Lesson 17

§3.10 – Derivatives of Inverse Trig Functions

Final Exam, Fall 2018 #11:

If $f(x)=\log_{10} x$, $f'(e)=?$

$\Rightarrow y=\log_{10} x \Rightarrow 10^y = x$ use Log. Diff.

$\ln(10^y)=\ln x$

$y \ln 10 = \ln x$

$\Rightarrow \frac{dy}{dx}(\ln 10)=\frac{1}{x} \Rightarrow f'(x)=\frac{1}{x \ln 10}$

$f'(e)=\frac{1}{e \ln 10}$

Ex1 Find $\frac{dy}{dx}$ if

(a) $\sin y = x$ ← Use Implicit Diff.

$$\frac{d}{dx}\{\sin y\} = \frac{d\{x\}}{dx}$$

$$\Rightarrow (\cos y)\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

If we restrict $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ then $y = \sin^{-1} x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\{\sin^{-1} x\} = \frac{1}{\cos y}$$

; Recall $\cos^2 y + \sin^2 y = 1$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \cos y = \pm\sqrt{1-x^2}$$

(b) $\tan y = x$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$(\sec^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

If we let $-\frac{\pi}{2} < y < \frac{\pi}{2} \Rightarrow y = \tan^{-1} x$

$$\therefore \frac{dy}{dx} = \frac{d(\tan^{-1} x)}{dx} = \frac{1}{\sec^2 y}$$; Recall $1 + \tan^2 y = \sec^2 y$

$$= \frac{1}{1+x^2}$$

Thm (Derivatives of Inverse Trig Fcns)

$\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$	$\frac{d(\cos^{-1}x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$ $(-1<x<1)$
$\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$	$\frac{d(\cot^{-1}x)}{dx} = -\frac{1}{1+x^2}$ $(-\infty<x<\infty)$
$\frac{d(\sec^{-1}x)}{dx} = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d(\csc^{-1}x)}{dx} = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ $(\lvert x\rvert>1)$

Ex 2 Find derivative

(a) $y = \sin^{-1}(x^2-3)$

$$\therefore y' = \frac{1}{\sqrt{1-(x^2-3)^2}}(2x) \quad \checkmark$$

$$= \frac{2x}{\sqrt{6x^2-x^4-8}} \quad \checkmark$$

(b) Find $\frac{dy}{dx}$ if $x^2y^3 = \tan^{-1}(2y)$ (5)

Use Implicit Diff:

$$x^2\left\{3y^2\frac{dy}{dx}\right\} + y^3\{2x\} = \frac{1}{1+(2y)^2}\left\{2\frac{dy}{dx}\right\}$$

$$\vdots$$

$$\frac{dy}{dx} = \frac{-2xy3}{\left[3x^2y^2 - \frac{2}{1+4y^2}\right]}$$

(c) $f(x) = (\sin^{-1}x)^x$ Use Log. Diff.

$$\ln f(x) = x\ln(\sin^{-1}x)$$ now diff. w.r.t. x

$$\frac{1}{f(x)}f'(x) = x\left[\frac{1}{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\right)\right] + \ln(\sin^{-1}x)$$

$$\therefore f'(x) = f(x)\left\{\frac{x}{(\sin^{-1}x)\sqrt{1-x^2}} + \ln(\sin^{-1}x)\right\}$$

Thm: (Derivative of Inverse Fcn)
If f is diff. and has inverse on I, x_0 is a pt in I, and $f'(x_0) \neq 0$, then

$$\boxed{(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}}$$

Ex3 Find $(f^{-1})'(3)$ if $f(x) = x^3 + x + 1$.

since $f(x_0) = 3 \Rightarrow x_0 = 1$

$$\therefore (f^{-1})'(3) = (f^{-1})'(f(1)) = \frac{1}{f'(1)} = \frac{1}{4}$$

Exam topics end here!