§7.2 Exponential Models

If $y(t)$ satisfies IVP $\begin{cases} \frac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$

then $\boxed{y(t) = y_0 e^{kt}}$

k = rate constant

y_0 = initial value

$y(t)$ has
- exponential growth if $k > 0$
- exponential decay if $k < 0$

Doubling Time: If $y(0) = y_0$ when $y(t) = 2y_0$?

$\therefore 2y_0 = y(t) = y_0 e^{kt}$

$\Rightarrow$ Doubling Time is $\boxed{T_2 = \frac{\ln 2}{k}}$

Ex1 In 2020, pop. in a city was 180,000. If pop. grows at 2.3% per year, when will pop. be 250,000?

Soln: $y(t) = y_0 e^{kt}$

year	pop.
(t=0) 2020	180,000
?	250,000

$\therefore y(t) = 180{,}000 e^{kt}$

Since $y(1) = 180{,}000(1.023) = 180{,}000 e^{k}$

$\therefore k = \ln(1.023)$

So $\boxed{y(t) = 180{,}000 e^{t\ln(1.023)}}$

$250{,}000 = y(t) = 180{,}000 e^{t\ln(1.023)}$

$\vdots$

$$t = \frac{\ln\left(\frac{250{,}000}{180{,}000}\right)}{\ln(1.023)} \approx \underline{14.45} \text{ yrs}$$

Doubling Time: $T_2 = \frac{\ln 2}{k} = \frac{\ln 2}{\ln 1.023}$

$\approx \underline{29.97}$ yrs

Ex 2 The # of E. coli doubles every 8 hrs, starting with 6 bacteria. How long does it take to have 100 bacteria?

Soln: $y(t) = 6e^{kt}$; $T_2 = \frac{\ln 2}{k} = 8$

$\therefore \boxed{y(t) = 6e^{t\left[\frac{\ln 2}{8}\right]}}$ $\quad k = \frac{\ln 2}{8}$

$100 = y(t) = 6e^{t\left[\frac{\ln 2}{8}\right]}$

$\vdots$

$$t = \frac{\ln\left(\frac{100}{6}\right)}{\left(\frac{\ln 2}{8}\right)} \approx \underline{32.47} \text{ hrs}$$

Exp. Decay: $y(t) = y_0 e^{kt}$

Half-life: If $y(0) = y_0$, $y(t) = \frac{1}{2} y_0$

$y_0 e^{kt} = \frac{1}{2} y_0$ ∴ Half-Life $\boxed{T_{\frac{1}{2}} = \frac{\ln(\frac{1}{2})}{k}}$

Ex3 Half-Life of C-14 is 5730 yrs. A fossilized bone has 20% of C-14 as a live bone. How old is bone?

Soln: $y(t) = y_0 e^{kt}$; $T_2 = \frac{\ln(\frac{1}{2})}{k} = 5730$

∴ $k = \frac{\ln(\frac{1}{2})}{5730}$

$\boxed{y(t) = y_0 e^{\frac{t \ln(\frac{1}{2})}{5730}}}$

Find t so that $y(t) = 0.20 y_0$

$y_0 e^{\frac{t \ln(\frac{1}{2})}{5730}} = 0.20 y_0$

⋮

$t = 5730 \frac{\ln(0.20)}{\ln(\frac{1}{2})} \approx 13{,}304.6$ yrs