§3.9 - Derivatives of Log & Exp Functions

Find y'' if $4y + y^2 = x^2$

Soln: Use Implicit Diff.

$$\frac{d}{dx}\{4y + y^2\} = \frac{d}{dx}\{x^2\}$$

$$\Rightarrow 4\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x$$

$$\vdots$$

$$\frac{dy}{dx} = \frac{x}{2+y} \quad \checkmark$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{2+y}\right) = \frac{(2+y)(1) - (x)\left(0 + \frac{dy}{dx}\right)}{(2+y)^2}$$

Find $\frac{dy}{dx}$ if $e^y = x$

Implicit Diff. $\frac{d}{dx}(e^y) = \frac{d}{dx}(x)$

$e^y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$

$\ln(e^y) = \ln x$

$y \ln e = \ln x \quad \therefore y = \ln x \Rightarrow \frac{d(\ln x)}{dx} = \frac{1}{x}$

Thm (Derivative of Natural Log)

① $\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$

② $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, for $x \neq 0$

③ $\frac{d}{dx}\left(\ln|u(x)|\right) = \frac{u'(x)}{u(x)}$, for $u(x) \neq 0$

Reason for ② If $x>0$ then ② is ①

If $x<0 \Rightarrow |x| = -x$

$\therefore \frac{d}{dx}\left(\ln|x|\right) = \frac{d}{dx}\left(\ln(-x)\right) = \frac{1}{(-x)} \cdot (-1) = \frac{1}{x}$

Ex1 Find derivative

(a) $y = \ln(3x) + 2\ln|\cos x|$

Soln: $y' = \frac{1}{(3x)}(3) + 2\left(\frac{1}{\cos x}\right)(-\sin x)$ ✓

Recall, $\left(\ln(3x)\right)' = \frac{1}{3x}(3) = \frac{1}{x}$

or, $\left(\ln(3x)\right)' = \left(\ln 3 + \ln x\right)' = \frac{1}{x}$

(b) $y = b^x$ $(b>0,\ b\neq 1)$

$\ln y = \ln(b^x) = x \ln b$

$$\frac{d(\ln y)}{dx} = \frac{d(x \ln b)}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln b \Rightarrow \frac{dy}{dx} = y \ln b = b^x \ln b$$

Thm:

$$\boxed{\frac{d(b^x)}{dx} = b^x \ln b}$$

Ex 2 Find $\frac{dy}{dx}$ if $\ln\left(\frac{x^3}{y}\right) = x^2 + 2^x$

$\Rightarrow 3\ln x - \ln y = x^2 + 2^x$ use Implicit Diff

diff. w.r.t. x

$$3\left(\frac{1}{x}\right) - \frac{1}{y}\frac{dy}{dx} = 2x + 2^x \ln 2$$

$$\frac{dy}{dx} = -y\left[2x + 2^x \ln 2 - \frac{3}{x}\right] \checkmark$$

Logarithmic Differentiation Method

Step 1 - take natural log of both sides & simplify using Law of Logarithms

Step 2 - differentiate both sides implicitly w.r.t. x

Step 3 - Solve for $\frac{dy}{dx}$

Ex3 Find $\frac{dy}{dx}$

(a) $y = x^{2x}$ (Use Log. Diff)

$$\ln y = \ln(x^{2x}) = 2x \ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(2x \ln x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\left[x\left(\frac{1}{x}\right) + (\ln x)(1)\right]$$

$$\frac{dy}{dx} = 2y\left[1 + \ln x\right] = 2x^{2x}\left[1 + \ln x\right]$$

(b) $y = \dfrac{(2x+1)^3 e^{\sin x}}{\sqrt{x^2-1}}$ Use Log. Diff.

$\ln y = 3\ln(2x+1) + \sin x - \frac{1}{2}\ln(x^2-1)$

$\frac{1}{y}\frac{dy}{dx} = 3\left(\frac{1}{2x+1}\right)(2) + \cos x - \frac{1}{2}\left(\frac{1}{x^2-1}\right)(2x)$

$\frac{dy}{dx} = y\left[\frac{6}{2x+1} + \cos x - \frac{x}{x^2-1}\right]$ ✓

Fall 2018 Final #11

If $f(x) = \log_{10} x$, $f'(e) = ?$