Lesson 34

$\boxed{\S 5.5}$ (cont'd)

Substitution Rule : $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

$$\boxed{u = g(x)}$$

For eg, $\int \frac{(\ln x)^3}{x}\,dx$, $u = (\ln x)^3$ $\times$

$\frac{du}{dx} = 3(\ln x)^2\left(\frac{1}{x}\right)$

$du = 3(\ln x)^2\left(\frac{1}{x}\right)dx$

let $\boxed{u = \ln x}$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\therefore$ $\boxed{du = \frac{1}{x}\,dx}$

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(\ln x)^4}{4} + C$$

$\boxed{\text{Ex 1}}$ Evaluate

(a) $I = \displaystyle\int \frac{e^{3x}}{1+e^{3x}}\,dx$; $J = \displaystyle\int \frac{e^{3x}}{1+e^{6x}}\,dx$

Soln:

$$I = \int \frac{e^{3x}}{1+e^{3x}}\,dx = \int \frac{1}{u}\left(\frac{1}{3}\,du\right)$$

let $\boxed{u = 1+e^{3x}}$

$\frac{du}{dx} = 3e^{3x}$

$\therefore \boxed{du = 3e^{3x}\,dx}$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|1+e^{3x}| + C$$

$$J = \int \frac{e^{3x}}{1+e^{6x}}\,dx = \int \frac{1}{1+u^2}\left(\frac{1}{3}\,du\right)$$

let $\boxed{u = e^{3x}}$

$\boxed{du = 3e^{3x}\,dx}$

$$= \frac{1}{3}\tan^{-1}u + C$$

$$= \frac{1}{3}\tan^{-1}\left(e^{3x}\right) + C$$

(b) $I = \int 2 \sin x \cos x \, dx = \int 2u \, du$

$\boxed{u = \sin x}$

$du = \cos x \, dx$

$$= u^2 + C$$

$$= \underline{\sin^2 x + C} \checkmark$$

or, let $\boxed{u = \cos x}$

$du = -\sin x \, dx$

$\therefore \; I = \int 2u(-du) = \int -2u \, du$

$$= -u^2 + K$$

$$= \underline{\underline{-\cos^2 x + K}}$$

Same answer:

$$\sin^2 x + C = (1 - \cos^2 x) + C$$

$$= -\cos^2 x + \underbrace{(1 + C)}_{K}$$

Substitution Rule (definite integrals)

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$\boxed{u = g(x)}$

$\boxed{\text{Ex 2}}$ Compute

(a) $\displaystyle\int_0^4 \frac{1}{16+x^2}\,dx = \int_0^4 \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx$

$\displaystyle = \frac{1}{16}\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$ let $\boxed{u = \frac{x}{4}}$

$\frac{du}{dx} = \frac{1}{4}$

$du = \frac{1}{4}dx$

$\displaystyle = \frac{1}{16}\int_0^1 \frac{1}{1+u^2}(4\,du)$

$\displaystyle = \frac{1}{4}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{4}\tan^{-1}u \Big|_{u=0}^{u=1}$

$= \left(\frac{1}{4}\tan^{-1}1\right) - \left(\frac{1}{4}\tan^{-1}0\right)$

$= \frac{1}{4}\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$ ✓

(b) $\int_0^3 2^x \, dx = \int_0^3 e^{\ln\{2^x\}} dx$

$= \int_0^3 e^{x(\ln 2)} dx = \int_0^{3\ln 2} e^u \left(\frac{du}{\ln 2}\right)$

let $\boxed{u = x \ln 2}$

$\frac{du}{dx} = \ln 2$

$\therefore \boxed{du = (\ln 2) dx}$

$= \frac{1}{\ln 2} \int_0^{\ln 8} e^u \, du$

$= \frac{1}{\ln 2} e^u \Big|_{u=0}^{u=\ln 8}$

$= \left(\frac{1}{\ln 2} e^{\ln 8}\right) - \left(\frac{1}{\ln 2} e^0\right)$

$= \frac{1}{\ln 2}(8) - \frac{1}{\ln 2} = \frac{7}{\ln 2}$ ✓

(c) $$\int_{\frac{1}{4}}^{1} \frac{\sin \pi\sqrt{t}}{\sqrt{t}}\,dt = \int_{\frac{\pi}{2}}^{\pi} \sin u \left(\frac{2}{\pi}\,du\right)$$ (6)

let $\boxed{u = \pi\sqrt{t}}$

$\Rightarrow \boxed{du = \frac{\pi}{2\sqrt{t}}\,dt}$

$$= \left(-\frac{2}{\pi}\cos u\right)\Bigg|_{u=\frac{\pi}{2}}^{\pi}$$

$$= \left(-\frac{2}{\pi}\cos \pi\right) - \left(-\frac{2}{\pi}\cos\frac{\pi}{2}\right)$$

$$= \frac{2}{\pi}$$ ✓