Lesson 28

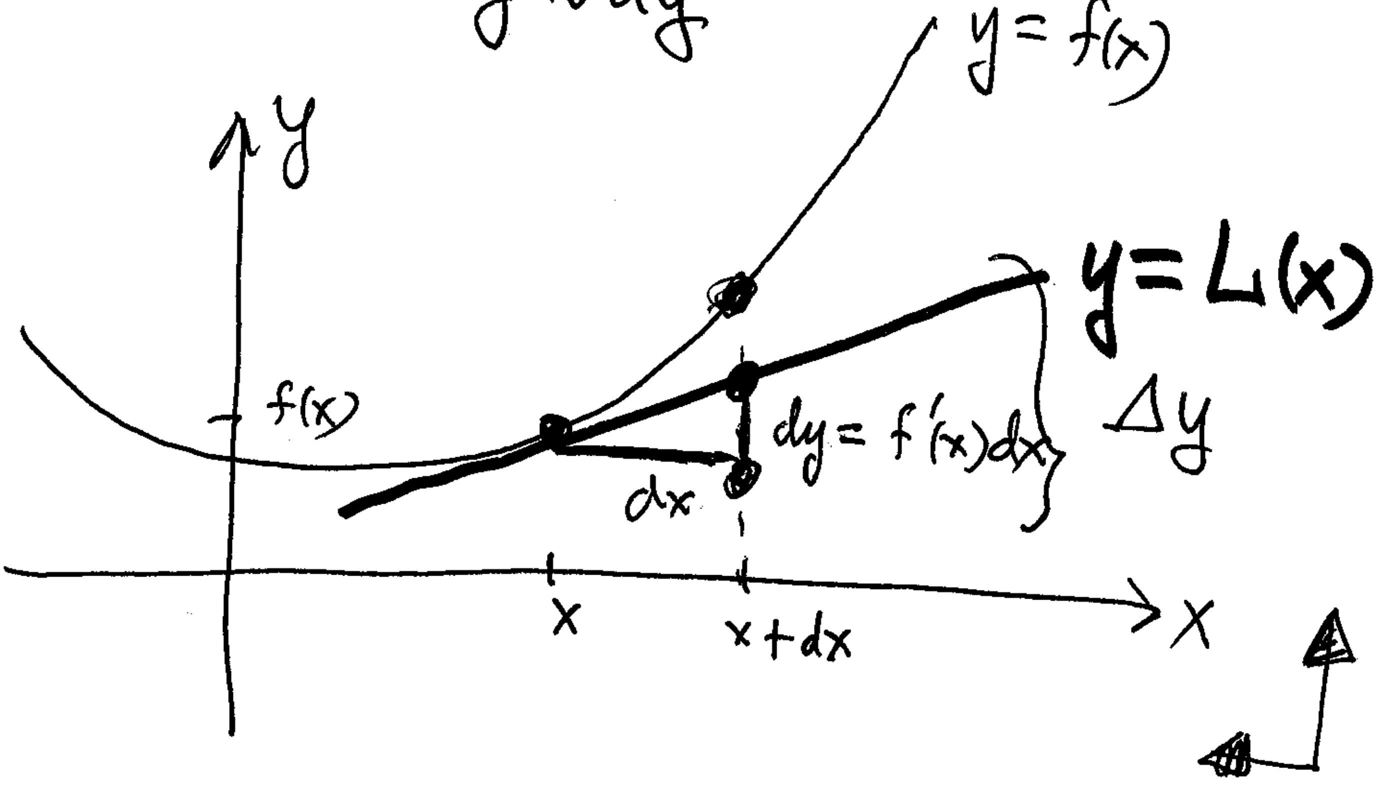
§4.7 – L'Hôpital Rule

Last time: If $y = f(x)$

① $\Delta x = dx$

② $\Delta y = f(x+dx) - f(x) \approx f'(x)\,dx = dy$

$$\Delta y \approx dy$$



$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

f, f', g, g' cont. at a, $g'(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

<u>L'Hôpital's Rule</u> (LR) Suppose f, g are diff. on open interval I, a belongs to I, and $g'(x) \neq 0$ on I when $x \neq a$.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or

if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

$$\Rightarrow \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}},$$

provided limit exist or is $\pm\infty$.

[LR holds if $x\to\pm\infty$, $x\to a^+$, $x\to a^-$]

For eg $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\tan \pi x} \underset{\underset{LR}{\uparrow}}{=} \lim\limits_{x\to 0} \dfrac{4\cos 4x}{\pi \sec^2 \pi x} = \dfrac{4}{\pi}$ ✓

$\lim\limits_{x\to\infty} \ln\left(\dfrac{6x+4}{\pi x - 1}\right) = \ln\left(\lim\limits_{x\to\infty} \dfrac{6x+4}{\pi x-1}\right)$

$\underset{\underset{LR}{\uparrow}}{=} \ln\left(\lim\limits_{x\to\infty} \dfrac{6}{\pi}\right) = \ln\left(\dfrac{6}{\pi}\right)$ ✓

Indeterminate Forms

(1) $\boxed{\frac{0}{0}}$, $\boxed{\frac{\infty}{\infty}}$ Apply LR directly

(2) $\boxed{0\cdot\infty}$ $f(x)g(x) = \dfrac{f(x)}{\frac{1}{g(x)}} \Rightarrow \boxed{\frac{0}{0}}$

(3) $\boxed{\infty - \infty}$ $f(x) - g(x)$ use algebra to reduce to another form

(4) $\boxed{1^\infty}$ $\boxed{0^0}$ $\boxed{\infty^0}$ $f(x)^{g(x)}$

$f(x)^{g(x)} = e^{\ln\left[f(x)^{g(x)}\right]} = e^{[g(x)\ln f(x)]}$

$\boxed{\text{Ex 1}}$ Compute

(a) $\lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+} e^{\ln[x^{2x}]}$

$\boxed{0^0}$

$= \lim_{x\to 0^+} e^{[2x\ln x]}$

$= e^{\lim_{x\to 0^+} 2x\ln x} = e^{\lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}}}$

$\underset{\uparrow \text{LR}}{=} e^{\left[\lim_{x\to 0^+} \frac{2\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}\right]} = e^{\lim_{x\to 0^+} -2x} = e^0 = \underline{1}$

(b) $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

$\boxed{\infty - \infty}$

$= \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \underset{\uparrow \text{LR}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{(x-1)\left(\frac{1}{x}\right) + \ln x}$

$\boxed{\frac{0}{0}}$

$= \lim_{x\to 1^+} \frac{1-x}{(x-1) + x\ln x} \overset{\text{LR}}{\underset{\downarrow}{=}} \lim_{x\to 1^+} \frac{-1}{2+\ln x} = -\frac{1}{2}$ ✓

Suppose $f(x), g(x) \to \infty$ as $x \to \infty$

If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \begin{cases} 0 \Rightarrow g(x) \text{ grows faster than } f(x) \\ \infty \Rightarrow f(x) \text{ grows faster than } g(x) \\ M \Rightarrow f(x), g(x) \text{ comparable growth} \end{cases}$

x^2, $e^{0.001x}$ faster:

<u>Check</u>: $\lim_{x\to\infty} \dfrac{x^2}{e^{0.001x}} = 0$

$\boxed{e^x + 2^x,\ 5e^x}$ faster?