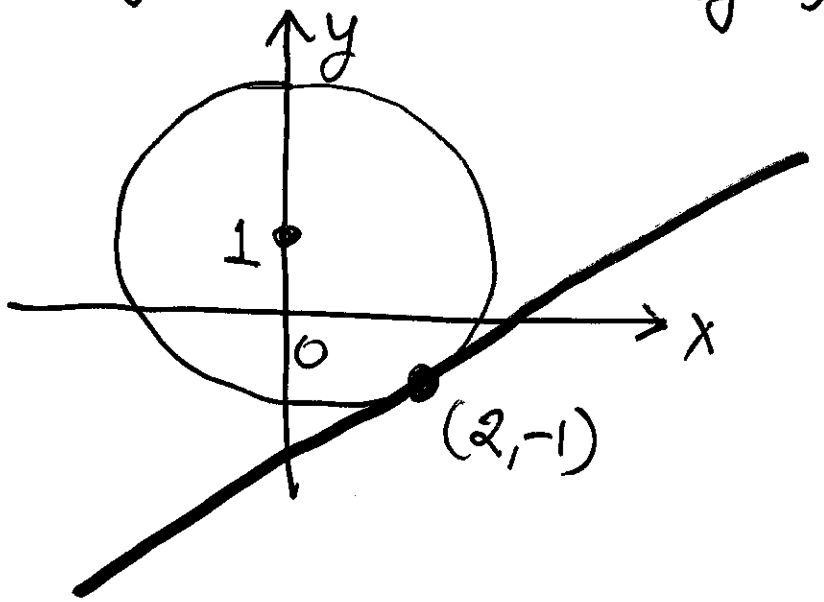
Lesson 15

§3.8 – Implicit Differentiation

Find slope of tangent line to $x^2+(y-1)^2=8$ at $(2,-1)$.



Soln 1 – Explicit diff. (solve for y, then diff.)

$(y-1)^2=8-x^2 \Rightarrow y-1=\pm\sqrt{8-x^2}$

$y=1\pm\sqrt{8-x^2}$. Since $y=-1$ when $x=2$

$\Rightarrow y=1-\sqrt{8-x^2}$

$\therefore \left.\dfrac{dy}{dx}\right|_{(2,-1)}=0-\frac{1}{2}(8-x^2)^{-1/2}(-2x)\Big|_{(2,-1)}=1$ ✓

Soln 2 – Implicit Differentiation

diff. both sides of eqn w.r.t. x
solve for $\frac{dy}{dx}$. $(y = y(x))$

$$x^2 + (y-1)^2 = 8$$

$$\frac{d}{dx}\left\{x^2 + (y(x) - 1)^2\right\} = \frac{d}{dx}\{8\}$$

$$\Rightarrow \quad 2x + 2(y-1)\left(\frac{dy}{dx} - 0\right)\Big\} = 0$$

$$2x + 2(y-1)\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(2,-1)} = \left.\frac{-2x}{2(y-1)}\right|_{(2,-1)} = \frac{-2(2)}{2(-1-1)} = 1 \checkmark$$

Implicit Differentiation Method

If $y = y(x)$ is defined implicitly by an equation $F(x,y)=0$ then

Step 1 – Differentiate $F(x,y)=0$ w.r.t. x

Step 2 – Solve for $\frac{dy}{dx}$

Note: $\frac{d}{dx}\{y^n\} = n y^{n-1} \frac{dy}{dx}$

$\boxed{\text{Ex1}}$ Compute $\frac{dy}{dx}$ if

$$x^2 + y^2 + e^{3y} = x^2 y^4 + 3x$$

Soln: $$\frac{d}{dx}\{x^2 + y^2 + e^{3y}\} = \frac{d}{dx}\{x^2 y^4 + 3x\}$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} + e^{3y}\left(3\frac{dy}{dx}\right) = x^2\left(4y^3\frac{dy}{dx}\right) + y^4(2x) + 3$$

$$\left(\frac{dy}{dx}\right)\left[2y + 3e^{3y} - 4x^2y^3\right] = 2xy^4 + 3 - 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{2xy^4+3-2x}{2y+3e^{3y}-4x^2y^3}$$ ✓

$\boxed{\text{Ex 2}}$ Find eqn of tangent line & normal line @ $(1,-1)$:

$$x^3+y^3 = 2xy+2$$

Soln: $\frac{d}{dx}\left\{x^3+y^3\right\} = \frac{d}{dx}\left\{(2x)y+2\right\}$

$$\Rightarrow \quad 3x^2+3y^2\frac{dy}{dx} = (2x)\frac{dy}{dx}+y(2)+0$$

$$\vdots$$

$$\frac{dy}{dx} = \frac{2y-3x^2}{3y^2-2x}$$ ✓

$$\left.\frac{dy}{dx}\right|_{(1,-1)} = -5 = m_{tan}$$

$\therefore$ <u>Tangent Line:</u> $y-(-1) = -5(x-1)$ ✓

<u>Normal Line:</u> $y-(-1) = \frac{1}{5}(x-1)$ ✓

$$m_{normal} = -\frac{1}{m_{tan}}$$

$\boxed{\text{Ex 3}}$ Compute $\frac{dw}{du}$ if $u^2 + \tan w = 3u$.

<u>Soln:</u> Want $\frac{dw}{du}$ $\Rightarrow$ $\frac{d}{du}\left\{u^2 + \tan w = 3u\right\}$

$(\therefore w = w(u))$

$$2u + (\sec^2 w)\left(\frac{dw}{du}\right) = 3$$

$$\Rightarrow \frac{dw}{du} = \frac{3-2u}{\sec^2 w}$$ ✓

Want $\dfrac{du}{dw}$ $\Rightarrow \dfrac{d}{dw}\left\{u^2 + \tan w = 3u\right\}$

$(\therefore\ u = u(w))$

$$2u\,\frac{du}{dw} + \sec^2 w = 3\,\frac{du}{dw}$$

$$\frac{du}{dw} = \frac{\sec^2 w}{3-2u} = \frac{1}{\left(\frac{dw}{du}\right)}$$