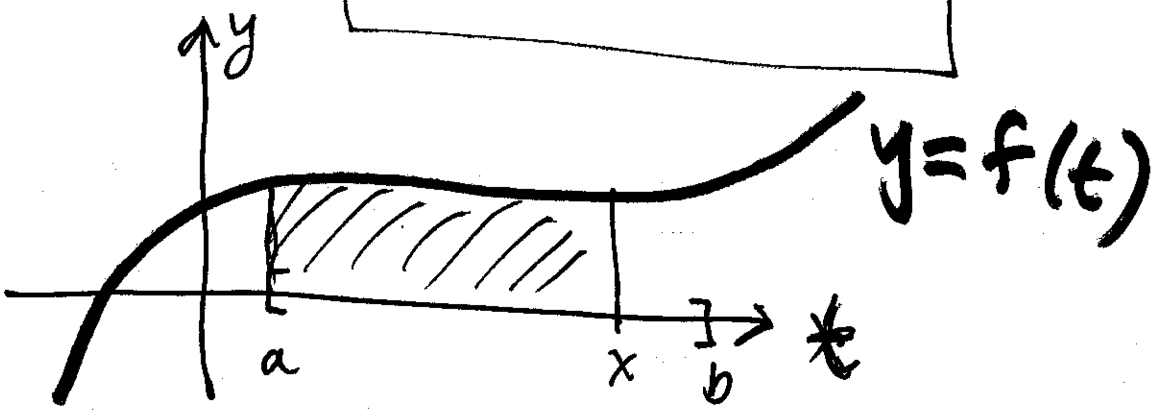
Lesson 32

§5.3 Fundamental Thm of Calculus

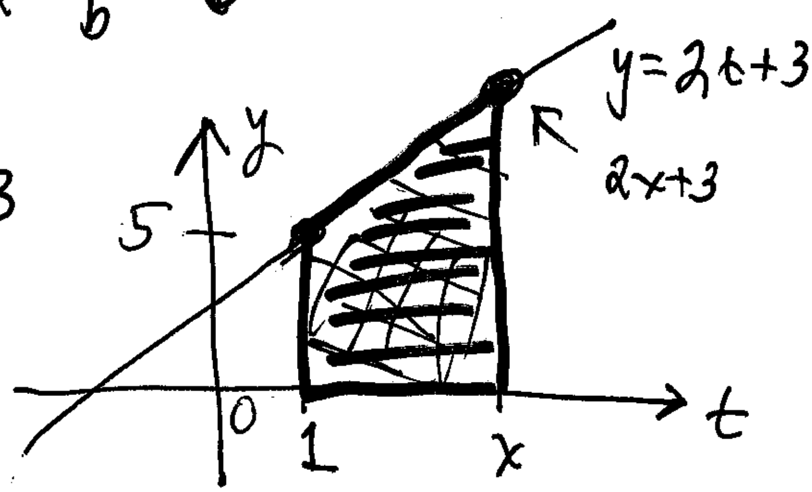
$y = f(x)$ define area function

$[a, b]$

$$\boxed{A(x) = \int_a^x f(t)\,dt}$$

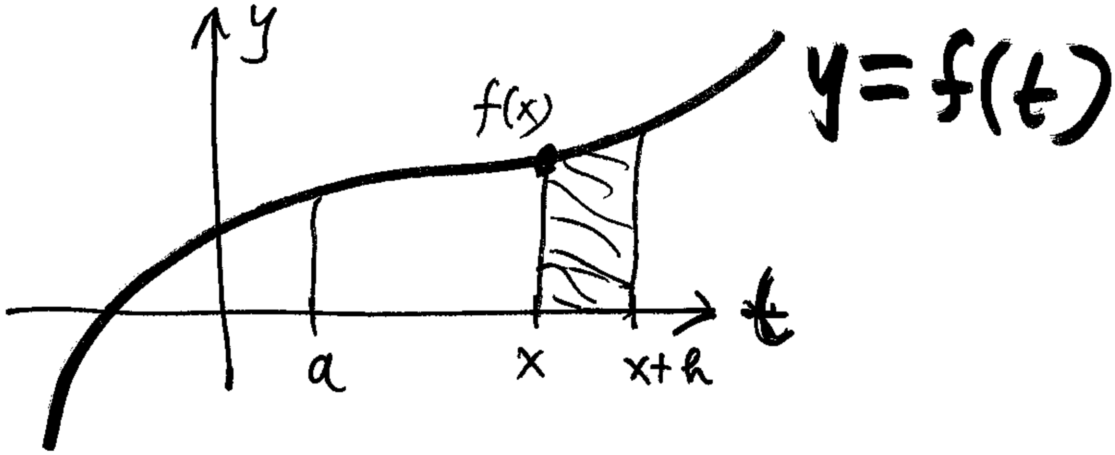


For eg $y = f(t) = 2t+3$



$$A(x) = \int_1^x (2t+3)\,dt$$

$$= \frac{1}{2}\left[(2x+3)+5\right](x-1) = x^2+3x-4$$



$$A(x+h) - A(x) \approx f(x)\,h$$

$$\Rightarrow \quad A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

<u>Fundamental Thm of Calculus (Part I)</u>

If f is cont. $[a,b]$ and diff. on (a,b)

then $$\boxed{\frac{d}{dx}\left\{\int_a^x f(t)\,dt\right\} = f(x)}$$ <u>Leibnitz' Rule</u>

<u>Remark</u>: i.e. $A(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$, since $A'(x) = f(x)$

If $F(x)$ is any antiderivative of $f(x)$

then $F(x) = A(x) + C$

$$F(b)-F(a) = \{A(b)+C\} - \{A(a)+C\}$$
$$= A(b) - A(a)$$
$$= \int_a^b f(t)\,dt - \cancelto{0}{\int_a^a f(t)\,dt}$$

Fundamental Thm of Calculus (Part II)

If $f(x)$ is cont. on $[a,b]$ and $F(x)$ any antiderivative of $f(x)$

then $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Notation: $$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$$

Ex1 Evaluate

(a) $\int_0^1 (\sqrt{x} + e^x)\,dx = \int_0^1 (x^{1/2} + e^x)\,dx$

Note: $F(x) = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + e^x = \dfrac{2}{3}x^{3/2} + e^x$

check: $F'(x) = \sqrt{x} + e^x$?

$= \left(\dfrac{2}{3}x^{3/2} + e^x\right)\Big|_{x=0}^{x=1}$

$= \left(\dfrac{2}{3} + e^1\right) - \left(0 + e^0\right)$

$= e - \frac{1}{3}$ ✓

(b) $\int_1^2 \dfrac{4}{x}\,dx = 4\ln|x|\,\Big|_{x=1}^{x=2}$

$= (4\ln 2) - (4\ln 1)$

$= 4\ln 2$ ✓

(c) $\displaystyle\int_0^{\pi/6} 8\sin w\,dw = \left(-8\cos w\right)\Big|_{w=0}^{w=\pi/6}$

$= \left(-8\cos\frac{\pi}{6}\right) - \left(-8\cos 0\right)$

$= \left(-8\left(\frac{\sqrt{3}}{2}\right)\right) + 8$ ✓

$\boxed{\text{Ex 2}}$ Differentiate

(a) $\displaystyle\int_1^{x} \sin^3(2t)\,dt$

$\displaystyle\frac{d}{dx}\left(\int_1^{x} \sin^3(2t)\,dt\right) = \sin^3(2x)$ ✓

(b) $\displaystyle\int_{(x^2+1)}^{7} \frac{1}{p+2}\,dp$

$\displaystyle\frac{d}{dx}\left(\int_{x^2+1}^{7} \frac{1}{p+2}\,dp\right) = -\frac{d}{dx}\left(\int_7^{x^2+1} \frac{1}{p+2}\,dp\right)$

let $u = x^2+1$

$$\Rightarrow = -\frac{d}{dx}\left(\int_7^{x^2+1} \frac{1}{p+2}\,dp\right) = -\frac{d}{du}\left(\int_7^{u} \frac{1}{p+2}\,dp\right)\frac{du}{dx}$$

$$= -\left(\frac{1}{u+2}\right)(2x) = \frac{-2x}{(x^2+1)+2} \checkmark$$

$\boxed{\text{Ex3}}$ If $g(x) = \int_1^x 6p(p+2)^2(4-p)\,dp$

where is $g \nearrow, \searrow$?

<u>Soln</u>: $g'(x) = 6x(x+2)^2(4-x)$ ✓