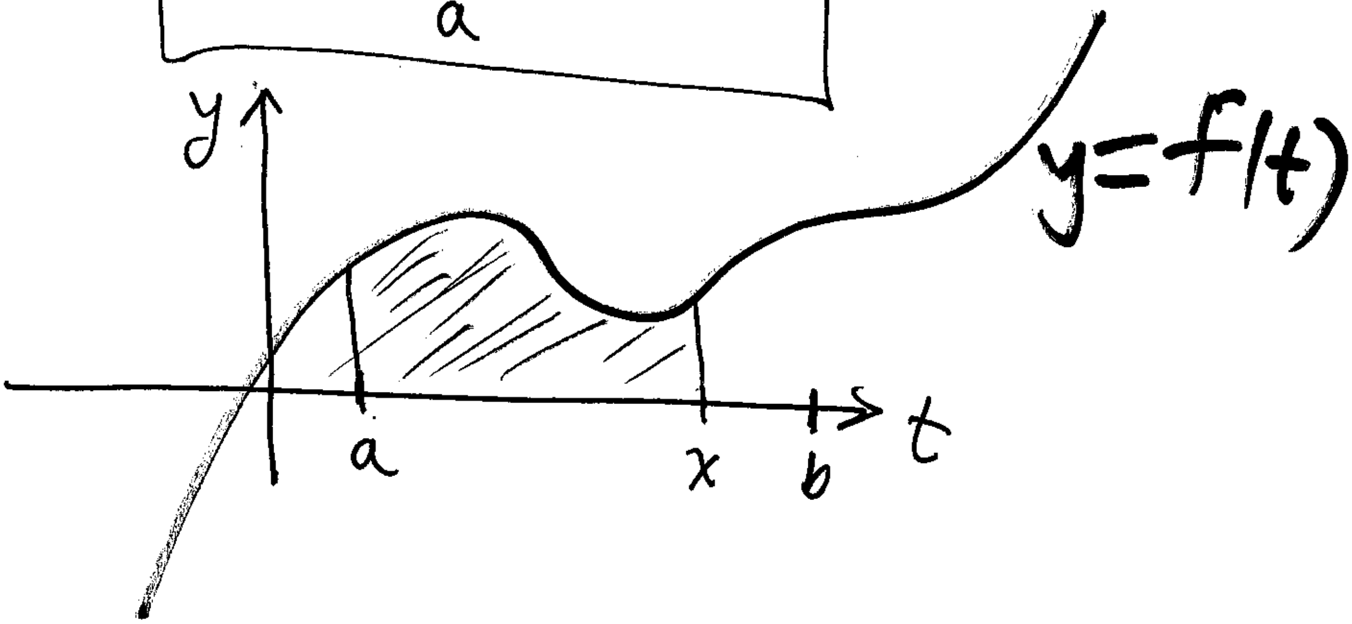
Lesson 32

$\boxed{\S 5.3}$ Fundamental Thm of Calculus

$y = f(t)$ define the area function of f

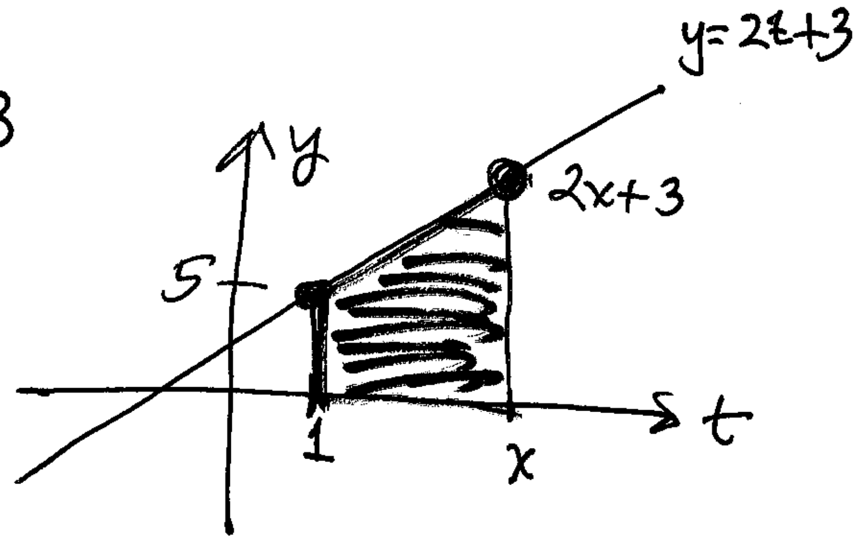
$[a, b]$

$$\boxed{A(x) = \int_a^x f(t)\,dt}$$



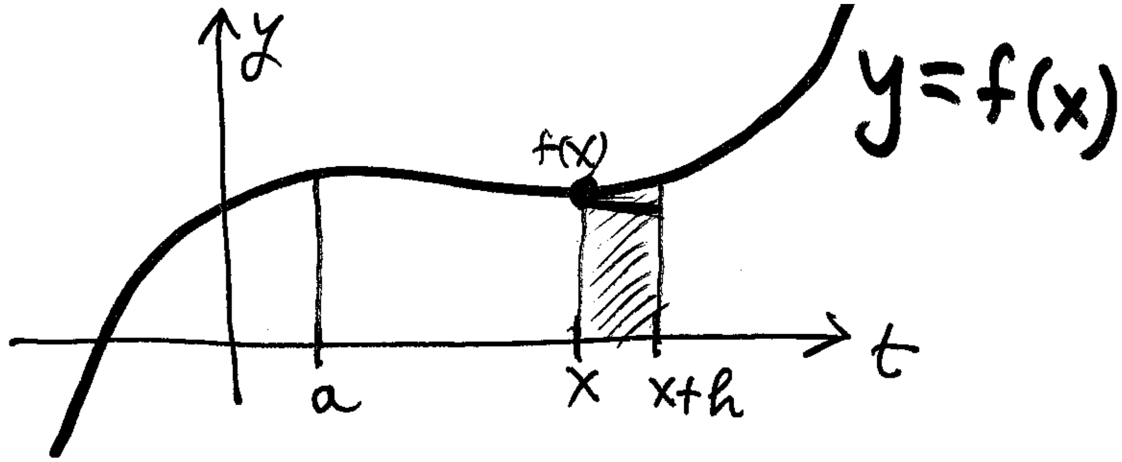
For eg, $y = f(t) = 2t + 3$

$$A(x) = \int_1^x (2t+3)\,dt$$



$$= \frac{1}{2}\left((2x+3) + 5\right)(x-1)$$

$$= x^2 + 3x - 4$$



$$\frac{A(x+h)-A(x)}{h} \approx \frac{f(x)\cancel{h}}{\cancel{h}}$$

$$\Rightarrow A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x)$$

Hence,

Fundamental Thm of Calculus (Part I)

If f cont. on $[a,b]$, diff. on (a,b) then

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

Leibnitz' Rule

Remark: i.e. $A(x)=\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$; since $A'(x)=f(x)$

If $F(x)$ is any antiderivative of f then $F(x)=A(x)+C$.

$$\therefore F(b)-F(a)=\{A(b)+\cancel{C}\}-\{A(a)+\cancel{C}\}$$

$$=A(b)-A(a)$$

$$=\int_a^b f(t)\,dt-\underbrace{\cancel{\int_a^a f(t)\,dt}}_{=0}$$

Fundamental Thm of Calculus (Part II)

If $f(x)$ is cont. on $[a,b]$ and $F(x)$ any antiderivative of $f(x)$ then

$$\boxed{\int_a^b f(x)\,dx=F(b)-F(a)}$$

Notation: $\int_a^b f(x)\,dx=F(x)\Big|_{x=a}^{x=b}=F(b)-F(a)$

$\boxed{\text{Ex1}}$ (a) $\int_0^1 (\sqrt{x} + e^x)\,dx$

$$= \int_0^1 (x^{1/2} + e^x)\,dx = F(x)\Big|_{x=0}^{x=1} = F(1) - F(0)$$

$$F(x) = \frac{x^{1/2+1}}{1/2+1} + e^x = \frac{2}{3}x^{3/2} + e^x$$

<u>check</u>: $F'(x) = \sqrt{x} + e^x$

$$= \left(\frac{2}{3} + e^1\right) - \left(\frac{2}{3}\cdot 0 + e^0\right) = e - \frac{1}{3}$$ ✓

(b) $\int_1^2 \frac{4}{x}\,dx = \left(4\ln|x|\right)\Big|_{x=1}^{x=2}$

$$= (4\ln 2) - (4\ln 1)$$

$$= 4\ln 2$$ ✓

(c) $\displaystyle\int_0^{\pi/6} 8\sin w\,dw = \left(-8\cos w\right)\Big|_{w=0}^{w=\pi/6}$

$= \left(-8\cos\frac{\pi}{6}\right) - \left(-8\cos 0\right)$

$= \left(-8\,\frac{\sqrt{3}}{2}\right) + 8$ ✓

$\boxed{\text{Ex2}}$ Differentiate

(a) $\displaystyle\int_1^{x} \sin^3(2t)\,dt$

$\therefore \displaystyle\frac{d}{dx}\left(\int_1^{x} \sin^3(2t)\,dt\right) = \sin^3(2x)$ ✓

(b) $\displaystyle\int_{(x^2+1)}^{7} \frac{1}{p+2}\,dp$

$\displaystyle\frac{d}{dx}\left(\int_{x^2+1}^{7} \frac{1}{p+2}\,dp\right) = -\frac{d}{dx}\left(\int_{7}^{x^2+1} \frac{1}{p+2}\,dp\right)$

let $u = x^2+1$

$$= -\frac{d}{du}\left(\int_7^u \frac{1}{p+2}\,dp\right)\frac{du}{dx}$$

$$= -\left(\frac{1}{u+2}\right)(2x) = -\frac{2x}{(x^2+1)+2}$$ ✓