{d}{dw}\left\{ u^2 + \tan w = 3u \right\}$$

$$\Rightarrow \quad 2u\frac{du}{dw} + \sec^2 w = 3\frac{du}{dw}$$

$$\vdots$$

$$\frac{du}{dw} = \frac{\sec^2 w}{3-2u} = \frac{1}{\left(\frac{dw}{du}\right)}$$

$\boxed{\text{Ex4}}$ Find y'' if $4y + y^2 = x^2$