Lesson 31

§5.2 Definite Integrals

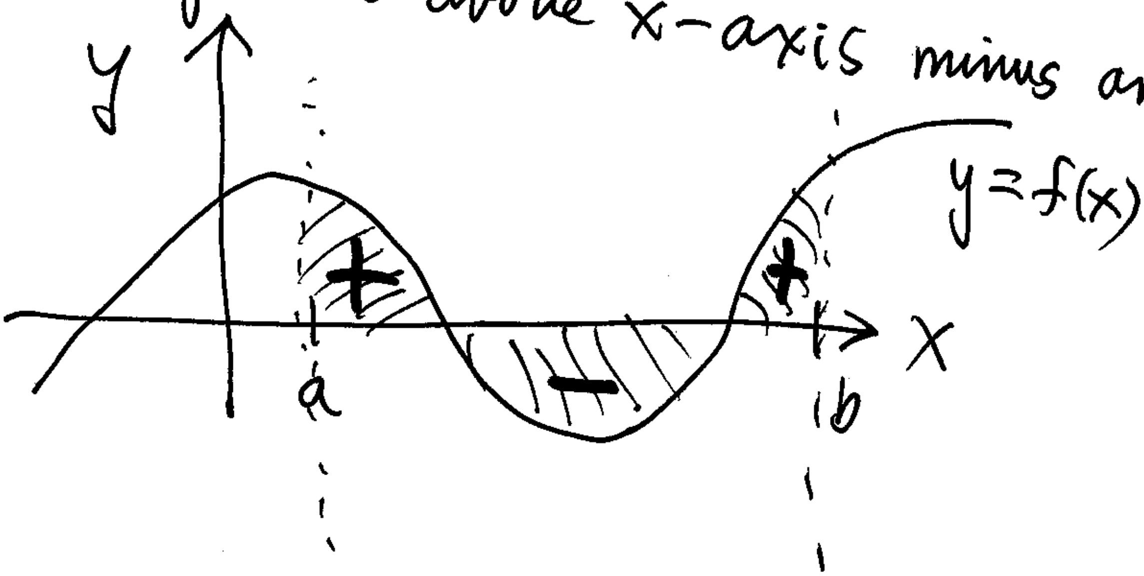
Sigma Notation: $\sum_{k=1}^{3} \frac{2k}{k+1} = \frac{2(1)}{2} + \frac{2(2)}{3} + \frac{2(3)}{4}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{(Gauss)}$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$$

$$f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

Def: The net area of region R bounded by $y=f(x)$, x-axis, $x=a$, $x=b$ is sum of areas above x-axis minus areas below:



$y = f(x)$; $[x_0, x_1], \ldots, [x_{n-1}, x_n]$

any partition of $[a, b]$ where $\Delta x_k = x_k - x_{k-1}$

let x_k^* be any pt in $[x_{k-1}, x_k]$, then

$\sum_{k=1}^{n} f(x_k^*) \Delta x_k$ <u>general Riemann sum of f</u>

Let $\Delta = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$

<u>Def</u>: $f(x)$ is <u>integrable</u> on $[a, b]$ if over all partitions of $[a, b]$ and for all possible x_k^*,

$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$ is a unique number.

Moreover

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k = \boxed{\int_a^b f(x)\, dx}$$

<u>Definite integral</u> of f over $[a, b]$

Recall,

Functions

$$\underbrace{\int f(x)\,dx}_{\text{indefinite integral of } f}$$

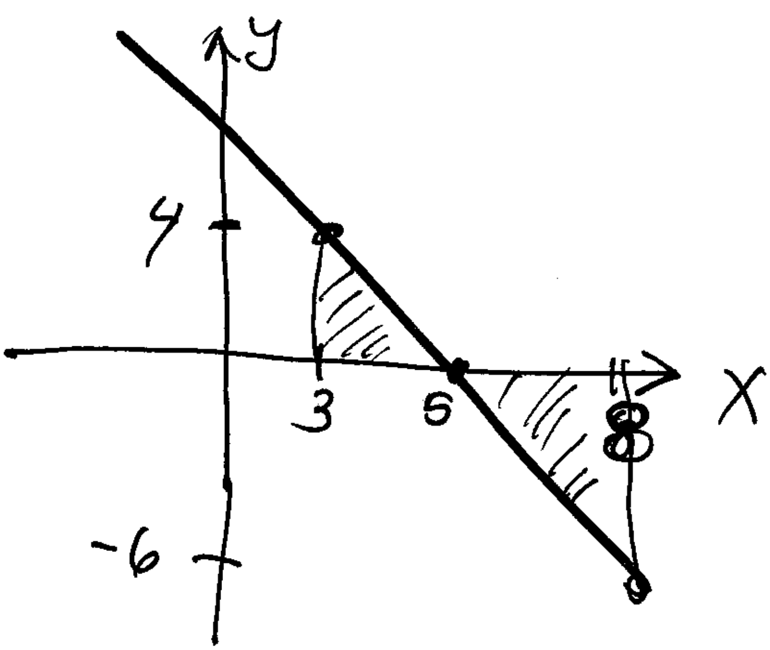
indefinite integral of f (i.e. antiderivative of f)

$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral of } f} \leftarrow \text{Number}$$

definite integral of f (net area under $y = f(x)$)

$\boxed{\text{Ex1}}$ Using geometry, evaluate

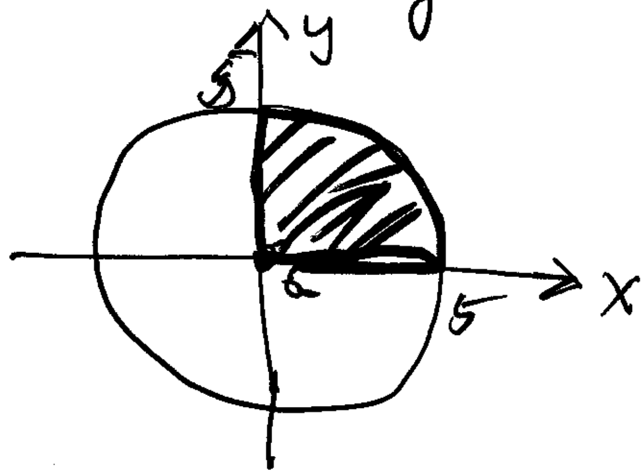
(a) $\displaystyle\int_3^8 \underbrace{(10-2x)}_{f(x)}\,dx = \frac{1}{2}(2)(4) - \frac{1}{2}(3)(6) = -5$ ✓



(b) $\displaystyle\int_0^5 \underbrace{\sqrt{25-x^2}}_{y=f(x)}\,dx$

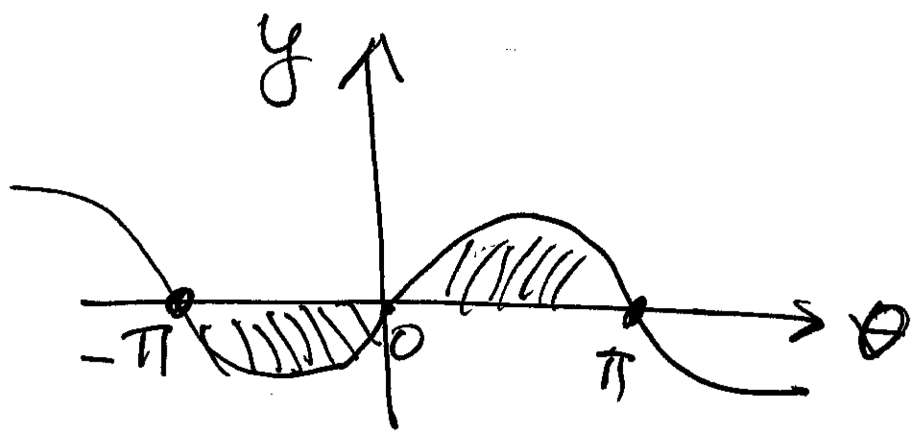
$y = \sqrt{25-x^2}$

$x^2+y^2 = 25$



$= \frac{1}{4}\left[\pi(5)^2\right]$

(c) $\displaystyle\int_{-\pi}^{\pi} \sin\theta\, d\theta = 0$



Basic Properties

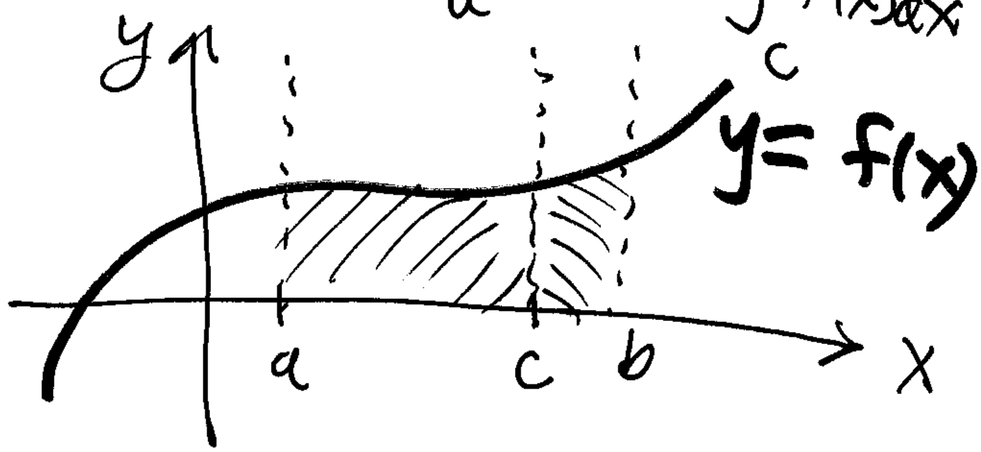
① $\int_b^a f(x)dx = -\int_a^b f(x)dx$

② $\int_a^a f(x)dx = 0$

③ $\int_a^b \{kf(x)+g(x)\}dx = k\int_a^b f(x)dx + \int_a^b g(x)dx$

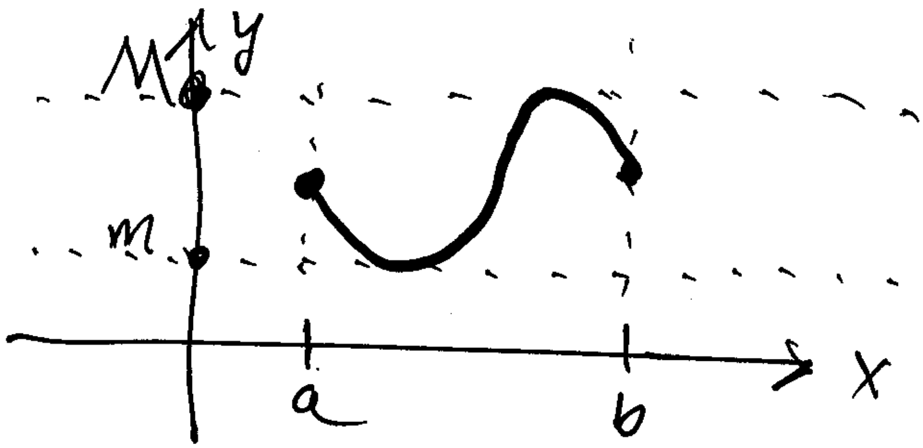
④ If $a \le c \le b$ then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



(5) If $m \le f(x) \le M$ on $[a,b]$

then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$



Ex2: Find upper & lower bounds

$$\int_1^3 (\underbrace{x^3 - 12x + 20}_{f(x)})\,dx$$

on $[1,3]$

$m = 4$

$M = 11$