# Lesson 35

## §2.2 Exponential Models

If $y(t)$ satisfies IVP $\begin{cases} \frac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$

then $\boxed{y(t) = y_0 e^{kt}}$ $k$ = rate constant

$y_0$ = initial value of $y$

$y(t)$ has
- <u>exponential growth</u> if $k > 0$
- <u>exponential decay</u> if $k < 0$

<u>Doubling Time</u>: If $y(0) = y_0$, $y(t) = 2y_0$?

$$2y_0 = y(t) = y_0 e^{kt}$$

$\Rightarrow$ <u>Doubling Time</u> $\boxed{T_2 = \frac{\ln 2}{k}}$

**Ex1** In 2020, pop. in a city 180,000. If pop. grows at 2.3% per year, when will pop. be 250,000?

Soln:

| year | pop. |
|---|---|
| (t=0) 2020 | 180,000 |
| ? | 250,000 |

$$y(t) = 180{,}000e^{kt}$$

Since $y(1) = 180{,}000(1.023) = 180{,}000e^{k}$

$\therefore k = \ln(1.023)$ ✓

Hence $\boxed{y(t) = 180{,}000e^{t\ln(1.023)}}$

$$250{,}000 = y(t) = 180{,}000\,e^{t\ln(1.023)}$$

$$\vdots$$

$$t = \frac{\ln\left(\frac{250{,}000}{180{,}000}\right)}{\ln(1.023)} \approx \underline{14.45 \text{ yrs}}$$

Doubling Time: $T_2 = \frac{\ln 2}{k} = \frac{\ln 2}{\ln(1.023)} \approx \underline{29.97\text{ yrs}}$

**Ex 2** The # of E. coli doubles every 8 hrs, starting with 6 bacteria. How long does it take to have 100 bacteria?

Soln: $y(t) = 6e^{kt}$ ; $T_2 = \frac{\ln 2}{k} = 8$

$k = \frac{\ln 2}{8}$

$\therefore$ $$y(t) = 6e^{t\left(\frac{\ln 2}{8}\right)}$$

$$100 = y(t) = 6\,e^{t\left(\frac{\ln 2}{8}\right)}$$

$\vdots$

$$t = \frac{\ln\left(\frac{100}{6}\right)}{\left(\frac{\ln 2}{8}\right)} \approx \underline{\underline{32.47}} \text{ hrs}$$

**Exp. Decay** : $y(t) = y_0 e^{kt}$

**Half-Life** : If $y(0) = y_0$, $y(t) = \frac{1}{2} y_0$ ?

$$y_0 e^{kt} = y(t) = \frac{1}{2} y_0$$

**Half-Life**: $$\boxed{T_{1/2} = \frac{\ln(\frac{1}{2})}{k}}$$

**Ex3** Half-life of C-14 is 5730 yrs.
A fossilized bone has 20% C-14 as a live bone.
How old is bone?

Soln: $y(t) = y_0 e^{kt}$ ; $T_{\frac{1}{2}} = \frac{\ln(\frac{1}{2})}{k} = 5730$

$k = \frac{\ln(\frac{1}{2})}{5730}$

$$\therefore \boxed{y(t) = y_0 e^{\frac{t \ln(\frac{1}{2})}{5730}}}$$

$$0.20 y_0 = y(t) = y_0 e^{\frac{t \ln(\frac{1}{2})}{5730}}$$

$$\Rightarrow t = \frac{5730 \ln(0.20)}{\ln(\frac{1}{2})} = 13{,}304.6 \text{ years}$$