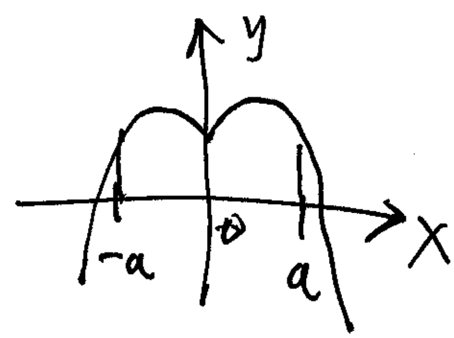
Lesson 33

$\boxed{\S\S\, 5.4 + 5.5}$

$$\underbrace{\int_a^b f(x)\,dx}_{\text{net area under } y=f(x) \text{ over } [a,b]} = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a) \qquad F'(x) = f(x)$$

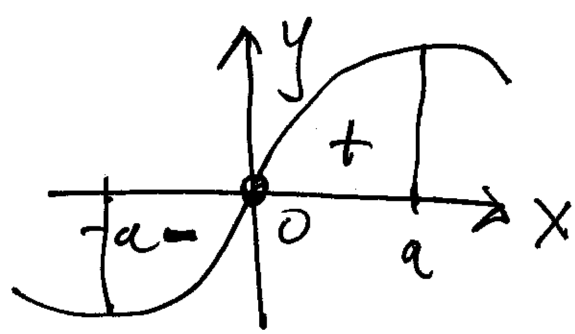
If $f(x)$ is an even fcn i.e. $f(-x) = f(x)$

then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



If $f(x)$ is an odd fcn i.e. $f(-x) = -f(x)$

then $\int_{-a}^{a} f(x)\,dx = 0$



For eg $\displaystyle\int_{-5}^{5}(|x|+4x^3)\,dx = \underbrace{\int_{-5}^{5}|x|\,dx}_{} + \underbrace{\int_{-5}^{5}4x^3\,dx}_{0}$

$$= 2\int_0^5 |x|\,dx$$

$$= 2\int_0^5 x\,dx = 25$$

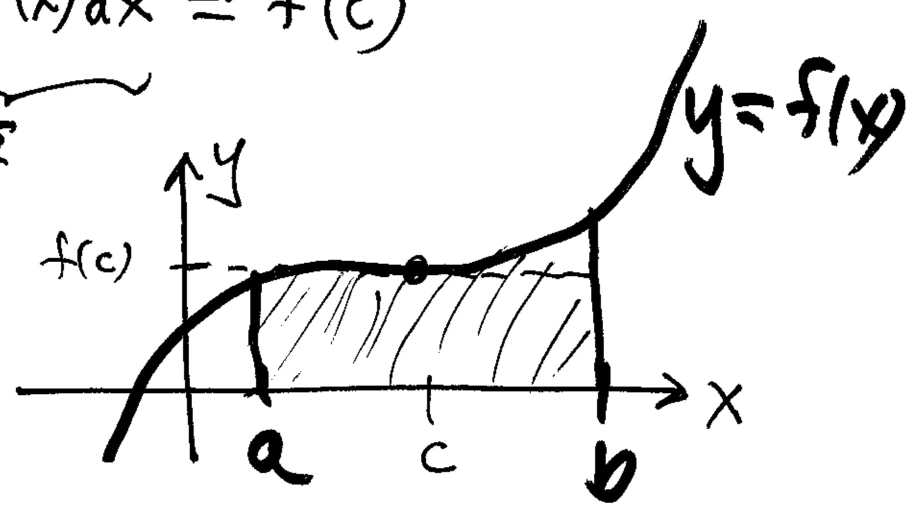
Def: The average value of $f(x)$ over $[a,b]$ is

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Mean Value Thm for Integrals:

If f is cont. on $[a,b]$ then there is a c in (a,b) such that

$$\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\bar{f}} = f(c)$$



$\boxed{\text{Ex 1}}$ Find average value $\bar{f}$ of
$f(x) = 2x(3x-1)$ over $[0,4]$.
Find c in $(0,4)$ where $f(c) = \bar{f}$

<u>Soln</u>: $\bar{f} = \frac{1}{4-0}\int_0^4 2x(3x-1)\,dx$

$= \frac{1}{4}\int_0^4 (6x^2 - 2x)\,dx$

$= \frac{1}{4}\left[2x^3 - x^2\right]\Big|_{x=0}^{x=4}$

$= \frac{1}{4}\left[2(4)^3 - 4^2\right] - \frac{1}{4}[0] = 28$ ✓

Solve $f(x) = 28$

$2x(3x-1) = 28$

$6x^2 - 2x - 28 = 0$

$\Rightarrow \quad x = 7/3 \quad$ or $\quad \cancel{x = -2}$

$c = 7/3$

Suppose $F'(x) = f(x)$ (i.e $F(x)$ is antiderivative of $f(x)$)

By <u>Chain Rule</u>

$$\frac{d}{dx}\{F(g(x))\} = F'(g(x))g'(x) = f(g(x))g'(x)$$

$$\therefore \int f(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int f(u)du$$

let $\boxed{u = g(x)}$

<u>Substitution Rule:</u>

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

<u>where</u> $\boxed{u = g(x)}$

$\boxed{\text{Ex 2}}$ Evaluate

(a) $\int 2x(x^2+1)^{10}\,dx = \int u^{10}\,du$

let $\boxed{u = x^2+1}$

$\frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x\,dx}$

$= \frac{u^{11}}{11} + C$

$= \frac{(x^2+1)^{11}}{11} + C$

(b) $\int \frac{4x^2}{x^3-8}\,dx = \int \frac{4}{u}\left(\frac{du}{3}\right)$

let $\boxed{u = x^3-8}$

$\frac{du}{dx} = 3x^2$

$\boxed{du = 3x^2\,dx}$

$= \frac{4}{3}\int \frac{1}{u}\,du$

$= \frac{4}{3}\ln|u| + C$

$= \frac{4}{3}\ln|x^3-8| + C$

c) $\int \frac{\sqrt{\ln(4p)}}{p}\,dp = \int \sqrt{u}\,du$

let $\boxed{u = \ln(4p)}$ $\qquad = \int u^{1/2}\,du$

$\frac{du}{dp} = \frac{1}{p} \therefore \boxed{du = \frac{1}{p}\,dp}$ $\qquad = \frac{u^{3/2}}{3/2} + C$

$= \frac{2}{3}\left(\ln(4p)\right)^{3/2} + C$

d) $\int \frac{5e^x}{1+e^{2x}}\,dx = \int \frac{5}{1+u^2}\,du$

let $\boxed{u = e^x}$

$du = e^x\,dx$

$= 5\tan^{-1} u + C$

$= 5\tan^{-1}(e^x) + C$