$\boxed{\S 3.9}$ Derivatives of Log & Exp. Functions

Find y'' if $4y + y^2 = x^2$

Soln: $\frac{d}{dx}\{4y + y^2\} = \frac{d}{dx}\{x^2\}$

$$4\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x$$

$$\vdots$$

$$\frac{dy}{dx} = \frac{x}{2+y}$$

Hence $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{2+y}\right)$

$$= \frac{(2+y)(1) - (x)\left(0 + \frac{dy}{dx}\right)}{(2+y)^2}$$

Find $\frac{dy}{dx}$ if $e^y = x$

$$\Rightarrow \frac{d(e^y)}{dx} = \frac{d(x)}{dx}$$

$$e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Since $e^y = x \Rightarrow \ln(e^y) = \ln x$

$$\Rightarrow y \ln e = \ln x \Rightarrow y = \ln x$$

Hence $\frac{d(\ln x)}{dx} = \frac{1}{x}$

Thm: (Derivative of Natural Log)

① $\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$

② $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, for $x \neq 0$

③ $\frac{d}{dx}(\ln|u(x)|) = \frac{u'(x)}{u(x)}$, for $u(x) \neq 0$

Reason for 2 If $x>0$ then 2 is 1

If $x<0 \Rightarrow |x| = -x$

$$\therefore \frac{d}{dx}\{\ln|x|\} = \frac{d}{dx}\{\ln(-x)\} = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

$\boxed{\text{Ex 1}}$ Find derivative

(a) $y = \ln(3x) + 2\ln|\cos x|$

$$\frac{dy}{dx} = \left(\frac{1}{\cancel{3}x}\right)(\cancel{3}) + 2\,\frac{1}{(\cos x)}(-\sin x)$$ ✓

$$(\ln(3x))' = \frac{1}{x} \quad \text{not} \quad \frac{1}{3x}$$

$$(\ln(3x))' = (\ln 3 + \ln x)' = \frac{1}{x}$$

(b) $y = b^x$ $(b>0,\ b\neq 1)$

$$\ln y = \ln(b^x) = x\ln b$$

$$\frac{d(\ln y)}{dx} = \frac{d}{dx}(x\ln b)$$

$$\frac{1}{y}\frac{dy}{dx} = \ln b \Rightarrow \frac{dy}{dx} = y\ln b = b^x \ln b$$

Hence,

Thm: $\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$

Ex2 Find $\frac{dy}{dx}$ if $\ln\left(\frac{x^3}{y}\right) = x^2 + 2^x$

Soln: $3\ln x - \ln y = x^2 + 2^x$ Use Implicit Diff

diff. w.r.t. x

$$\Rightarrow 3\left(\frac{1}{x}\right) - \frac{1}{y}\frac{dy}{dx} = 2x + 2^x \ln 2$$

$$\frac{dy}{dx} = -y\left[2x + 2^x\ln 2 - \frac{3}{x}\right]$$ ✓

Logarithmic Differentiation Method

Step 1 - take natural log of both sides + simplify using Law of Logarithms

Step 2 - Differentiate both sides implicitly w.r.t. x

Step 3 - Solve for $\frac{dy}{dx}$

$\boxed{\text{EX3}}$ Find $\frac{dy}{dx}$

ⓐ $y = x^{2x}$ Use Log. Diff.

$$\ln y = \ln(x^{2x}) = 2x \ln x$$

$$\frac{d(\ln y)}{dx} = \frac{d(2x \ln x)}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = 2\left[x\left(\frac{1}{x}\right) + (\ln x)(1)\right]$$

$$\therefore \frac{dy}{dx} = 2y[1 + \ln x]$$

$$= 2x^{2x}[1 + \ln x]$$

(b) $y = \dfrac{(2x+1)^3 e^{\sin x}}{\sqrt{x^2-1}}$ Use Log Diff.

$\ln y = 3\ln(2x+1) + \sin x - \frac{1}{2}\ln(x^2-1)$

$\therefore \frac{1}{y}y' = 3\left(\frac{1}{2x+1}\right)(2) + \cos x - \frac{1}{2}\left(\frac{1}{x^2-1}\right)(2x)$

$y' = y\left[\frac{6}{2x+1} + \cos x - \frac{x}{x^2-1}\right]$ ✓

(c) Fall 2018 Final #11

If $f(x) = \log_{10} x$, $f'(e) = ?$

Soln: $y = \log_{10} x$

$10^y = x$ Use Log. Diff.

$\ln(10^y) = \ln x$

$y \ln 10 = \ln x$ Diff. w.r.t. x.