Lesson 28

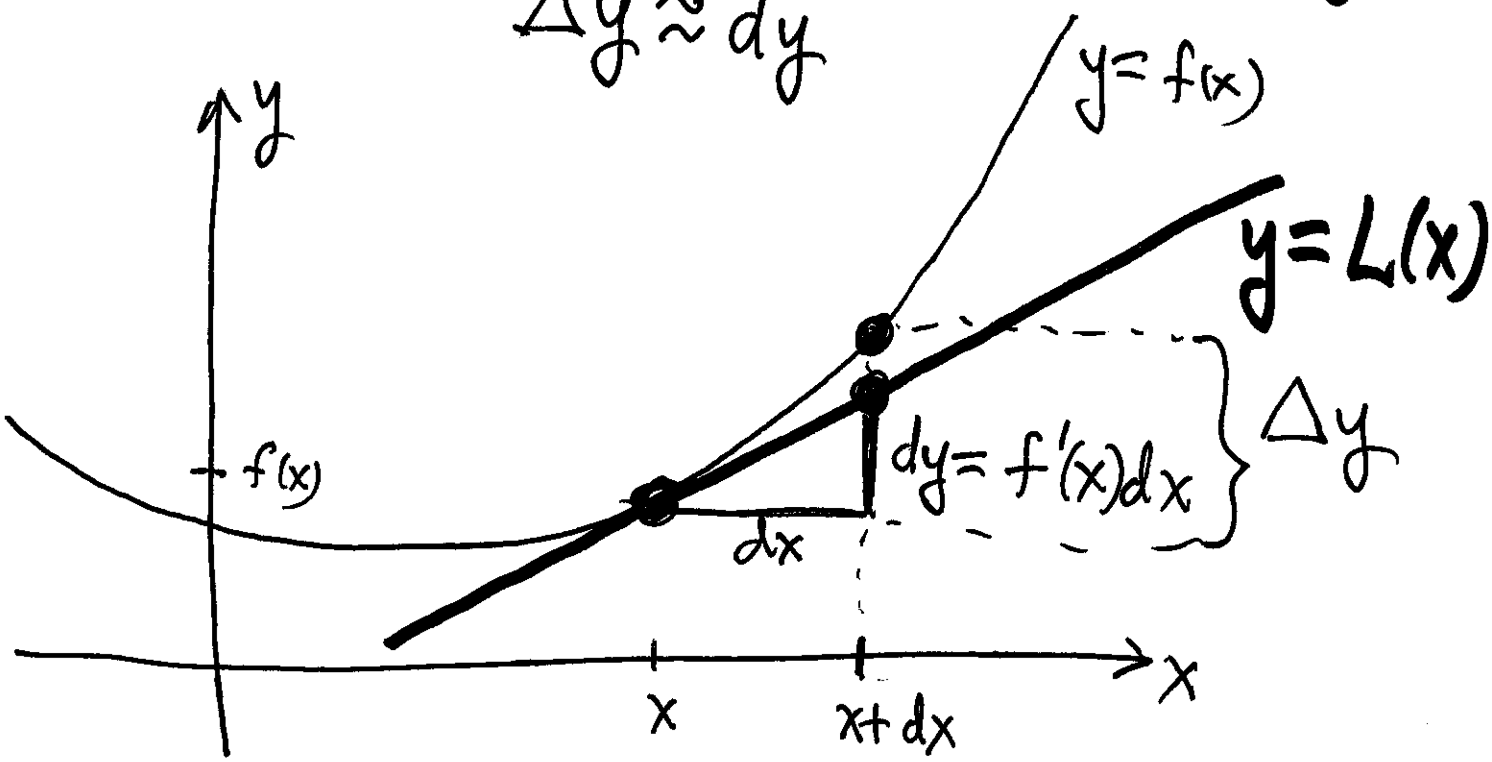
§4.7 – L'Hôpital's Rule

Last time: $y = f(x)$

① $\Delta x = dx$

② $\Delta y = f(x+dx) - f(x) \approx f'(x)dx = dy$

$$\Delta y \approx dy$$



$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$;

f, f', g, g' cont. at a, $g'(a) \neq 0$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

<u>L'Hôpital's Rule</u> (LR) Suppose f, g are diff. on open interval I, a is a point in I, and $g'(x) \neq 0$ on I when $x \neq a$.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or

if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

$$\Rightarrow \boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}},$$

provided limit exists or is $\pm\infty$

[LR holds when $x \to \pm\infty$, $x \to a^+$, $x \to a^-$]

For eg, $\lim_{x\to 0} \frac{\sin 4x}{\tan \pi x} \underset{LR}{=} \lim_{x\to 0} \frac{4\cos 4x}{\pi \sec^2 \pi x} = \frac{4}{\pi}$ ✓

$$\lim_{x\to\infty} \ln\left(\frac{6x+4}{\pi x - 1}\right) = \ln\left[\lim_{x\to\infty} \frac{6x+4}{\pi x-1}\right]$$

$$\underset{LR}{=} \ln\left[\lim_{x\to\infty} \frac{6}{\pi}\right] = \ln\left(\frac{6}{\pi}\right)$$ ✓

Indeterminate Forms

(1) $\boxed{\frac{0}{0}}$, $\boxed{\frac{\infty}{\infty}}$ Apply LR directly

(2) $\boxed{0\cdot\infty}$ $f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}}$ $\boxed{\frac{0}{0}}$

(3) $\boxed{\infty - \infty}$ $f(x) - g(x)$ use algebra to convert another form...

(4) $\boxed{1^\infty}$ $\boxed{0^0}$ $\boxed{\infty^0}$

$$f(x)^{g(x)} = e^{\ln[f(x)^{g(x)}]} = e^{[g(x)\ln f(x)]}$$ ✓

$\boxed{\text{Ex 1}}$ Compute

(a) $\lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+} e^{\ln[x^{2x}]}$

$\boxed{0^0}$

$$= \lim_{x\to 0^+} e^{[2x\ln x]} = e^{\lim_{x\to 0^+} 2x\ln x}$$

$$= e^{\lim_{x\to 0^+} \frac{2\ln x}{\frac{1}{x}}} \underset{LR}{=} e^{\left[\lim_{x\to 0^+} \frac{\frac{2}{x}}{-\frac{1}{x^2}}\right]}$$

$\boxed{\frac{\infty}{\infty}}$

$$= e^{\lim_{x\to 0^+} -2x} = e^0 = 1 \checkmark$$

(b) $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x}$ $\boxed{\frac{0}{0}}$

$\boxed{\infty - \infty}$

$$\underset{LR}{=} \lim_{x\to 1^+} \frac{1-x}{(x-1) + x\ln x} \underset{LR}{=} \lim_{x\to 1^+} \left(\frac{-1}{2+\ln x}\right) = -\frac{1}{2}$$

$\boxed{\frac{0}{0}}$

Suppose $f(x), g(x) \to \infty$ as $x \to \infty$

If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \begin{cases} 0 \Rightarrow g(x) \text{ grows faster than } f(x). \\ \infty \Rightarrow f(x) \text{ grows faster} \\ M \Rightarrow f, g \text{ comparable} \end{cases}$