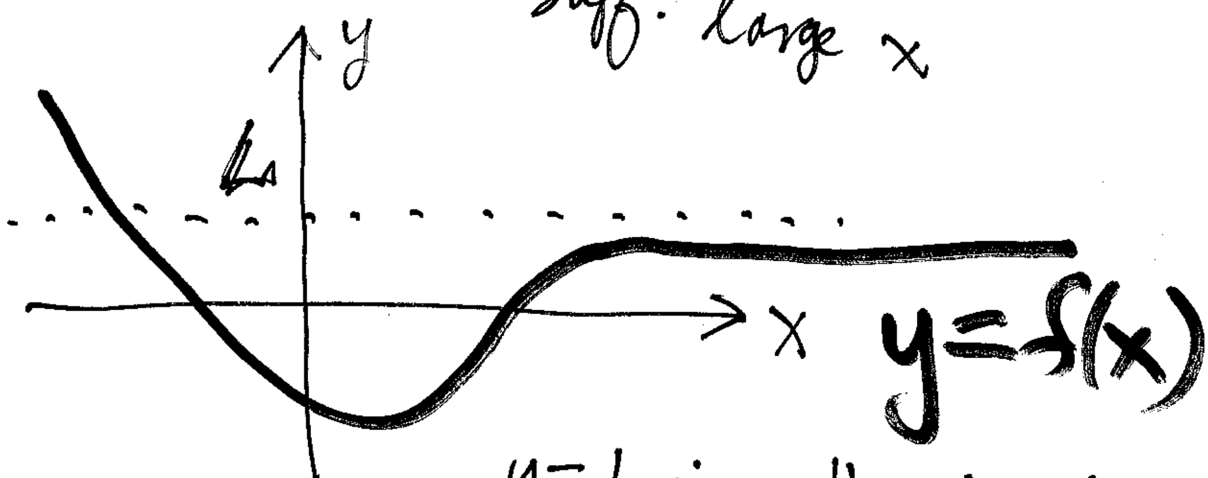
Lesson 6

§2.5 – Limits at ∞

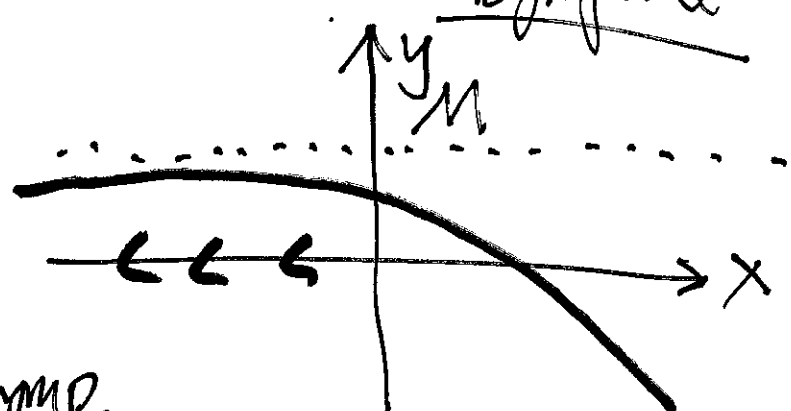
Last time: $\lim_{\substack{x \to 2^+ \\ (x>2)}} \frac{1-x^2}{x-2} = -\infty$ ($1-x^2 < 0$, $x-2 > 0$) $\therefore x=2$ is vertical asymp

Def: $\boxed{\lim_{x \to \infty} f(x) = L}$ means $f(x)$ is arbitrarily close to L for all suff. large x



$y=L$ is a Horizontal Asymptote of f

$\boxed{\lim_{x \to -\infty} f(x) = M}$



$y=M$
Horizontal Asymp.

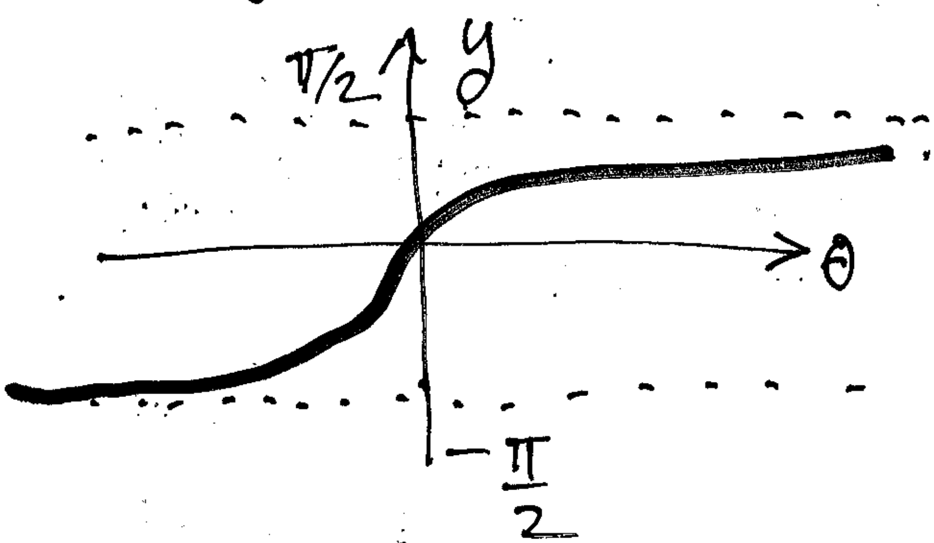
$\boxed{\text{Ex1}}$ Compute (if they exist)

(a) $\displaystyle\lim_{x\to\infty}\left(3+\frac{\cos x}{x}\right) = 3+0 = 3$

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$$

$-\frac{1}{x} \downarrow 0$ as $x\to\infty$ $\qquad$ $\frac{1}{x} \downarrow 0$ as $x\to\infty$

(b) $\displaystyle\lim_{\theta\to-\infty} \tan^{-1}\theta = -\frac{\pi}{2}$

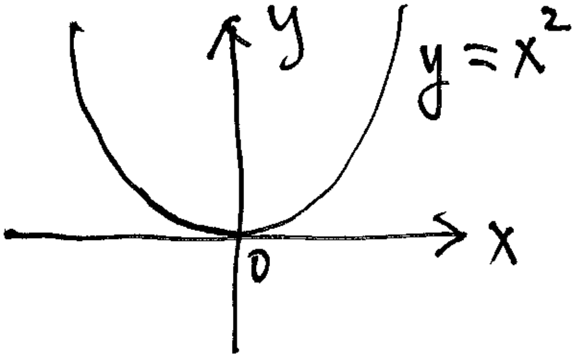


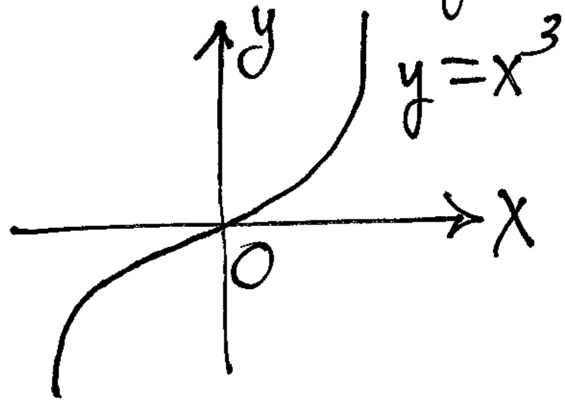
$y = \tan^{-1}\theta$

(c) $\displaystyle\lim_{x\to\infty} \sin x$ (DNE <u>and</u> is neither $\pm\infty$)

Def: (Infinite limit at ∞)

$\boxed{\lim_{x\to\infty} f(x) = \infty}$ means f grows arb. large as x grows arb. large





Theorem (Limits of Powers) If $n = 1, 2, 3, \cdots$ then

① $\lim_{x\to\pm\infty} x^n = \infty$ (n even)

② $\left.\begin{array}{l} \lim_{x\to\infty} x^n = \infty \\ \lim_{x\to-\infty} x^n = -\infty \end{array}\right\}$ (n odd)

③ $\lim_{x\to\pm\infty} \frac{1}{x^n} = 0$

dominant term

④ If $p(x) = a_n x^n + \cdots + a_1 x + a_0$ then

$$\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n = \pm\infty$$

The behavior of $f(x)$ ~~as~~ as $x \to \pm\infty$ is the <u>end behavior</u> of $f(x)$.

<u>Def</u>: If graph of f approaches a line that is neither horizontal nor vertical as $x \to \pm\infty$, the line is a <u>slant/oblique asymptote</u>.

From page 97: $f(x) = \dfrac{2x^2+6x-2}{x+1}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{2x^2+6x-2}{x^2}}{\frac{x+1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2+\frac{6}{x}-\frac{2}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \infty$$

Also $\lim_{x\to-\infty} f(x) = -\infty$

Slant Asympt: "Long Division"

$$\frac{2x^2+6x-2}{x+1} = (2x+4) + \underbrace{\left(\frac{-6}{x+1}\right)}_{\to 0 \text{ as } x\to\infty}$$

$$\begin{array}{r|l} & \;\; 2x+4 \\ \hline (x+1) & 2x^2+6x-2 \\ & 2x^2+2x \\ \hline & \qquad 4x-2 \\ & \qquad 4x+4 \\ \hline & \qquad\quad -6 \end{array}$$

$y = 2x+4$
slant asymp

Thm (End Behavior & Asymp. for Rational Functions)

Let $f(x) = \frac{p(x)}{q(x)}$, where $p(x) = a_m x^m + \cdots + a_0$
$q(x) = b_n x^n + \cdots + b_0$

1. If $\deg p(x) < \deg q(x) \Rightarrow \lim_{x \to \pm\infty} f(x) = 0 \quad \therefore y = 0$ Horiz. Asymp
2. If $\deg p(x) = \deg q(x) \Rightarrow \lim_{x \to \pm\infty} f(x) = \frac{a_m}{b_n} \quad \therefore y = \frac{a_m}{b_n}$ Hor. Asym
3. If $\deg p(x) > \deg q(x) \Rightarrow \lim_{x \to \pm\infty} f(x) = \infty$ (or $-\infty$) NO Horiz. Asym
4. If $\deg p(x) = 1 + \deg q(x) \Rightarrow$ slant asymptote
5. If $f(x) = \frac{p(x)}{q(x)}$ reduced form ← then zeros are vert. asympt.