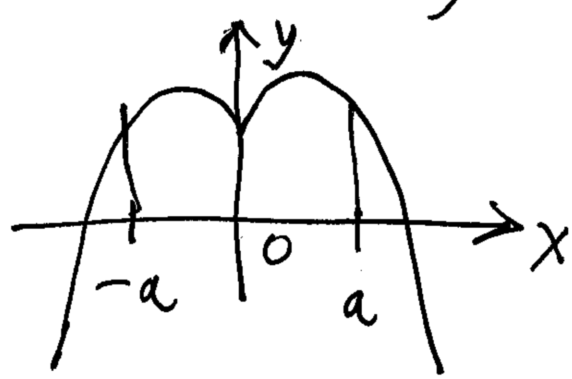
Lesson 33

$\boxed{\S\S 5.4 + 5.5}$

$$\underbrace{\int_a^b f(x)\,dx}_{\substack{\text{Net area} \\ \text{under } y=f(x) \\ \text{over } [a,b]}} = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a) \qquad F'(x) = f(x)$$

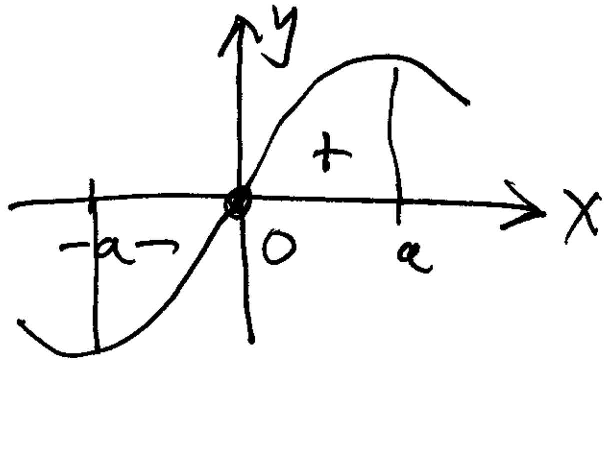
If $f(x)$ is an <u>even</u> fcn i.e. $f(-x) = f(x)$

then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



If $f(x)$ is an <u>odd</u> fcn i.e., $f(-x) = -f(x)$

then $\int_{-a}^{a} f(x)\,dx = 0$



For eg, $\int_{-5}^{5} (|x| + 4x^3)\,dx$

$$= \int_{-5}^{5} |x|\,dx + \underbrace{\int_{-5}^{5} 4x^3\,dx}_{=0}$$

$$= 2\int_{0}^{5} |x|\,dx$$

$$= 2\int_{0}^{5} x\,dx = 25$$

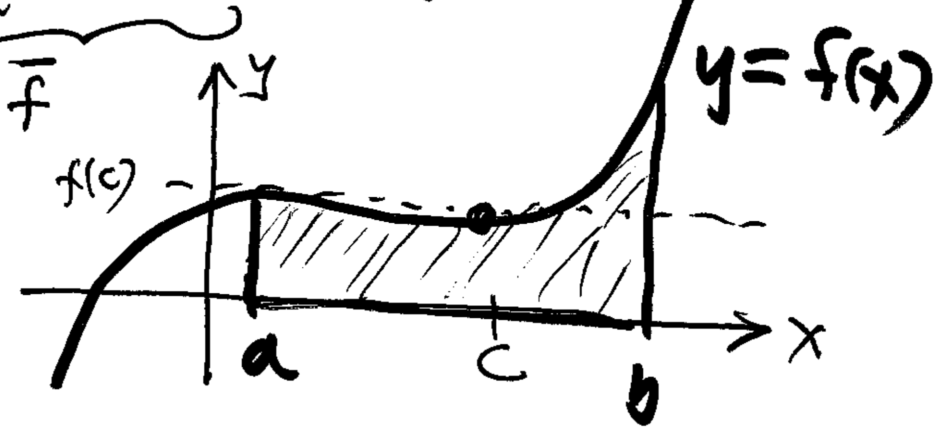
Def: The average value of $f(x)$ over $[a, b]$ is

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

Mean Value Thm For Integrals:

If f is cont. on $[a, b]$, then there is a c in (a, b) such that

$$\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\bar{f}} = f(c)$$



$\boxed{\text{Ex1}}$ Find average value $\bar{f}$ of

$f(x) = 6x^2 - 2x$ over $[0, 4]$.

Find c in $(0,4)$ where $f(c) = \bar{f}$

Soln: $\bar{f} = \frac{1}{4-0}\int_0^4 (6x^2 - 2x)\,dx$

$= \frac{1}{4}\left[2x^3 - x^2\right]\Big|_{x=0}^{x=4} = \frac{1}{4}\left[2(4)^3 - 4^2\right] - \frac{1}{4}[0]$

$= 28$ ✓

Solve $f(\cancel{x}) = \bar{f}$

$6x^2 - 2x = 28$

$6x^2 - 2x - 28 = 0$

$x = 7/3$, $\cancel{x = -2}$

$\therefore$ $c = 7/3$

Suppose $F'(x) = f(x)$ i.e $F(x)$ is antiderivative of $f(x)$

Chain Rule $\Rightarrow$

$$\frac{d}{dx}\{F(g(x))\} = F'(g(x))g'(x) = f(g(x))g'(x)$$

$$\therefore \int f(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int f(u)du$$

let $\boxed{u = g(x)}$

<u>Substitution Rule:</u>

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

where $\boxed{u = g(x)}$

$\boxed{\text{Ex2}}$ Evaluate

ⓐ $\int 2x(x^2+1)^{10}\,dx = \int u^{10}\,du = \dfrac{u^{11}}{11} + C$

let $\boxed{u = x^2+1}$

$\dfrac{du}{dx} = 2x \therefore \boxed{du = 2x\,dx}$

$= \dfrac{(x^2+1)^{11}}{11} + C$

ⓑ $\int \dfrac{4x^2}{x^3-8}\,dx = \int \dfrac{1}{u}(4)\dfrac{du}{3}$

let $\boxed{u = x^3-8}$

$\dfrac{du}{dx} = 3x^2 \therefore \boxed{du = 3x^2\,dx}$

$= \dfrac{4}{3}\int \dfrac{1}{u}\,du$

$= \dfrac{4}{3}\ln|u| + C = \dfrac{4}{3}\ln|x^3-8| + C$

(c) $\int \frac{\sqrt{\ln(4p)}}{p}\,dp = \int \sqrt{u}\,du$

let $\boxed{u = \ln(4p)}$ $= \int u^{1/2}\,du$

$\frac{du}{dp} = \frac{1}{p} \therefore \boxed{du = \frac{1}{p}\,dp}$ $= \frac{u^{3/2}}{3/2} + C$

$= \frac{2}{3}\left(\ln(4p)\right)^{3/2} + C$

(d) $\int \frac{5e^x}{1+e^{2x}}\,dx = \int \frac{5}{1+u^2}\,du$

let $\boxed{u = e^x}$ $= 5\tan^{-1} u + C$

$du = e^x\,dx$ $= 5\tan^{-1}(e^x) + C$