Lesson 34

$\boxed{\S 5.5}$ (cont'd)

Substitution Rule: $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

let $\boxed{u = g(x)}$

For eg, $I = \int \frac{(\ln x)^3}{x}\,dx$

if $u = (\ln x)^3$

$\frac{du}{dx} = 3(\ln x)^2\left(\frac{1}{x}\right)$

$du = \frac{3(\ln x)^2}{x}\,dx$

NO

let $\boxed{u = \ln x}$

$\frac{du}{dx} = \frac{1}{x}$ $\therefore$ $\boxed{du = \frac{1}{x}\,dx}$

Then $I = \int u^3\,du$

$= \frac{u^4}{4} + C$

$= \frac{(\ln x)^4}{4} + C$ ✓

$\boxed{\text{Ex 1}}$ Evaluate

ⓐ $I=\int \frac{e^{3x}}{1+e^{3x}}\,dx$; $J=\int \frac{e^{3x}}{1+e^{6x}}\,dx$

Soln: $I=\int \frac{e^{3x}}{1+e^{3x}}\,dx=\int \frac{1}{1+e^{3x}}\,e^{3x}dx$

let $\boxed{u=1+e^{3x}}$

$\frac{du}{dx}=3e^{3x}$

$\therefore\ \boxed{du=3e^{3x}dx}$

$=\int \frac{1}{u}\left(\frac{1}{3}du\right)$

$=\frac{1}{3}\ln|u|+C$

$=\frac{1}{3}\ln|1+e^{3x}|+C$

$J=\int \frac{e^{3x}}{1+e^{6x}}\,dx=\int \frac{1}{1+u^2}\left(\frac{du}{3}\right)$

let $\boxed{u=e^{3x}}$

$\frac{du}{dx}=3e^{3x}$

$\therefore\ \boxed{du=3e^{3x}dx}$

$=\frac{1}{3}\tan^{-1}u+C$

$=\frac{1}{3}\tan^{-1}(e^{3x})+C$

(b) $I = \int 2 \sin x \cos x \, dx = \int 2u \, du$

let $\boxed{u = \sin x}$

$\boxed{du = \cos x \, dx}$

$= u^2 + C$

$= \sin^2 x + C$ ✓

Or, let $\boxed{u = \cos x}$

$\boxed{du = -\sin x \, dx}$

$\therefore I = \int 2 \sin x \cos x \, dx = \int 2u(-du)$

$= \int -2u \, du = -u^2 + K$

$= -\cos^2 x + K$ ✓

Same answer: $\sin^2 x + C = (1 - \cos^2 x) + C$

$= -\cos^2 x + \underbrace{(1 + C)}_{K}$

Substitution Rule (definite integrals)

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

$$\boxed{u = g(x)}$$

$\boxed{\text{Ex2}}$ Compute

ⓐ $$\int_0^4 \frac{1}{16+x^2}\,dx = \int_0^4 \frac{1}{16\left\{1+\frac{x^2}{16}\right\}}\,dx$$

$$= \frac{1}{16}\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$$

let $\boxed{u = \frac{x}{4}}$

$\frac{du}{dx} = \frac{1}{4}$

$\therefore \boxed{du = \frac{1}{4}\,dx}$

$$= \frac{1}{16}\int_0^1 \frac{1}{1+u^2}(4\,du)$$

$$= \frac{1}{4}\tan^{-1}u \Big|_{u=0}^{u=1}$$

$$= \left(\frac{1}{4}\tan^{-1}1\right) - \left(\frac{1}{4}\tan^{-1}0\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16} \checkmark$$

(b) $\int_0^3 2^x\,dx = \int_0^3 e^{\ln\{2^x\}}\,dx$

$= \int_0^3 e^{x(\ln 2)}\,dx = \int_0^{3\ln 2} e^u \frac{du}{\ln 2}$

let $\boxed{u = x(\ln 2)}$

$\frac{du}{dx} = \ln 2$

$\therefore \boxed{du = (\ln 2)\,dx}$

$= \frac{1}{\ln 2}\int_0^{\ln 8} e^u\,du$

$= \left(\frac{1}{\ln 2} e^u\right)_{u=0}^{u=\ln 8}$

$= \left(\frac{1}{\ln 2} e^{\ln 8}\right) - \left(\frac{1}{\ln 2} e^0\right)$

$= \frac{8}{\ln 2} - \frac{1}{\ln 2} = \frac{7}{\ln 2}$ ✓